TAIWANESE JOURNAL OF MATHEMATICS
Vol. 16, No. 5, pp. 1707-1717, October 2012
This paper is available online at http://journal.taiwanmathsoc.org.tw

THE CHAOS OF THE SOLUTION SEMIGROUP FOR THE QUASI-LINEAR LASOTA EQUATION

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Abstract. This paper is concerned with the solution semigroup of a quasi-linear Lasota equation. We show the existence and uniqueness of a solution semigroup for the quasi-linear Lasota equation. We also find a necessary and sufficient condition for the solution semigroup of the equation to be chaotic.

1. Introduction

First-order partial differential equations are frequently used as mathematical models for the population of cells. In this paper we consider the quasi-linear Lasota equation

$$\frac{\partial}{\partial t}u + c\,(x)\,\frac{\partial}{\partial x}u = g\,(x,u)\,, t \geq 0, 0 \leq x \leq 1, \tag{1.1}$$

with an initial condition

$$u\,(0,x) = v\,(x)\,, 0 \leq x \leq 1, \tag{1.1a}$$

where v is a continuously differentiable function, and c is a continuous function defined on $[0,1]$ with

$$c\,(0) = 0, c\,(x) > 0 \text{ for } x \in (0,1], \text{and } \int_0^1 \frac{dx}{c(x)} = \infty. \tag{1.2}$$

In this model, a cell is characterized by a single, scalar variable x to represent maturity, which is normalized to have values in $[0,1]$. The state of the population at time t is characterized by a density function $u\,(t,\cdot)$, i.e.,

$$\int_{x_1}^{x_2} u\,(t,x)\,dx$$

Received August 10, 2011, accepted November 17, 2011.
Communicated by Francois Hamel.
2010 *Mathematics Subject Classification*: 35K30, 47D03.
Key words and phrases: Lasota equation, C_0-semigroup.

which measures the quantity of cells that have a maturity between x_1 and x_2 at time t. The coefficient $c(x)$ is the velocity of cell differentiation and the function $g(x, u)$ is defined by

$$g(x, u) = u\left(p(x, u) - \frac{d}{dx}v(x)\right)$$

where $p(x, u)$ is the proliferation rate (the relative increase of number of cells per unit time).

The quasi-linear Lasota equation (1.1) has been developed as a model for the dynamics of a self-reproducing cell population, such as the population of developing red blood cell (erythrocyte precursors). It has also been applied to a conceptualization of abnormal blood cell production such as leukemia. Acute leukemia arises when the normal process of cell maturation is destabilized and immature dysfunctional precursor blood cells accumulate in the bone marrow. Normal blood cells population contains no more than 5% immature blast cells in bone marrow and none in the circulating blood. In leukemia blood cells population $30\% - 100\%$ of bone marrow cells are immature blast cells, and they are also presents in the circulating blood. The distinction between normal (stable) and abnormal (unstable, hypercyclic or chaotic) production of cells in the model lies in the initial presence of a sufficient supply of the most immature cells (cell of maturity 0). If $v(x) > 0$, then the population has the capacity to stabilize, but if $v(x) = 0$, then extreme instability can occur. The following example illustrates these behaviors. The mathematical model has also been discussed by Mackey and Schwegle in [5].

Consider the first-order partial differential equation

$$\frac{\partial}{\partial t}u + rx\frac{\partial}{\partial x}u = \alpha u(1-u), t \geq 0, 0 \leq x \leq 1 \tag{1.3}$$

with the initial condition

$$u(0, x) = \varphi(x), 0 \leq x \leq 1.$$

In (1.3), rx (with $r > 0$) is the velocity of aging and $\alpha \in R$ is a constant related to the relative proliferation and death rate of the cells.

When $\alpha = 0$, the solution of (1.3) is $u(t, x) = \varphi\left(xe^{-t}\right)$ and $\lim_{t\to\infty} u(t, x) = \varphi(0)$. In general, the solution of (1.3) is given by

$$u(t, x) = \frac{\varphi\left(xe^{-t}\right)e^{\alpha t}}{1 - \varphi\left(xe^{-t}\right)\left[1 - e^{\alpha t}\right]}. \tag{1.4}$$

As long as $\varphi(0) > 0$, the solution $u(t, x)$, given by (1.4), is globally stable, and in fact it has the property

$$\lim_{t\to\infty} u(t, x) = \begin{cases} 0, & \text{for } \alpha < 0; \\ 1, & \text{for } \alpha > 0. \end{cases}$$

However, if $\varphi(0) = 0$, the solutions are chaotic. This can be demonstrated by picking an initial function of the form

$$\varphi(y) = \beta y^n for y \in [0, 1], \beta \in (0, 1).$$

Then the solution $u(t, x)$ of (1.3) has the explicit form

$$u(t, x) = \frac{\beta x^n e^{(\alpha - nr)t}}{1 - \beta x^n e^{-nrt} + \beta x^n e^{(\alpha - nr)t}}. \tag{1.5}$$

From this explicit form, we see that

$$\lim_{t\to\infty} u(t, x) = \begin{cases} 0, & \alpha < nr \\ \frac{\beta x^n}{1+\beta x^n}, & \alpha = nr \\ 1, & \alpha > nr \end{cases} \tag{1.6}$$

This demonstrates the multistablility that may be exhibited by the chaotic solution (1.5) of (1.3) when $\varphi(0) = 0$.

If $c(x) = x$ and $g(x, u) = \lambda u$,where λ is the constant in (1.1), then (1.1)-(1.1)a can be written as

$$\frac{\partial}{\partial t}u + x\frac{\partial}{\partial x}u = \lambda u, t \geq 0, 0 \leq x \leq 1 \tag{1.7}$$

with the initial condition

$$u(0, x) = v(x), 0 \leq x \leq 1. \tag{1.8}$$

This initial value problem is so the called Lasota equation, which has been studied by many authors (see e.g., [2], [3], [4], [6] and the references therein).

In this paper we consider two special cases of (1.1): In first case, we consider $g(x, u) = f(u)$, where f satisfies a local Lipschitz condition. In the second case, we consider $g(x, u) = h(x)u$, where $h(x)$ is a bounded continuous function on an interval I of R. The second case was also considered by Fukiko Takeo in [7] and [8]. He gave some sufficient conditions for the solution semigroup to be hypercyclic or chaotic when $c(x) = rx$ and $r > 0$ is a constant. In this paper, we consider $c(x)$ as a continuous function that satisfies the condition (1.2), which is not limited to be linear.

2. Preliminaries

We call a family of continuous linear operators $\{T(t)\}_{t\geq 0} \subset L(X)$ in a Banach space X a C_0-semigroup if it satisfies following three conditions:

(1) $T(t)(T(s)x) = T(t + s)x$ for all s, $t \geq 0$ and $x \in X$;

(2) $T(0)x = x$ for $x \in X$;

(3) $T(t)x \to x$ as $t \downarrow 0$, for every $x \in X$.

A C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called hypercyclic if there exists $x \in X$ such that the set $\{T(t)x : t \geq 0\}$ is dense in X. The C_0-semigroup $\{T(t)\}_{t\geq 0}$ is called chaotic if $\{T(t)\}_{t\geq 0}$ is hypercyclic and the set of periodic points

$$X_{per} = \{x \in X : \text{ there exists a } t > 0 \text{ such that } T(t)x = x\}$$

is dense in X.

We use some notations which were introduced by Batty [1] who considered the differential equation

$$\frac{\partial T}{\partial t} = \lambda \circ T$$

where $\lambda : R \to R$ is a continuous function. Batty showed that the operator $A = \lambda(x)D$, where D denotes the differentiation operator on $C^1(R)$, generated a C_0-semigroup on $C_0(R)$. We will use this semigroup to consider the solution of (1.1) for the first case $g(x,u) = f(u)$.

Let

$$\begin{aligned} Z(\lambda) &= \{x \in R : \lambda(x) = 0\}, \\ U(\lambda) &= R \setminus Z(\lambda) = \{x \in R : \lambda(x) \neq 0\}, \end{aligned}$$

and let $A_l^+(\lambda)$ (respectively, $A_l^-(\lambda)$) be the set of all points x in $Z(\lambda) \cup \{\infty\}$ such that for some $y < x$, $\lambda(\tau) \geq 0$ (respectively, $\lambda(\tau) \leq 0$) for every τ in the interval (y,x), and $\frac{1}{\lambda(\tau)}$ is integrable over (y,x). Let $A_r^+(\lambda)$ (respectively, $A_r^-(\lambda)$) be the set of all points x in $Z(\lambda) \cup \{-\infty\}$ such that for some $z > x$, $\lambda(\tau) \geq 0$ (respectively, $\lambda(\tau) \leq 0$) for every τ in the interval in (x,z), and $\frac{1}{\lambda(\tau)}$ is integrable over (x,z).

Let

$$\begin{aligned} A_l(\lambda) &= A_l^+(\lambda) \cup A_l^-(\lambda), A_r(\lambda) = A_r^+(\lambda) \cup A_r^-(\lambda) \\ A(\lambda) &= A_r(\lambda) \cup A_l(\lambda). \end{aligned}$$

The following result from [1] (proposition 2.7 p. 218) is known.

Lemma 1. *Let $\lambda : R \to R$ be continuous. Then the following are equivalent:*

(1) $\overline{\lambda D \mid C_c^\infty(R)}$ *generates a C_0-semigroup on $C_0(R)$, where $C_c^\infty(R)$ is the space of all C^∞ functions on R whose support is compact and contained in R,*

(2) $A_r^+(\lambda) = A_l^-(\lambda) = \emptyset$,

where D denotes the differentiation operator on $C^1(R)$.

For the normalized case, R is replaced by $[0,1]$. Let $A_l^-(\lambda)$ consist of $x \neq 0$ in $Z(\lambda)$ such that for some $0 < y < x$, $\lambda(\tau) \leq 0$ for every τ in the interval (y,x) and $\frac{1}{\lambda(\tau)}$ is integrable over (y,x); while $A_r^+(\lambda)$ consist of $x \neq 1$ in $Z(\lambda)$ such that for some $1 > z > x$, $\lambda(\tau) \geq 0$ for every τ in the interval (x,z) and $\frac{1}{\lambda(\tau)}$ is integrable over (x,z). The equivalent conditions of Lemma 1 are:

(1) $\overline{\lambda D}$ generates a C_0-semigroup on $C[0,1]$,

(2) $A_r^+(\lambda) = A_l^-(\lambda) = \emptyset$, $\lambda(0) \geq 0$, $\lambda(1) \leq 0$.

3. The Case $g(x,u) = f(u)$

In this section we consider a special case of (1.1) with $g(x,u) = f(u)$, where f satisfies a local Lipschitz condition. We transform (1.1) into the following Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}u = Au + Fu; \\ u(0) = u_0 \end{cases} \tag{3.1}$$

where $A = -cD$ with $D(A) = \{f : f \in C^1[0,1]\}$, $(cD)f(x) = c(x)f'(x)$ and $Fu = f(u)$.

To solve problem (3.1), we first consider the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}u = Au; \\ u(0) = u_0. \end{cases}$$

Let $c(x) = -\lambda(x)$. From (1.2), we have $Z(\lambda) = \{0\}$ and $A_r^+(-c) = \{z \in (0,1),\ -c(z) \geq 0\}$. Since $c(z) \geq 0$ for $0 < z < 1$ we see that $A_r^+(-c) = \emptyset$. On the other hand, $A_l^-(-c) = \{y : y < 0, -c(y) \geq 0\}$. Since c is only defined on $[0,1]$, this implies $A_l^-(-c) = \emptyset$. Now we have that $\lambda(0) = -c(0) = 0$, $\lambda(1) = -c(1) \leq 0$ and $A_r^+(-c) = A_l^-(-c) = \emptyset$. This implies that c satisfied the condition (2) of Lemma 1 and $\overline{A} = \overline{-(cD)}$ generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $C[0,1]$.

To represent the C_0-semigroup $\{T(t)\}_{t\geq 0}$ in explicit form we define a function $q(x)$ by

$$q(x) = -\int_x^1 \frac{ds}{c(s)} \ \ forall\ 0 < x \leq 1. \tag{3.2}$$

Since $c(x) > 0$ we see that for any $0 < x_1 < x_2 < 1$,

$$-\int_{x_1}^1 \frac{ds}{c(s)} < -\int_{x_2}^1 \frac{ds}{c(s)}.$$

Therefore, the function q is strictly increasing on the interval $(0,1)$. It follows that q is an one-to-one mapping, and hence q^{-1} exists on $[0,\infty)$. From the definition of $q(x)$, it is obvious that $q'(x) = \frac{1}{c(x)}$.

Define $h_t : [0,1] \to [0,1]$ in the following way

$$h_t(0) = 0, \text{ if } c(x) = 0 \text{ at } x = 0,$$
$$h_t(x) = q^{-1}(q(x) - t), \text{ if } c(x) \neq 0 \text{ for } 0 < x \leq 1, t \geq 0.$$

We then defined $T(t)$ on $C[0,1]$ by

$$(T(t)f)(x) = f(h_t(x)). \tag{3.3}$$

Since $\int_0^1 \frac{ds}{c(s)} = \infty$, this implies that h_t is defined for all $t \in [0, \infty)$, and hence $T(t)$ is also defined on the interval $[0, \infty)$. Notice that if $\lim_{x_0 \to 0}\left(\int_{x_0}^1 \frac{ds}{c(s)}\right) = t_{\max} < \infty$, then h_t can only be defined on $[0, t_{\max}]$, and hence $T(t)$ can only be defined on $[0, t_{\max}]$. In this case we need not to discuse the chaos of the solution semigroup.

According to this definition , we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(T(t)f)(x) = \frac{\partial}{\partial t} f(h_t(x)) \\
&= f'(h_t(x))\left(\frac{\partial}{\partial t}(h_t(x))\right) \\
&= f'(h_t(x))\frac{-1}{q'(h_t(x))} \\
&= -c(h_t(x)) f'(h_t(x)) = \lambda(h_t(x)) f'(h_t(x)) \\
&= -c(D)(T(t)f)(x), \text{ forall } t > 0 \text{ and } f \text{ in } C^1[0,1].
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x}(T(t)f)(x) &= \frac{\partial}{\partial x} f(h_t(x)) \\
&= f'(h_t(x))\left(\frac{\partial}{\partial x}(h_t(x))\right) = f'(h_t(x))\frac{q'(x)}{q'(h_t(x))}.
\end{aligned}$$

This shows that the operator $\overline{A} = \overline{-(cD)}$ generates a C_0-semigroup $\{T(t)\}_{t \geq 0}$ on $C[0,1]$, and $(T(t)f)(x) = u(t,x)$ is the solution of the initial value problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u + c(x)\dfrac{\partial}{\partial x}u = 0; \\ u(0,x) = u_0(x). \end{cases}$$

By the general perturbation theorem, (3.1) has a unique mild solution $u(t,x)$ that satisfies

$$u(t,x) = T(t)u_0 + \int_0^t T(t-s) f(u(s,x))\, ds.$$

4. The Case $g(x,u) = h(x)\,u$

In [7] and [8], Takeo considered the following initial value problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u + c(x)\dfrac{\partial}{\partial x}u = h(x)\,u; \\ u(0,x) = f(x). \end{cases} \tag{4.1}$$

where $c(x) = r$ or $c(x) = rx$ with r a positive constant. Here we consider $c(x)$ satisfying (1.2) and $I = [0,1]$.

The following lemma is one of Takeo's results which will be used later [8, Theorem 2.1].

Lemma 2. *Let X be the space $C_0([0,\infty), C)$ of all complex-valued functions on $[0,\infty)$ satisfying $\lim_{x\to\infty} f(x) = 0$ with the norm $\|f\|_\infty = \sup_{x\in[0,\infty)} |f(x)|$. Consider the initial value problem*

$$\begin{cases} \dfrac{\partial}{\partial t}u - \dfrac{\partial}{\partial x}u = h(x)\,u, \ x \in [0,\infty), t > 0; \\ u(0,x) = f(x) \qquad x \in [0,\infty). \end{cases} \tag{4.2}$$

where $h \in C([0,\infty), C)$ is bounded and $f \in X$.

Then the solution semigroup $\{Q(t)\}_{t\geq 0}$,

$$(Q(t)f)(x) = \exp\left(\int_x^{x+t} h(s)\,ds\right) f(x+t)$$

is a strongly continuous semigroup on X. Moreover, (1) $\{Q(t)\}_{t\geq 0}$ is hypercyclic if only if $\limsup_{x\to\infty} \int_0^x \Re h(s)\,ds = \infty$, where $\Re h$ is the real part of h; (2) if $X = C_0([0,\infty), C)$, then $\{Q(t)\}_{t\geq 0}$ is chaotic if and only if $\int_0^x \Re h(s)\,ds = \infty$.

By renaming $y = x + t$, the differential equation in (4.2) becomes

$$\frac{d}{dt}u(t, y-t) = u_t - u_x = h(y-t)\,u$$

with the initial data f. Then we have the solution semigroup

$$(Q(t)f)(x) = \exp\left(\int_x^{x+t} h(s)\,ds\right) f(x+t).$$

Suppose $u(t,y)$ is the solution of (4.2), where $y \in \{-q(x) : 0 < x \leq 1\} = [0,\infty)$, and $y \to \infty$ as $x \to 0$. By the assumption (1.2), the range of $-q$ is $[0,\infty)$.

Let $y = -q(x)$,

$$\begin{aligned} v(t,x) &= u(t,y) \text{ for } 0 < x \leq 1, \text{and} \\ v(t,0) &= \lim_{z\to 0} v(t,z) = \lim_{z\to 0} u(t,-q(z)) = \lim_{y\to\infty} u(t,y) = 0. \end{aligned}$$

where q is defined by (3.2).

Then we get

$$\frac{\partial}{\partial t}u = \frac{\partial}{\partial t}v$$

and

$$\frac{\partial}{\partial y}u = \frac{\partial}{\partial x}v \cdot \frac{\partial x}{\partial y} = (-c\,(x)) \cdot \frac{\partial}{\partial x}v.$$

This implies that v is the solution of the initial value problem

$$\text{(4.3)} \qquad \begin{cases} \dfrac{\partial}{\partial t}v + c\,(x)\,\dfrac{\partial}{\partial x}v = k\,(x)\,v, & x \in [0,1], t > 0; \\ u\,(0,x) = f\,(x) & x \in [0,1]. \end{cases}$$

where $c\,(x)$ satisfied (1.2) and $k\,(x) = h\,(-q\,(x))$.

The unique solution semigroup $\{S\,(t)\}_{t\geq 0}$ of (4.3) is given by

$$\text{(4.4)} \qquad (S\,(t)\,v)\,(x) = \sum_{n=0}^{\infty} T_n\,(t)\,v\,(x) \qquad \text{for } v\,(x) \in C[0,1],$$

where $T_n\,(\cdot)$ is defined recursively as

$$T_0\,(t)\,v\,(x) = T\,(t)\,v\,(x) = v\,(h_t(x))$$

where $T\,(t)$ and $h_t(x)$ are defined by (3.3), and

$$T_{n+1}\,(t) = \textstyle\int_0^t T_0\,(t-s)\,BT_n\,(s)\,ds,$$

where B is a bounded operator defined by

$$Bv\,(x) = k\,(x)\,v\,(x)\,.$$

By induction and integration by part we get

$$\text{(4.5)} \qquad \begin{aligned} T_n\,(t)\,v\,(x) &= \frac{1}{n!}\left(\int_0^t T\,(t-s)\,k\,(x)\,ds\right)^n T\,(t)\,v\,(x) \\ &= \frac{1}{n!}\left(\int_0^t k\,(h_{t-s}(x))\,ds\right)^n v\,(h_t(x)) \end{aligned}$$

In fact, if (4.5) is true for n, then

$$
\begin{aligned}
&T_{n+1}(t)\,v(x)\\
&=\int_0^t T_0(t-s)\left(k(x)\left(\frac{1}{n!}\int_0^s k(h_{s-\tau}(x))\,d\tau\right)^n v(h_t(x))\right)ds\\
&=v(h_t(x))\int_0^t k(h_{t-s}(x))\,\frac{1}{n!}\left(\int_0^s k(h_{t-\tau}(x))\,d\tau\right)^n ds\\
&=v(h_t(x))\,[\left(\int_0^s k(h_{t-\tau}(x))\,d\tau\right)\frac{1}{n!}\left(\int_0^s k(h_{t-\tau}(x))\,d\tau\right)^n |_{s=0}^{s=t}\\
&\quad-\int_0^t\left(\int_0^s k(h_{t-\tau}x)\,d\tau\right)\frac{1}{(n-1)!}\left(\int_0^s k(h_{t-\tau}x)\,d\tau\right)^{n-1} k(h_{t-s}x)\,ds]\\
&=v(h_t(x))\,[\frac{1}{n!}\left(\int_0^t k(h_{t-\tau}(x))\,d\tau\right)^{n+1}\\
&\quad-\int_0^t \frac{1}{(n-1)!}\left(\int_0^s k(h_{t-\tau}(x))\,d\tau\right)^n k(h_{t-s}(x))\,ds]\\
&=\frac{1}{n!}\left(\int_0^t k(h_{t-\tau}(x))\,d\tau\right)^{n+1} v(h_t(x))-nT_{n+1}(t)\,v(x)
\end{aligned}
$$

This implies that

$$
T_{n+1}(t)\,v(x)=\frac{1}{(n+1)!}\left(\int_0^t k(h_{t-\tau}(x))\,d\tau\right)^{n+1} v(h_t(x)),
$$

and hence (4.5) is also true for $n+1$. Thus (4.5) holds for all integer n. From (4.4) and (4.5), we have

$$
\begin{aligned}
(S(t)\,v)(x)&=\sum_{n=0}^{\infty}T_n(t)\,v(x)=\exp\left(\int_0^t k(h_{t-s}(x))\,ds\right)v(h_t(x))\\
&=\exp\left(\int_0^t T(t-s)\,k(x)\,ds\right)T(t)\,v(x).
\end{aligned}
$$

To apply Lemma 2, we need to prove the equivalence of $S(t)$ and $Q(t)$. We rewrite $S(t)$ and $Q(t)$ as follows:

$$
\begin{aligned}
(S(t)\,v)(x)&=\exp\left(\int_0^t k(h_{t-s}(x))\,ds\right)v(h_t(x))\\
&=\exp\left(\int_{h_t(x)}^x \frac{k(r)}{c(r)}dr\right)v(h_t(x))=\frac{\eta(x)}{\eta(h_t(x))}v(h_t(x))
\end{aligned}
$$

where $\eta(x)=\exp\left(-\int_x^1 \frac{k(r)}{c(r)}dr\right)$, and

$$
(Q(t)\,f)(x)=\exp\left(\int_x^{x+t} h(s)\,ds\right)f(x+t)=\frac{\rho(x)}{\rho(x+t)}f(x+t)
$$

with $\rho(x) = \exp\left(-\int_0^x h(s)\,ds\right)$. From $k(x) = h(-q(x))$ and by a change of variable, we get

$$\eta(x) = \rho(-q(x))$$

and $\eta(h_t(x)) = \rho(-q(x)+t)$. Define a linear map Ψ from $C_0([0,\infty),C)$ to $C_0([0,1],C)$ by

$$\Psi f(y) = f\left(q^{-1}(-y)\right).$$

From the above definition Ψ is 1-1 and onto. This implies the existence of Ψ^{-1} which is given by

$$\Psi^{-1} f(x) = f(-q(x)).$$

Since $y = -q(x)$, we have

$$\begin{aligned}
\Psi \circ Q(t) \circ \Psi^{-1}(f(x)) &= \Psi(Q(t) f(y)) = \Psi(u(t,y)) = \Psi\left(\frac{\rho(y)}{\rho(y+t)} f(y+t)\right) \\
&= \frac{\eta(x)}{\eta(h_t(x))} \Psi(f(x+t)) = \frac{\eta(x)}{\eta(h_t(x))} f\left(q^{-1}(-y-t)\right) \\
&= \frac{\eta(x)}{\eta(h_t(x))} f(h_t(x)) = S(t) f(x) = v(t,x) = u(t,y)
\end{aligned}$$

for every $f \in C_0([0,1],C)$. This shows that $\Psi \circ Q(t) \circ \Psi^{-1} = S(t)$ and the equivalence of $S(t)$ and $Q(t)$. According to the above result and Lemma 2 we obtain a necessary and sufficient condition for $\{S(t)\}_{t\geq 0}$ to be chaotic.

Theorem 3. *Let X be the space $C_0([0,1],C) = \{f \in C([0,1],C) \mid f(0) = 0\}$ endowed with the supremum norm. Consider the initial value problem:*

$$\begin{cases} \dfrac{\partial}{\partial t} u + c(x) \dfrac{\partial}{\partial x} u = k(x) u, & x \in [0,1], t > 0; \\ u(0,x) = f(x) & x \in [0,1]. \end{cases}$$

where $c(x)$ satisfies (1.2), $k \in C([0,\infty),C)$ is bounded and $f \in X$.

Then the solution semigroup $\{S(t)\}_{t\geq 0}$ is given by

$$(S(t) f)(x) = \exp\left(\int_0^t T(t-s) k(x)\,ds\right) T(t) f(x),$$

where $\{T(t)\}_{t\geq 0}$ is a semigroup defined by (3.3). Moreover, $\{S(t)\}_{t\geq 0}$ is chaotic if and only if

$$\lim_{x\to 0} \int_x^1 \frac{\Re k(s)}{c(s)} ds = \infty.$$

Therefore, if $\Re k(0) > 0$ and $c(x) = O(x^\alpha)$, $\alpha \geq 1$ as $x \to 0$, then $\{S(t)\}_{t\geq 0}$ is chaotic.

Proof. By using $k(x) = h(-q(x))$, the initial value problem (4.3) can be transform to the initial value problem (4.2). Since

$$\int_0^\infty h(s)\,ds = \lim_{x\to 0}\int_x^1 \frac{\Re h(-q(\tau))}{c(\tau)}d\tau = \lim_{x\to 0}\int_x^1 \frac{\Re k(s)}{c(s)}ds,$$

we see by applying Lemma 2 that $\{S(t)\}_{t\geq 0}$ is chaotic if and only if $\lim_{x\to 0}\int_x^1 \frac{\Re k(s)}{c(s)}ds = \infty$. Moreover, if $\Re k(0) > 0$, then $\{S(t)\}_{t\geq 0}$ is chaotic since $c(x) = O(x^\alpha)$, $\alpha \geq 1$ as $x \to 0$.

References

1. C. J. K. Batty, Derivations on the line and flow along orbits, *Pacific Journal of Mathmatics*, **126(2)** (1987), 209-225.
2. A. L. Dawidowicz, N. Haribash and A. Poskrobko, On the invariant measure for the quasi-linear Lasota equation, *Math. Meth. Appl. Sci.*, **30** (2007), 779-787.
3. A. L. Dawidowicz and A. Poskrobko, On chaotic and stable behaviour of the von Foerster-Lasota equation in some Orlicz spaces, *Proc. of the Est. Acad. Sci.*, **57(2)** (2008), 61-69.
4. A. Lasota and T. Szarek, Dimension of measures invariant with respect to the Wa'zewska partial differential equation, *J. Differential Equations*, **196** (2004), 448-465.
5. M. C. Mackey and H. Schwegle, Ensemble and trajectory statistics in a nonlinear partial differential equation, *Journal of Statistical Physics*, **70(1-2)** (1993).
6. R. Rudnicki, Chaos for some infinite-demensional dynamical systems, *Math. Meth. Appl. Sci.*, **27** (2004), 723-738.
7. Fukiko Takeo, Chaos and hypercyclicity for solution semigroups to some partial differential equations, *Nonlinear Analysis*, **63** (2005), e1943-e1953.
8. Fukiko Takeo, Chatoic or hypercyclic semigroups on a function space $C_0(I, C)$ or $L^p(I, C)$, *SUT Journal of Mathmatics*, **41(1)** (2005), 43-61.

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