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ROUGH SINGULAR INTEGRALS ASSOCIATED TO SUBMANIFOLDS

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Abstract. We investigate the L^p boundedness for a class of singular integral operators associated to submanifolds, including surfaces of revolution, under the $L(\log L)(S^{n-1})$ or Block space condition on the kernel functions. Our results improve known results.

1. INTRODUCTION

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$.

The L^p boundedness of singular integrals has attracted the attention of many authors. There are several papers concerning with rough kernels associated to surfaces of revolution. Kim, Wainger, Wright and Ziesler [15], Chen and Fan [5], Lu, Pan and Yang [20], Al-Salman and Pan [3], Al-Qassem and Pan [1], etc..

In 2001, Lu, Pan and Yang [20] gave the following theorem.

Theorem A. [[20]]. *Let $n \geq 2$. Let $\psi(t) : [0, \infty) \to \mathbb{R}$ be in $C^1(0, \infty)$ and satisfy $|\psi(t) - \psi(0)| \leq C_0 t^\alpha$ for some $\alpha > 0$ and small t. Let $\Omega \in H^1(S^{n-1})$, $b \in L^\infty(\mathbb{R}_+)$. Then $T_{\Omega,t,\psi,b}(f)(x, x_{n+1})$ defined by*

$$(1.1)\qquad T_{\Omega,t,\psi,b}(f)(x, x_{n+1}) = \mathrm{p.\,v.} \int_{\mathbb{R}^n} b(|y|) \frac{\Omega(y')}{|y|^n} f\big(x - y, x_{n+1} - \psi(|y|)\big)\, dy.$$

is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided the maximal function $\mathcal{M}_\psi$, defined by

$$(1.2)\qquad \mathcal{M}_\psi f(x_1, x_2) = \sup_{k\in\mathbb{Z}} \frac{1}{2^k} \int_{2^k}^{2^{k+1}} |f(x_1 - t, x_2 - \psi(t))|\, dt,$$

is bounded on $L^p(\mathbb{R}^2)$.

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However, for $\psi(t) = \log(1+t)$, $\mathcal{M}_\psi$ is not bounded for any $1 < p < \infty$, as is noted in Stein and Wainger [23, p. 1291]. So, we cannot apply the above theorem to this case.

In this paper, we will deal with singular integrals with two parameter functions, which are essentially singular integrals associated to surface of revolution. As a byproduct, using Example 1 and our Theorem 1 below, we can show that if $\Omega \in L\log L(S^{n-1}) \cup B_q^{0,0}(S^{n-1})$ $(q > 1)$, $b(t) = h\big(\log(1+t)\big)\, t/\big((1+t)\log(1+t)\big)$ with $h \in \Delta_\gamma$ $(\gamma > 1)$, then, for $\psi(t) = \log(1+t)$, the above operator $T_{\Omega,t,\psi,b}$ is L^p bounded provided $|1/p - 1/2| < \min\{1/2,\, 1/\gamma'\}$. The precise definitions of Δ_γ and the block spaces $B_q^{0,0}(S^{n-1})$ will be given soon.

We comment the following: Al-Salman and Pan [3], Al-Qassem and Pan [1] gave L^p boundedness results in the case $\Omega \in L\log L(S^{n-1}) \cup B_q^{0,0}(S^{n-1})$ $(q > 1)$, $b(t) \in L^\infty(\mathbb{R}_+)$, and $\psi(t)$ is increasing and convex, and satisfy $\psi(0) = 0$. However, $\log(1+t)$ is concave, and so we cannot apply their theorems to the above case, too.

Now, the singular integrals with two parameter functions are the following ones:

$$
\begin{aligned}
&T_{\Omega,\phi,\psi,b}(f)(x, x_{n+1}) \\
&:= \mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{b(|y|)\Omega(y')}{|y|^n} f\big(x - \phi(|y|)y', x_{n+1} - \psi(|y|)\big)\, dy.
\end{aligned}
\tag{1.3}
$$

Here, $b \in \Delta_\gamma$, $\Omega \in L\log L(S^{n-1}) \cup B_q^{0,0}(S^{n-1})$ $(q > 1)$. $\phi(t)$ and $\psi(t)$ are nonnegative $C^1(\mathbb{R}_+)$ functions satisfying $\phi(t)/(t\phi'(t)), \psi(t)/(t\psi'(t)) \in L^\infty(\mathbb{R}_+)$ and doubling or convexitylike conditions. Relating to this, we also consider two maximal operators $\mathcal{M}_{\Omega,\phi,\psi,h}$ and $\mathcal{T}^*_{\Omega,\phi,\psi,h} f$ defined by

$$
\begin{aligned}
&\mathcal{M}_{\Omega,\phi,\psi,h} f(x, x_{n+1}) \\
&= \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} \big| f\big(x - \phi(|y|)y', x_{n+1} - \psi(|y|)\big)\big| |\Omega(y') h(|y|)|\, dy,
\end{aligned}
\tag{1.4}
$$

and

$$
\begin{aligned}
&\mathcal{T}^*_{\Omega,\phi,\psi,h} f(x, x_{n+1}) \\
&= \sup_{0<\varepsilon<A} \left| \int_{\varepsilon<|y|<A} f\big(x - \phi(|y|)y', x_{n+1} - \psi(|y|)\big) \frac{\Omega(y')}{|y|^n} h(|y|)\, dy \right|.
\end{aligned}
\tag{1.5}
$$

Precise conditions on ϕ and ψ are the following assumptions (A-1) and (A-2).

(A-1) ϕ is a nonnegative $C^1(\mathbb{R}_+)$ function and $\phi(t)/\big(t\phi'(t)\big) \in L^\infty(\mathbb{R}_+)$.

(A-2) ϕ satisfies one of the following conditions:

(i) ϕ is increasing, and $\phi(2t) \le c_1 \phi(t)$.

(ii) ϕ is increasing, and $t\phi'(t)$ is increasing.

(iii) ϕ is decreasing, and $\phi(t) \le c_2\phi(2t)$.

(iv) ϕ is decreasing and convex.

Remark 1. Under the condition (A-1), if ϕ is increasing and convex, then $t\phi'(t)$ is increasing. And if ϕ is decreasing and $-t\phi'(t)$ is decreasing, then ϕ is convex.

For $1 \le \gamma \le \infty$, $\Delta_\gamma(\mathbb{R}_+)$ is the collection of all measurable functions $b : [0,\infty) \to \mathbb{C}$ satisfying

$$\|b\|_{\Delta_\gamma} = \sup_{R>0}\bigg(\frac{1}{R}\int_0^R |b(t)|^\gamma dt\bigg)^{1/\gamma} < \infty.$$

We note that

$$L^\infty(\mathbb{R}_+) \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\alpha(\mathbb{R}_+) \quad \text{for } \alpha < \beta,$$

$$L^\gamma(\mathbb{R}_+,\, dt/t) \subset \Delta_\gamma(\mathbb{R}_+) \quad \text{for } 1 \le \gamma < \infty,$$

and all these inclusions are proper.

We also note that our operator $T_{\Omega,\phi,\psi,b}$ is a singular integral of type in (1.1) with rough kernel associated to surface of revolution. In fact, by using the polar coordinates, changing the variable $s = \phi(r)$, and then changing the polar coordinates to the usual ones, we get

$$\begin{aligned}
&T_{\Omega,\phi,\psi,b}(f)(x,x_{n+1})\\
&= \text{p. v.}\int_{\mathbb{R}^n} b(|y|)\frac{\Omega(y')}{|y|^n} f\big(x-\phi(|y|)y', x_{n+1}-\psi(|y|)\big)\,dy\\
&= \text{p. v.}\int_{S^{n-1}}\int_0^\infty \frac{b(r)\Omega(y')}{r} f\big(x-\phi(r)y', x_{n+1}-\psi(r)\big)\,dr\,d\sigma\\
&= \text{p. v.}\int_{S^{n-1}}\int_0^\infty \frac{b(\phi^{-1}(s))s}{\phi^{-1}(s)\phi'(\phi^{-1}(s))}\Omega(y')\, f\big(x-sy', x_{n+1}-\psi(\phi^{-1}(s))\big)\,\frac{ds}{s}\,d\sigma\\
&= \text{p. v.}\int_{\mathbb{R}^n} b(\phi^{-1}(|y|))\frac{|y|}{\phi^{-1}(|y|)\phi'(\phi^{-1}(|y|))}\frac{\Omega(y')}{|y|^n}\, f\big(x-y, x_{n+1}-\psi(\phi^{-1}(|y|))\big)\,dy.
\end{aligned}$$

So, setting

$$\tilde{b}(t) = b(\phi^{-1}(t))\frac{t}{\phi^{-1}(t)\phi'(\phi^{-1}(t))} = \bigg[b(t)\,\frac{\phi(s)}{s\phi'(s)}\bigg]_{s=\phi^{-1}(t)}, \tag{1.6}$$

we see that

$$\begin{aligned}
T_{\Omega,\phi,\psi,b}(f)(x,x_{n+1}) &= T_{\Omega,t,\psi(\phi^{-1}),\tilde{b}}(f)(x,x_{n+1})\\
&= \text{p. v.}\int_{\mathbb{R}^n} \tilde{b}(|y|)\frac{\Omega(y')}{|y|^n}\, f\big(x-y, x_{n+1}-\psi(\phi^{-1}(|y|))\big)dy,
\end{aligned} \tag{1.7}$$

i.e., $T_{\Omega,\phi,\psi,b}$ is a singular integral $T_{\Omega,t,\psi(\phi^{-1}),\tilde{b}}$ with rough kernel associated to surface of revolution $\{x_{n+1} = \psi(\phi^{-1}(|y|))\}$. Here, we see that

$$\|\tilde{b}\|_{\Delta_\gamma} \le \big(1 + \|\phi(t)/\big(t\phi'(t)\big)\|_\infty\big)\|b\|_{\Delta_\gamma}. \tag{1.8}$$

However, it seems to be very complicated to study our operator $T_{\Omega,\phi,\psi,b}$ in the form of $T_{\Omega,t,\psi(\phi^{-1}),\tilde{b}}$.

Example 1. Let $\phi(t) = e^t - 1$ and $\psi(t) = t$. Then ϕ satisfies $(A-1)$ and $(A-2)$ (i) and (ii), and ψ satisfies $(A-1)$ and $(A-2)$ (ii). Since $\phi^{-1}(t) = \log(1+t)$ and $\phi'(t) = e^t$, we have

$$\begin{aligned} &T_{\Omega,\phi,\psi,b}(f)(x, x_{n+1}) \\ &= \mathrm{p.\,v.} \int_{\mathbb{R}^n} b(\log(1+|y|)) \frac{|y|}{(1+|y|)\log(1+|y|)} \frac{\Omega(y')}{|y|^n} \\ &\quad f\big(x-y, x_{n+1} - \log(1+|y|)\big)\, dy. \end{aligned} \tag{1.9}$$

That is, in this case $T_{\Omega,\phi,\psi,b}(f)(x, x_{n+1}) = T_{\Omega,t,\log(1+t),\tilde{b}}(f)(x, x_{n+1})$ in (1.1) with $\tilde{b}(t) = b\big(\log(1+t)\big)\, t/\big((1+t)\log(1+t)\big)$.

To state our theorem, we use two function spaces $L(\log L)(S^{n-1})$ and the block spaces $B_q^{(0,0)}(S^{n-1})$. Let $L(\log L)^\alpha(S^{n-1})$ (for $\alpha > 0$) denote the class of all measurable functions Ω which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(S^{n-1})} = \int_{S^{n-1}} |\Omega(y')| \log^\alpha(2 + |\Omega(y')|)\, d\sigma(y') < \infty.$$

For $q \ge 1$, let $B_q^{(0,\gamma)}(S^{n-1})$ denote the block space generated by q-blocks (its precise definition will be given in Section 3).

Now, we can state our main theorem.

Theorem 1. *Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, and $\Omega \in L(\log L)(S^{n-1}) \cup \big(\cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})\big)$, satisfying the cancelation condition*

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1.10}$$

Let ϕ and ψ satisfy the assumptions (A-1) *and* (A-2). *Then*

(i) *for every p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, there exists a positive constant C_p such that*

$$\|T_{\Omega,\phi,\psi,h} f\|_{L^p(\mathbb{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \tag{1.11}$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

(ii) *for all* $\gamma' < p \le \infty$, *there exists a positive constant* C_p *such that*

$$\|\mathcal{M}_{\Omega,\phi,\psi,h} f\|_{L^p(\mathbb{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \tag{1.12}$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

(iii) *for all* $\gamma' < p < 1/(1/2 - \min\{1/2, 1/\gamma'\})$ *there exists a positive constant* C_p *such that*

$$\|T^*_{\Omega,\phi,\psi,h} f\|_{L^p(\mathbb{R}^{n+1})} \le C_p \|f\|_{L^p(\mathbb{R}^{n+1})} \tag{1.13}$$

for every $f \in L^p(\mathbb{R}^{n+1})$. *We use the convention* $1/0 = +\infty$.

In the case of $L \log L$, Al-Salman and Pan [3] showed Theorem 1 (i) and (iii) for $\phi(t) = t$ (i.e., in the case of surface of revolution) and ψ under the condition that it is nonnegative, increasing, convex and $\psi(0) = 0$, and (ii) under the additional condition on $h \in L^\infty(\mathbb{R}_+)$.

In the case of the block spaces, Al-Qassem and Pan [1] showed Theorem 1 (i) and (iii) for $\phi(t) = t$ and ψ under the condition that it is nonnegative, increasing, convex and $\psi(0) = 0$, and (ii) under the additional condition on $h \in L^\infty(\mathbb{R}_+)$.

To understand the relationship in the above results, we remark the following proper inclusion relations:

$$L^q(S^{n-1}) \subset L(\log L)(S^{n-1}) \subset H^1(S^{n-1}) \subset L^1(S^{n-1}) \quad (q \ge 1), \tag{1.14}$$

$$L(\log L)^\beta(S^{n-1}) \subset L(\log L)^\alpha(S^{n-1}) \quad \text{if } 0 < \alpha < \beta, \tag{1.15}$$

$$L(\log L)^\alpha(S^{n-1}) \subset H^1(S^{n-1}) \quad \text{for all } \alpha \ge 1, \tag{1.16}$$

where $H^1(S^{n-1})$ is the Hardy space on the unit sphere.
Note that $L(\log L)^{1+\varepsilon}(S^{n-1})$ does not contain $B_q^{(0,0)}(S^{n-1})$ for any $\varepsilon > 0$, and $B_q^{(0,0)}(S^{n-1}) \subset H^1(S^{n-1})$ for any $q > 1$.

As is easily checked, from the conditions in [2] it follows that $\phi(t)/\big(t\phi'(t)\big)$, $\psi(t)/\big(t\psi'(t)\big) \le 1$. Hence, our results are improvements of theirs. In particular, we can cover the case where $\phi(\cdot)$, $\psi(\cdot)$ are positive, increasing and concave, such as $\phi(t) = t^a$ and $\psi(t) = t^b$ $(0 < a, b < 1)$. We can also cover the case $\phi(t) = t^a$ $(0 < t < 1)$, $\psi(t) = at^b/b$ $(t \ge 1)$, where $0 < a < 1 < b$.

Remark 2. Taking $\Omega(y') = y_1/|y|$, $\phi(t) = \log^{1/n}(1+t)$ and $\psi(t) = 0$, our singular integral has the form

$$T_{\Omega,\phi,\psi,1} f(x, x_{n+1}) = \text{p. v.} \int_{\mathbb{R}^n} f\big(x - y' \log^{1/n}(1+|y|), x_{n+1}\big) \frac{y_1}{|y|^{n+1}}\, dy, \tag{1.17}$$

To this operator, L^p boundedness does not hold for any $1 < p < \infty$. In this case, ϕ satisfies (A-2) (i) but $\phi(t)/\big(t\phi'(t)\big) \notin L^\infty(\mathbb{R}_+)$, i.e., ϕ does not satisfy (A-1). This gives a reason why we treat the assumption (A-1).

Finally, we note that unfortunately we could not get similar results for $H^1(S^{n-1})$ kernels Ω, besides $L^2(\mathbb{R}^{n+1})$ boundedness.

The main tools in this paper come from Al-Qassem and Pan [2]. Besides their ideas, we use two observations. One is about relations between monotonic functions and the directional Hardy-Littlewood maximal function (Lemma 2.2). The other is about behaviors of the Fourier transform of measures arising from our singular integral operator (Lemma 3.1).

This paper is organized as follows. In Section 2, we recall some properties of monotonic functions satisfying (A-1) and (A-2), and state Lemma 2.2, and that $\{\Phi(a^k)\}_{k\in\mathbb{Z}}$ is a lacunary sequence. We also give Fourier transform estimates of some measures in this section. In Section 3, we prepare necessary lemmas to prove our theorems, in the framework by Al-Qassem and Pan [2], such as Lemma 3.1. In Section 4, we discuss the proof of Theorem 1 in the case of $\Omega \in L\log L(S^{n-1})$. The proof of Theorem 1 in the case of $\Omega \in \cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})$ is given in Section 5. In the last section, we shall give a proof of our claim in Remark 2.

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Preliminaries

In this section, we recall fundamental properties between monotonic functions and the directional Hardy-Littlewood maximal function. All lemmas in this section are given in our paper [7], [17]. We begin with stating fundamental properties of positive and monotone C^1 functions $\Phi(t)$ satisfying the condition (A-1), i.e. $\Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$.

Lemma 2.1. ([7, 17]). (i) *Suppose Φ is positive and increasing. Then $\Phi(t)/(t\Phi'(t)) \le b$ $(t>0)$, if and only if $\Phi(at)/\Phi(t) \ge a^{1/b}$ for all $a>1$ and $t>0$. Hence, if $a>1$, $\Phi(a^{k+1})/\Phi(a^k) \ge a^{1/b}$ for $k\in\mathbb{Z}$, i.e. $\{\Phi(a^k)\}_{k\in\mathbb{Z}}$ is a lacunary sequence. Moreover,*

$$\Phi(t) \le \Phi(1)t^{1/b} \quad (0<t\le 1), \quad \Phi(t) \ge \Phi(1)t^{1/b} \quad (t\ge 1),$$
$$t\Phi'(t) \ge \frac{\Phi(1)}{b}t^{1/b} \quad (t\ge 1),$$

and hence $\lim_{t\to 0}\Phi(t)=0$, $\lim_{t\to\infty}\Phi(t)=+\infty$. Also, $t\Phi'(t)$ cannot be a decreasing function on $(0,\infty)$.

(ii) *Suppose Φ is positive and decreasing. Then $-\Phi(t)/(t\Phi'(t)) \le b$ $(t>0)$ if and only if $\Phi(t)/\Phi(at) \ge a^{1/b}$ for all $a>1$ and $t>0$. Hence, if $a>1$, $\Phi(a^{-(k+1)})/\Phi(a^{-k}) \ge a^{1/b}$ for $k\in\mathbb{Z}$, i.e. $\{\Phi(a^{-k})\}_{k\in\mathbb{Z}}$ is a lacunary sequence. Moreover,*

$$\Phi(t) \geq \Phi(1)t^{-1/b} \quad (0 < t \leq 1), \qquad \Phi(t) \leq \Phi(1)t^{-1/b} \quad (t \geq 1),$$
$$- t\Phi'(t) \geq \frac{\Phi(1)}{b} t^{-1/b} \quad (0 < t \leq 1),$$

and hence $\lim_{t\to 0} \Phi(t) = +\infty$, $\lim_{t\to\infty} \Phi(t) = 0$. *Also,* $-t\Phi'(t)$ *cannot be an increasing function on* $(0,\infty)$.

Now we recall several properties between monotonic functions and the directional Hardy-Littlewood maximal functions.

Lemma 2.2. ([7, 17]). *Let* $\Omega \in L^1(S^{n-1})$. *Suppose* Φ *is a positive function on* $(0,\infty)$ *satisfying* $|\Phi(t)/(t\Phi'(t))| \leq b$ *and satisfies one of the following conditions:*

(i) Φ *is increasing, and* $\Phi(2t) \leq c_1\Phi(t)$.

(ii) Φ *is increasing, and* $t\Phi'(t)$ *is increasing.*

(iii) Φ *is decreasing, and* $\Phi(t) \leq c_2\Phi(2t)$.

(iv) Φ *is decreasing and convex.*

Then there exists $C > 0$ *such that*

$$\left| \int \int_{t/2<|y|<t} \frac{\Omega(y')f\big(x - \Phi(|y|)y'\big)}{|y|^n}\, dy \right| \leq C(1+b) \int_{S^{n-1}} |\Omega(y')| M_{y'}f(x)\, d\sigma(y'),$$

where $M_{y'}f(x)$ *is the directional Hardy-Littlewood maximal function of* f, *defined by*

$$\sup_{r>0} \frac{1}{2r} \int_{|t|<r} |f(x - ty')|\, dt.$$

Remark 3. (i) If Φ is positive, increasing, and $\Phi(t)/(t\Phi'(t))$ is decreasing, then $t\Phi'(t)$ is increasing on $(0,\infty)$.

(ii) If Φ is positive, increasing and convex, then $t\Phi'(t)$ is increasing on $(0,\infty)$.

(iii) If Φ is positive, decreasing, and $-t\Phi'(t)$ is decreasing on $(0,\infty)$, then $\Phi(t)$ is convex.

(iv) If Φ is positive, decreasing, and $-\Phi(t)/(t\Phi'(t))$ is increasing, then $-t\Phi'(t)$ is decreasing, and hence $\Phi(t)$ is convex.

Next, we prepare the following estimates about Fourier transforms of some measures on $\mathbb{R}^{n+1}$.

Lemma 2.3. [17]. *Let* $1 < q \leq \infty$, $\Omega \in L^q(S^{n-1})$ *and* ψ *be a real valued function on* $(0,\infty)$.

(i) *If* Φ *is positive, increasing,* $\Phi(2t) \leq c_1\Phi(t)$, *and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t} \Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2 \frac{ds}{s} \leq \frac{C_\alpha 2^\alpha(\log c_1)^{1-\alpha}\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t/2)\xi|^\alpha}.$$

(ii) *If* Φ *is positive, decreasing,* $\Phi(t) \leq c_2\Phi(2t)$, *and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t} \Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2 \frac{ds}{s} \leq \frac{C_\alpha 2^\alpha(\log c_2)^{1-\alpha}\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}.$$

Lemma 2.4. ([17]). *Let* $1 < q \leq \infty$, $\Omega \in L^q(S^{n-1})$ *and* ψ *be a real valued function on* $(0,\infty)$.

(i) *If* Φ *is positive, increasing,* $t\Phi'(t)$ *is increasing, and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t} \Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2 \frac{ds}{s} \leq \frac{C_\alpha 4^\alpha(\log 2)^{1-\alpha}\|\varphi\|^\alpha_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t/2)\xi|^\alpha}.$$

(ii) *If* Φ *is positive, decreasing and convex, and* $\varphi(t) := \Phi(t)/(t\Phi'(t)) \in L^\infty(0,\infty)$, *then it holds for any* $0 < \alpha < 1/q'$

$$\int_{t/2}^{t} \Big|\int_{S^{n-1}} \Omega(x')e^{-i(\Phi(s)\xi\cdot x'+\eta\psi(s))}d\sigma(x')\Big|^2 \frac{ds}{s} \leq \frac{C_\alpha 8^\alpha(\log 2)^{1-\alpha}\|\varphi\|^\alpha_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\Phi(t)\xi|^\alpha}.$$

Finally in this section, we will note the Littlewood-Paley operator for a lacunary sequence.

Let $\{a_j\}_{j\in\mathbb{Z}}$ be a lacunary sequence of positive numbers satisfying

$$\frac{a_{j+1}}{a_j} \geq a > 1, \quad j \in \mathbb{Z}.$$

Take a non-increasing $C^\infty\big([0,\infty)\big)$ function $\varphi(t)$ such that

$$0 \leq \varphi(t) \leq 1 \ \big(t \in [0,\infty)\big), \quad \varphi(t) = 1 \ (0 \leq t \leq 1), \quad \varphi(t) = 0 \ (t \geq a).$$

We define functions ψ_j on $(0,\infty)$ by

$$\psi_j(t) = \varphi\Big(\frac{t}{a_{j+1}}\Big) - \varphi\Big(\frac{t}{a_j}\Big).$$

We set

$$\Psi_j(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} \psi_j(|\xi|)e^{ix\cdot\xi}d\xi. \tag{2.18}$$

Then we can use the Littlewood-Paley theory and get

Lemma 2.5. ([17]). *Let* $a_0 > 1$ *and* $1 < p < \infty$. *Let* Ψ_j *be as above. Then there exists a positive constant* C_p *such that*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\Psi_j * f(x)|^2 \right)^{1/2} \right\|_p \le C_p \|f\|_p, \quad f \in L^p(\mathbb{R}^n), \tag{2.19}$$

where C_p *is independent of* $a \ge a_0$.

3. Some Definitions and Lemmas

In this section, we give some definitions and prepare some lemmas to prove our theorems.

The block spaces originated in the work of Taibleson and Weiss on the convergence of the Fourier series in connection with the developments of the real Hardy spaces. We will recall the definition of block spaces on S^{n-1}. For further information about the theory of spaces generated by blocks and its applications to harmonic analysis, see the book [21] and a survey article [18].

Definition 1. A q-block on S^{n-1} is an $L^q(S^{n-1})$ $(1 < q \le \infty)$ function $b(x)$ that satisfies

$$\begin{aligned} &\text{(i)} \quad \operatorname{supp} b \subset I; \\ &\text{(ii)} \quad \|b\|_q \le |I|^{-1/q'}, \end{aligned} \tag{3.1}$$

where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) \cap S^{n-1}$ is a cap on S^{n-1} for some $x_0' \in S^{n-1}$ and $\theta_0 \in (0, 1]$.

Jiang and Lu [13] introduced the class of block spaces $B_q^{(0,v)}(S^{n-1})$ $(v > -1)$ concerning the study of homogeneous singular integral operators.

Definition 2. For $1 < q \le \infty$ and $v > -1$, the block space $B_q^{(0,v)}(S^{n-1})$ is defined by

$$B_q^{(0,v)}(S^{n-1}) = \Big\{ \Omega \in L^1(S^{n-1}); \, \Omega = \sum_{j=1}^{\infty} \lambda_j b_j, \, M_q^{(0,v)}(\{\lambda_j\}) < \infty \Big\}, \tag{3.2}$$

where each λ_j is a complex number, each b_j is a q-block supported on a cap I_j on S^{n-1}, and

$$M_q^{(0,v)}(\{\lambda_j\}) = \sum_{j=1}^{\infty} |\lambda_j| \big\{ 1 + \log^{(v+1)}\big(|I_j|^{-1}\big) \big\}. \tag{3.3}$$

Let $\|\Omega\|_{B_q^{(0,v)}(S^{n-1})} = \inf\{M_q^{(0,v)}(\{\lambda_j\}); \Omega = \sum_{j=1}^{\infty} \lambda_j b_j$ and each b_j is a q-block supported on a cap I_j on $S^{n-1}\}$. Then $\|\cdot\|_{B_q^{(0,v)}(S^{n-1})}$ is a norm on the space $B_q^{(0,v)}(S^{n-1})$, and $\left(B_q^{(0,v)}(S^{n-1}), \|\cdot\|_{B_q^{(0,v)}(S^{n-1})}\right)$ is a Banach space.

The following inclusion relations are known.

$$(3.4)\quad \begin{aligned} &B_q^{(0,v_1)}(S^{n-1}) \subset B_q^{(0,v_2)}(S^{n-1}) \quad \text{if } v_1 > v_2 > -1;\\ &B_{q_1}^{(0,v)}(S^{n-1}) \subset B_{q_2}^{(0,v)}(S^{n-1}) \quad \text{if } 1 < q_2 < q_1 \text{ for any } v > -1;\\ &\bigcup_{p>1} L^p(S^{n-1}) \subset B_q^{(0,v)}(S^{n-1}) \quad \text{for any } q > 1,\ v > -1;\\ &\bigcup_{q>1} B_q^{(0,v)}(S^{n-1}) \not\subset \bigcup_{q>1} L^q(S^{n-1}) \quad \text{for any } v > -1;\\ &B_q^{(0,v)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+v}(S^{n-1}) \quad \text{for any } q > 1,\ v > -1. \end{aligned}$$

Definition 3. For arbitrary real-valued functions $\phi(\cdot)$ and $\psi(\cdot)$ on $(0,\infty)$, a measurable function $h : (0,\infty) \to \mathbb{C}$ and $\Omega : S^{n-1} \to \mathbb{C}$, we define the family $\{\sigma_{t,h};\, t \in (0,\infty)\}$ of measures and the maximal operator σ_h^* on $\mathbb{R}^{n+1}$ by

$$(3.5)\qquad \int_{\mathbb{R}^{n+1}} f\, d\sigma_{t,h} = \int_{t/2<|y|\le t} f(\phi(|y|)y', \psi(|y|))h(|y|)\frac{\Omega(y')}{|y|^n}\, dy,$$

$$(3.6)\qquad \sigma_h^* f(x, x_{n+1}) = \sup_{t>0}\big||\sigma_{t,h}| * f(x, x_{n+1})\big|,$$

where $|\sigma_{t,h}|$ is defined in the same way as $\sigma_{t,h}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

Furthermore, for $m \in \mathbb{N}$, we define the family $\{\tilde{\sigma}_{m,k,h};\, k \in \mathbb{Z}\}$ of measures and the maximal operator $\tilde{\sigma}_{m,h}^*$ on $\mathbb{R}^{n+1}$ by

$$(3.7)\qquad \begin{aligned} \int_{\mathbb{R}^{n+1}} f\, d\tilde{\sigma}_{m,k,h} &= \int_{2^{mk}<|y|\le 2^{m(k+1)}} f(\phi(|y|)y', \psi(|y|))h(|y|)\frac{\Omega(y')}{|y|^n}\, dy\\ &= \sum_{l=mk+1}^{m(k+1)} \int_{\mathbb{R}^{n+1}} f\, d\sigma_{2^l,h}, \end{aligned}$$

$$(3.8)\qquad \tilde{\sigma}_{m,h}^* f(x, x_{n+1}) = \sup_{k\in\mathbb{Z}}\big||\tilde{\sigma}_{m,k,h}| * f(x, x_{n+1})\big|,$$

where $|\tilde{\sigma}_{m,k,h}|$ is defined in the same way as $\tilde{\sigma}_{m,k,h}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

Lemma 3.1. *Let* $1 < q \le +\infty$, $m \in \mathbb{N}$, *and* $\Omega \in L^q(S^{n-1})$ *with* $\|\Omega\|_{L^1(S^{n-1})} \le 1$, $\|\Omega\|_{L^q(S^{n-1})} \le 2^m$, *satisfying the cancelation condition* $\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0$. *Let* $\psi(\cdot)$ *be an arbitrary real-valued function on* $(0, \infty)$, *and* $h \in \Delta_\gamma$ *for some* $1 < \gamma \le \infty$. *Assume that* ϕ *is a positive* $C^1(0, \infty)$ *function satisfying the assumptions* (A-1) *and* (A-2).

Then there exist positive constants C *and* $\alpha < 1/q'$ *such that in the case of increasing* ϕ

$$|\widehat{\sigma}_{t,h}(\xi, \eta)| \le C\|h\|_{\Delta_1}, \tag{3.9}$$

$$|\widehat{\sigma}_{t,h}(\xi, \eta)| \le \frac{C\|h\|_{\Delta_\gamma}(1 + \|\varphi\|_\infty)}{|\phi(t/2)\xi|^{\alpha/m}}, \tag{3.10}$$

$$|\widehat{\sigma}_{t,h}(\xi, \eta)| \le C\|h\|_{\Delta_1}|\phi(t)\xi|^{\alpha/m}, \tag{3.11}$$

and in the case of decreasing ϕ, $\phi(t/2)$ *is replaced by* $\phi(t)$ *in* (3.10) *and* $\phi(t)$ *is replaced by* $\phi(t/2)$ *in* (3.11).

Similarly, in the case of increasing ϕ

$$|\widehat{\tilde{\sigma}}_{m,k,h}(\xi, \eta)| \le Cm\|h\|_{\Delta_1}, \tag{3.12}$$

$$|\widehat{\tilde{\sigma}}_{m,k,h}(\xi, \eta)| \le \frac{Cm\|h\|_{\Delta_\gamma}(1 + \|\varphi\|_\infty)}{|\phi(2^{mk})\xi|^{\alpha/m}}, \tag{3.13}$$

$$|\widehat{\tilde{\sigma}}_{m,k,h}(\xi, \eta)| \le Cm\|h\|_{\Delta_1}|\phi(2^{m(k+1)})\xi|^{\alpha/m}, \tag{3.14}$$

and in the case of decreasing ϕ, $\phi(2^{mk})$ *is replaced by* $\phi(2^{m(k+1)})$ *in* (3.13) *and* $\phi(2^{m(k+1)})$ *is replaced by* $\phi(2^{mk})$ *in* (3.14).

Proof. From the definition we have

$$|\widehat{\sigma}_{t,h}(\xi, \eta)| \le \int_{t/2}^{t} \frac{|h(r)|}{r} dr \int_{S^{n-1}} |\Omega(y')|\, d\sigma(y') \le 2\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})} \le 2\|h\|_{\Delta_1}.$$

Next, we show (3.10). In the case $1 < \gamma \le 2$, by a change of variable, Hölder's inequality and $\|\Omega\|_{L^1(S^{n-1})} \le 1$ we have

$$\begin{aligned}
|\widehat{\sigma}_{t,h}(\xi, \eta)| &\le \int_{t/2}^{t} |h(r)| \left| \int_{S^{n-1}} \Omega(y') e^{-i(\phi(r)y'\cdot\xi + \psi(r)\eta)} d\sigma(y') \right| \frac{dr}{r} \\
&\le 2^{1/\gamma}\|h\|_{\Delta_\gamma} \Big(\int_{t/2}^{t} \Big| \int_{S^{n-1}} \Omega(y') e^{-i(\phi(r)y'\cdot\xi + \psi(r)\eta)} d\sigma(y') \Big|^{\gamma'} \frac{dr}{r} \Big)^{1/\gamma'} \\
&\le 2^{1/\gamma}\|h\|_{\Delta_\gamma} \left(\int_{t/2}^{t} \left| \int_{S^{n-1}} \Omega(y') e^{-i(\phi(r)y'\cdot\xi + \psi(r)\eta)} d\sigma(y') \right|^{2} \frac{dr}{r} \right)^{1/\gamma'}.
\end{aligned}$$

In the case $\gamma > 2$, using Cauchy-Schwarz' inequality in place of Hölder's inequality, we get a similar inequality. Together with, we have

$$|\widehat{\sigma}_{t,h}(\xi,\eta)| \leq 2\|h\|_{\Delta_\gamma}\left(\int_{t/2}^{t}\left|\int_{S^{n-1}}\Omega(y')e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}d\sigma(y')\right|^2\frac{dr}{r}\right)^{1/\max\{\gamma',2\}}.$$

So, if ϕ satisfies (A-2) (i), by Lemma 2.3 we have for $0<\alpha<1/q'$

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\leq 2\|h\|_{\Delta_\gamma}\left(\frac{C_\alpha\|\varphi\|_\infty\|\Omega\|^2_{L^q(S^{n-1})}}{|\phi(t/2)\xi|^\alpha}\right)^{1/\max\{\gamma',2\}}\\ &\leq C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)\left(\frac{2^{2m}}{|\phi(t/2)\xi|^\alpha}\right)^{1/\max\{\gamma',2\}}.\end{aligned}$$

From this and (3.9) we obtain

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\leq C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)\left(\frac{2^{2m}}{|\phi(t/2)\xi|^\alpha}\right)^{1/(m\max\{\gamma',2\})}\\ &\leq \frac{C\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)}{|\phi(t/2)\xi|^{\alpha/(m\max\{\gamma',2\})}}.\end{aligned}$$

Taking $\alpha/\max\{\gamma',2\}$ newly as α, we get (3.10). The other three cases can be proved in a similar way, using Lemmas 2.4 (i), 2.3 (ii) and 2.4 (ii), respectively.

Finally we prove (3.11). Using the cancelation property of Ω and the monotonicity of ϕ, we have

$$\begin{aligned}|\widehat{\sigma}_{t,h}(\xi,\eta)| &\leq \int_{t/2}^{t}|h(r)|\left|\int_{S^{n-1}}\Omega(y')\left(e^{-i(\phi(r)y'\cdot\xi+\psi(r)\eta)}-e^{-i\psi(r)\eta}\right)d\sigma(y')\right|\frac{dr}{r}\\ &\leq C\|h\|_{\Delta_1}\max\{|\phi(t)\xi|,|\phi(t/2)\xi|\}\|\Omega\|_{L^1(S^{n-1})}.\end{aligned}$$

Combining this with (3.9) yields the desired estimate (3.11).

Since

$$\begin{aligned}\widehat{\widehat{\sigma}}_{m,k,h}(\xi,\eta) &= \int_{2^{mk}\leq|y|<2^{m(k+1)}}e^{-i(\xi\cdot\phi(|y|)y'+\eta\psi(|y|))}h(|y|)\frac{\Omega(y')}{|y|^n}\,dy\\ &= \sum_{l=mk+1}^{m(k+1)}\widehat{\sigma}_{2^l,h}(\xi,\eta),\end{aligned}\tag{3.15}$$

we obtain (3.12), (3.13) and (3.14) from (3.9), (3.10) and (3.11), respectively. ■

Remark 4. It is worthwhile to note that in order to get (3.9) and (3.10), we do not need the cancelation condition on Ω.

We next state a variant of the Lemma 3.4 in [2].

Lemma 3.2. *Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with*

$$\frac{a_{k+1}}{a_k} \ge a^A \quad \text{for some } a > 1 \text{ and } A > 0.$$

Let $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a sequence of Borel measures on $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose that for all $\ell \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, and some $\alpha > 0$, $C_0 > 0$, ℓ_0, $\ell_1 \in \mathbb{N} \cup \{0\}$, and $p_0 \ge 2$, we have

(*i*) $|\widehat{\sigma_k}(\xi)| \le C_0 \max\{1, (a_{k+\ell_0}|L(\xi)|)^{\alpha/A}, (a_{k-\ell_1}|L(\xi)|)^{-\alpha/A}\}$,

(*ii*) $\Big\|\Big(\sum_{k\in\mathbb{Z}} |\sigma_k * g_k|^2\Big)^{1/2}\Big\|_{p_0} \le C_0 \Big\|(\sum_{k\in\mathbb{Z}} |g_k|^2)^{1/2}\Big\|_{p_0}$ *for arbitrary functions g_k on $\mathbb{R}^n$.*

Then for $p_0' < p < p_0$, there exists a positive constant C_p such that

$$\Big\|\sum_{k\in\mathbb{Z}} \sigma_k * f\Big\|_p \le C_p C_0 \|f\|_p$$

and

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |\sigma_k * f|^2\Big)^{1/2}\Big\|_p \le C_p C_0 \|f\|_p$$

for all $f \in L^p(\mathbb{R}^n)$. The constant C_p is independent of A and of the linear transformation L.

In Al-Qassem and Pan [2], this lemma is given in the case $\ell_0 = 1$ and $\ell_1 = 0$, but one can easily check that the above holds.

For the maximal function $\tilde{\sigma}^*_{m,h}(f)$, we can show the following lemma in the same way as in the proof of the corresponding Lemma 3.3 in [2], by using Lemmas 2.2, 3.1, and 3.2.

Lemma 3.3. *Let $1 < q \le +\infty$, $m \in \mathbb{N}$, $h \in \Delta^\gamma(\mathbb{R}^n)$ for some $1 < \gamma \le \infty$, and $\Omega \in L^q(S^{n-1})$ with $\|\Omega\|_{L^1(S^{n-1})} \le 1$, $\|\Omega\|_{L^q(S^{n-1})} \le 2^m$. Assume that ϕ and ψ are positive $C^1(0,\infty)$ functions satisfying the assumptions* (A-1) *and* (A-2)*.*

Then for every $\gamma' < p \le \infty$, there exists a positive constant C_p independent of m such that

$$\|\tilde{\sigma}^*_{m,h}(f)\|_p \le C_p m \|f\|_p \tag{3.16}$$

for every $f \in L^p(\mathbb{R}^{n+1})$.

Proof. We prove this only in the case of increasing ϕ. In the case $p = \infty$, we have clearly

$$\begin{aligned}&\big||\tilde{\sigma}_{m,k,h}| * f(x, x_{n+1})\big|\\&= \left|\int_{2^{mk}\le|u|<2^{m(k+1)}} f\big(x - \phi(|u|)u', x_{n+1} - \psi(|u|)\big)|h(|u|)|\frac{|\Omega(u')|}{|u|^n}\,du\right|\\&\le \|f\|_\infty \sum_{l=mk+1}^{m(k+1)} \int_{2^{l-1}}^{2^l} \frac{|h(r)|}{r}\,dr \int_{S^{n-1}} |\Omega(u')|\,d\sigma(u')\\&\le 2m\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})}\|f\|_\infty.\end{aligned}$$

From this (3.16) follows immediately.

Next, we shall prove the case (a) $\gamma = \infty$ and $1 < p < \infty$. Fix a $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{\varphi}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{\varphi}(\xi) = 0$ for $|\xi| \ge 2$. For each $t > 0$, let $\widehat{\varphi}_t(\xi) = \hat{\varphi}\big(\phi(t)\xi\big)$. Define the family of measures $\{\Upsilon_{m,t}\}_{t>0}$ and $\{\vartheta_{m,k}\}_{k\in\mathbb{Z}}$ by

$$\begin{aligned}\widehat{\Upsilon}_{m,t}(\xi,\eta) &= \widehat{\mu}_{m,t,h}(\xi,\eta) - \widehat{\mu}_{m,t,h}(0,\eta)\widehat{\varphi}_t(\xi),\\ \widehat{\vartheta}_{m,k}(\xi,\eta) &= \sum_{l=km+1}^{k(m+1)} \widehat{\Upsilon}_{m,2^l}(\xi,\eta),\end{aligned}\tag{3.17}$$

where $\mu_{m,t,h} = |\sigma_{m,t,h}|$. Then

$$|\tilde{\sigma}_{m,k,h}| = \sum_{l=km+1}^{k(m+1)} \mu_{m,2^l,h}\tag{3.18}$$

and

$$\begin{aligned}&\vartheta_{m,k} * f(x, x_{n+1}) = |\tilde{\sigma}_{m,k,h}| * f(x, x_{n+1})\\&= \sum_{l=km+1}^{k(m+1)} \int_{\mathbb{R}^n}\Big(\int_{2^{l-1}\le|u|<2^l} f\big(x-y, x_{n+1}-\psi(|u|)\big)|h(|u|)|\frac{|\Omega(u')|}{|u|^n}\,du\Big)\varphi_{2^l}(y)\,dy.\end{aligned}\tag{3.19}$$

Now, let

$$\begin{aligned}&g_m(f) = \Big(\sum_{k\in\mathbb{Z}} |\vartheta_{m,k} * f|^2\Big)^{1/2}, \quad \vartheta_m^*(f) = \sup_{k\in\mathbb{Z}}\big||\vartheta_{m,k}| * f|,\\&M_\psi f(x,\, x_{n+1}) = \sup_{t>0}\left|\int_{t/2}^t f\big(x, x_{n+1} - \psi(s)\big)\,\frac{ds}{s}\right|.\end{aligned}$$

By (3.19) we have

$$
\begin{aligned}
&\tilde{\sigma}^*_{m,h} f(x,x_{n+1}) = \sup_{k\in\mathbb{Z}} \big| |\tilde{\sigma}_{m,k,h}| * f \big| (x,x_{n+1}) \\
&\le \sup_{k\in\mathbb{Z}} \big| |\vartheta_{m,k}| * f| + \sup_{k\in\mathbb{Z}} \sum_{l=km+1}^{k(m+1)} \int_{\mathbb{R}^n} \\
&\bigg(\int_{2^{l-1}\le|u|<2^l} \big| f\big(x-y, x_{n+1}-\psi(|u|)\big)\big| |h(|u|)| \frac{|\Omega(u')|}{|u|^n}\, du \bigg) |\varphi_{2^l}(y)|\, dy \\
&\le g_m(f)(x,x_{n+1}) + \|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} \sup_{k\in\mathbb{Z}} \sum_{l=km+1}^{k(m+1)} \int_{\mathbb{R}^n} \\
&\bigg(\sup_{\varepsilon>0} \int_{\varepsilon/2}^{\varepsilon} \frac{\big| f\big(x-y, x_{n+1}-\psi(r)\big)\big|}{r}\, dr \bigg) |\varphi_{2^l}(y)|\, dy \\
&\le g_m(f)(x,x_{n+1}) + Cm\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} \big((M_{\mathbb{R}^n} \otimes \mathrm{id}_{\mathbb{R}^1}) \circ M_\psi \big) f(x,x_{n+1}).
\end{aligned} \tag{3.20}
$$

Hence, by (3.19) we get

$$
\begin{aligned}
&\vartheta^*_m(f)(x,x_{n+1}) \\
&\le g_m(f)(x,x_{n+1}) + 2Cm\|h\|_\infty \|\Omega\|_{L^1(S^{n-1})} \big((M_{\mathbb{R}^n} \otimes \mathrm{id}_{\mathbb{R}^1}) \circ M_\psi \big) f(x,x_{n+1}).
\end{aligned} \tag{3.21}
$$

On the other hand, we have

$$
\widehat{\Upsilon}_{m,t}(\xi,\eta) = \int_{t/2<|y|<t} \big\{ e^{-i\phi(|y|)y'\cdot\xi} - \hat{\varphi}\big(\phi(t)\xi\big) \big\} e^{-i\psi(|y|)\eta} |h(|y|)| \frac{|\Omega(y')|}{|y|^n}\, dy. \tag{3.22}
$$

So, if $|\phi(t)\xi| > 1$, we get

$$
\begin{aligned}
\big|\widehat{\Upsilon}_{m,t}(\xi,\eta)\big| &\le 2 \int_{t/2}^{t} \frac{|h(r)|}{r} dr \int_{S^{n-1}} |\Omega(y')|\, d\sigma(y') \\
&\le 4\|h\|_{\Delta_1} \|\Omega\|_{L^1(S^{n-1})} \le 4\|h\|_{\Delta_1} \|\Omega\|_{L^1(S^{n-1})} |\phi(t)\xi|.
\end{aligned}
$$

And, if $|\phi(t)\xi| \le 1$, we have

$$
\begin{aligned}
\big|\widehat{\Upsilon}_{m,t}(\xi,\eta)\big| &= \bigg| \int_{t/2<|y|<t} \big\{ e^{-i\phi(|y|)y'\cdot\xi} - 1 \big\} e^{-i\psi(|y|)\eta} |h(|y|)| \frac{|\Omega(y')|}{|y|^n}\, dy \bigg| \\
&\le \int_{S^{n-1}} |\Omega(y')| \int_{t/2}^{t} |(\phi(|y|)y'\cdot\xi| \frac{|h(r)|}{r} dr\, d\sigma(y') \\
&\le 2\|h\|_{\Delta_1} \|\Omega\|_{L^1(S^{n-1})} |\phi(t)\xi|.
\end{aligned}
$$

Hence, in any case, we have

$$
\big|\widehat{\Upsilon}_{m,t}(\xi,\eta)\big| \le 4\|h\|_{\Delta_1} \|\Omega\|_{L^1(S^{n-1})} |\phi(t)\xi|. \tag{3.23}
$$

Also, by (3.22) we have

$$\left|\widehat{\Upsilon}_{m,t}(\xi,\eta)\right| \le \left|\int_{t/2<|y|<t} e^{-i(\phi(|y|)y'\cdot\xi+\psi(|y|)\eta)}|h(|y|)|\frac{|\Omega(y')|}{|y|^n}\,dy\right| + \left|\hat{\varphi}\big(\phi(t)\xi\big)\right|\int_{t/2<|y|<t}\frac{|h(|y|)\Omega(y')|}{|y|^n}\,dy.$$

Using Lemma 3.1 to the first part and the assumption $\varphi\in\mathcal{S}(\mathbb{R}^n)$ to the second part, we see that there exist $C>0$ and $0<\alpha<1/q$ such that

$$\left|\widehat{\Upsilon}_{m,t}(\xi,\eta)\right| \le C\frac{1}{|\phi(t/2)\xi|^{\alpha/m}}. \tag{3.24}$$

From (3.22) it is clear that

$$\left|\widehat{\Upsilon}_{m,t}(\xi,\eta)\right| \le 4\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})}. \tag{3.25}$$

By (3.23), (3.24), (3.25) and the definition of $\vartheta_{m,k}$, we see that there exist $C>0$ and $0<\alpha<1/q$ such that

$$|\widehat{\vartheta}_{m,k}(\xi,\eta)| \le Cm\|h\|_{\Delta_1}, \tag{3.26}$$

$$|\widehat{\vartheta}_{m,k}(\xi,\eta)| \le \frac{Cm\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)}{|\phi(2^{mk})\xi|^{\alpha/m}}, \tag{3.27}$$

$$|\widehat{\vartheta}_{m,k}(\xi,\eta)| \le Cm\|h\|_{\Delta_1}|\phi(2^{m(k+1)})\xi|^{\alpha/m}. \tag{3.28}$$

By (3.26), (3.27), (3.28), the lacunarity of $\{\phi(2^{mk})\}_{k\in\mathbb{Z}}$ (Lemma 2.1), and Plancherel's theorem, we get

$$\|g_m(f)\|_2 \le Cm\|f\|_2. \tag{3.29}$$

By the boundedness of the Hardy-Littlewood maximal function on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$, Lemma 2.3 and (3.29) we get

$$\|\vartheta_m^*(f)\|_2 \le Cm\|f\|_2. \tag{3.30}$$

By (3.29), (3.30) and applying the proof of the lemma in [9, p. 544] with $p_0=4$ and $q=2$, we obtain

$$\Big\|\Big(\sum_{k\in\mathbb{Z}}|\vartheta_{m,k}*g_k|^2\Big)^{1/2}\Big\|_4 \le Cm\Big\|\Big(\sum_{k\in\mathbb{Z}}|g_k|^2\Big)^{1/2}\Big\|_4 \tag{3.31}$$

for arbitrary functions $\{g_k\}_{k\in\mathbb{Z}}$ on $\mathbb{R}^{n+1}$. By (3.26), (3.27), (3.28), (3.31) and applying Lemma 3.3 we get

$$\|g_m(f)\|_p \le Cm\|f\|_p \tag{3.32}$$

for all $p \in (4/3,\, 4)$ and $f \in L^p(\mathbb{R}^{n+1})$. By replacing $p = 2$ with $p = 4/3 + \varepsilon$ with $\varepsilon \to 0$ in (3.29) and repeating the preceding arguments, we obtain (3.31) for every $p \in (8/7,\, 8)$ and $f \in L^p(\mathbb{R}^{n+1})$. By continuing this process we finally get

$$\|g_m(f)\|_p \le Cm\|f\|_p \tag{3.33}$$

for all $p \in (1,\, \infty)$ and $f \in L^p(\mathbb{R}^{n+1})$. Therefore, by (3.33) and (3.20), we obtain (3.16), which completes the proof of the lemma in the case $\gamma = \infty$.

Now, we shall treat the case $1 < \gamma < \infty$ and $\gamma' < p \le \infty$. By Hölder's inequality we get

$$\begin{aligned}
&\big||\tilde{\sigma}_{m,k,h}| * f(x, x_{n+1})\big| \\
&\le \Big(\int_{2^{mk}<|y|\le 2^{m(k+1)}} \frac{|h(|y|)|^\gamma |\Omega(y')|}{|y|^n} dy\Big)^{1/\gamma} \\
&\quad \times \Big(\int_{2^{mk}<|y|\le 2^{m(k+1)}} \big|f\big(x-\phi(|y|)y', x_{n+1}-\psi(|y|)\big)\big|^{\gamma'} \frac{|\Omega(y')|}{|y|^n} dy\Big)^{1/\gamma'} \\
&\le \Big(\sum_{l=mk}^{m(k+1)} \int_{2^{l-1}}^{2^l} \frac{|h(r)|^\gamma}{r} dr\Big)^{1/\gamma} \|\Omega\|_{L^1(S^{n-1})}^{1/\gamma} \big(|\tilde{\sigma}_{m,k,1}| * |f|^{\gamma'}(x, x_{n+1})\big)^{1/\gamma'} \\
&\le (2m\|\Omega\|_{L^1(S^{n-1})})^{1/\gamma} \|h\|_{\Delta_\gamma} \big(|\tilde{\sigma}_{m,k,1}| * |f|^{\gamma'}(x, x_{n+1})\big)^{1/\gamma'},
\end{aligned}$$

and hence

$$\tilde{\sigma}^*_{m,h}(f)(x, x_{n+1}) \le (2m\|\Omega\|_{L^1(S^{n-1})})^{1/\gamma} \|h\|_{\Delta_\gamma} \big(\tilde{\sigma}^*_{m,1}(|f|^{\gamma'})(x, x_{n+1})\big)^{1/\gamma'}. \tag{3.34}$$

Thus, applying the case (a) to $\tilde{\sigma}^*_{m,1}(|f|^{\gamma'})$ for $\gamma' < p \le \infty$, we get

$$\begin{aligned}
\|\tilde{\sigma}^*_{m,h}(f)\|_p &\le (2m\|\Omega\|_{L^1(S^{n-1})})^{1/\gamma} \|h\|_{\Delta_\gamma} \|\tilde{\sigma}^*_{m,1}(|f|^{\gamma'})\|_{p/\gamma'}^{1/\gamma'} \\
&\le Cm\|h\|_{\Delta_\gamma} \|f\|_p.
\end{aligned}$$

This completes the proof of the lemma. ∎

Lemma 3.4. *Let $h \in \triangle_\gamma$ for some $1 < \gamma \le 2$ and $2 \le p < 2\gamma/(2-\gamma)$. Again let m, Ω, ϕ, ψ be as in Lemma 3.3. Then there exists a positive constant C_p such that*

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |\tilde{\sigma}_{m,k,h} * g_k|^2\Big)^{1/2}\Big\|_p \le C_p m^{1/2} \Big\|(\sum_{k\in\mathbb{Z}} |g_k|^2)^{1/2}\Big\|_p \tag{3.35}$$

for any sequence $\{g_k\}$ of functions on $\mathbb{R}^{n+1}$.

Proof. By duality there exists a nonnegative function $f \in L^{(p/2)'}(\mathbb{R}^{n+1})$ with $\|f\|_{(p/2)'} = 1$ such that

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |\tilde{\sigma}_{m,k,h} * g_k|^2\Big)^{1/2}\Big\|_p^2 = \sum_{k\in\mathbb{Z}} |\tilde{\sigma}_{m,k,h} * g_k(x, x_{n+1})|^2 f(x, x_{n+1})\, dx dx_{n+1}.$$

By Schwarz's inequality

$$\begin{aligned}&|\tilde{\sigma}_{m,k,h} * g_k(x, x_{n+1})|^2 \\ &\le C \int_{2^{mk}}^{2^{m(k+1)}} \int_{S^{n-1}} \big|g_k\big(x - \phi(r)y', x_{n+1} - \psi(r)\big)\big|^2 |\Omega(y')||h(r)|^{2-\gamma}\, d\sigma(y')\,\frac{dr}{r}.\end{aligned}$$

Thus, by a change of variable we have

$$\begin{aligned}(3.36)\quad &\Big\|\Big(\sum_{k\in\mathbb{Z}} |\tilde{\sigma}_{m,k,h} * g_k|^2\Big)^{1/2}\Big\|_p^2 \\ &\le C \int_{\mathbb{R}^{n+1}} \Big(\sum_{k\in\mathbb{Z}} |g_k(x, x_{n+1})|^2\Big) \tilde{\sigma}^*_{m,|h|^{2-\gamma}}(\tilde{f})(-x, -x_{n+1})\, dx dx_{n+1},\end{aligned}$$

where $\tilde{f}(x, x_{n+1}) = f(-x, -x_{n+1})$. By Lemma 3.3 and noting $|h(\cdot)|^{2-\gamma} \in \Delta_{\gamma/(2-\gamma)}(\mathbb{R}_+)$ and $(p/2)' = \big(\gamma/(2-\gamma)\big)'$ we obtain

$$(3.37)\qquad \|\tilde{\sigma}^*_{m,|h|^{2-\gamma}}(\tilde{f})\|_{L^{(p/2)'}(\mathbb{R}^{n+1})} \le C_p m \|f\|_{L^{(p/2)'}(\mathbb{R}^{n+1})} = C_p m.$$

Therefore, by (3.36), (3.37) and Hölder's inequality we get (3.35) for $2 \le p < 2\gamma/(2-\gamma)$. ∎

4. Proof of Theorem 1 in the case of $\Omega \in L(\log L)(S^{n-1})$

Using Lemma 3.2 and Lemma 3.4 in the case $q = 2$, we shall show our Theorem 1 in a quite similar way in the proof of Theorems 1.1 in [2].

Proof of Theorem 1. We first show (i). Assume that Ω satisfies (1.1) and belongs to $L\log L(S^{n-1})$. For $m \in \mathbb{N}$, set $E_m = \{y' \in S^{n-1} : 2^{m-1} \le |\Omega(y')| < 2^m\}$, and $E_0 = \{y' \in S^{n-1}; |\Omega(y')| < 1\}$. For $m \in \mathbb{N} \cup \{0\}$, set $A_m = \|\Omega\chi_{E_m}\|_{L^1(S^{n-1})}$, and $\Lambda = \{m \in \mathbb{N} : A_m > 2^{-m}\}$. For $m \in \Lambda$ define the sequence $\{\Omega_m\}_{m\in\Lambda}$ of functions by

$$\Omega_m(y') = \frac{1}{A_m}\bigg(\Omega(y')\chi_{E_m}(y') - \frac{1}{|S^{n-1}|}\int_{E_m} \Omega(x')\, d\sigma(x')\bigg),$$

and

$$\Omega_0(y') = \Omega(y') - \sum_{m\in\Lambda} A_m \Omega_m(y').$$

Then it is easy to verify that for all $m \in \Lambda \cup \{0\}$ and some positive constant C,

$$\|\Omega_m\|_{L^2(S^{n-1})} \le C2^m, \quad \|\Omega_m\|_{L^1(S^{n-1})} \le C, \tag{4.1}$$

$$A_0 + \sum_{m\in\Lambda} m\, A_m \le C\|\Omega\|_{L\log L(S^{n-1})}, \tag{4.2}$$

$$\int_{S^{n-1}} \Omega_m(y')\, d\sigma(y') = 0, \quad \Omega = \Omega_0 + \sum_{m\in\Lambda} A_m\Omega_m. \tag{4.3}$$

From the above, we see that

$$\|T_{\Omega,\phi,\psi,h}f\|_p \le \|T_{0,\phi,\psi,h}f\|_p + \sum_{m\in\Lambda} A_m\|T_{m,\phi,\psi,h}f\|_p, \tag{4.4}$$

where

$$T_{m,\phi,\psi,h}f(x,x_{n+1}) = \text{p. v.}\int_{\mathbb{R}^n} f(x-\phi(|y|)y', x_{n+1}-\psi(|y|))\frac{\Omega_m(y')}{|y|^n}h(|y|)dy.$$

However,

$$\begin{aligned}
&T_{m,\phi,\psi,h}f(x,x_{n+1})\\
&= \sum_{k\in\mathbb{Z}}\int_{2^{mk}<|y|\le 2^{m(k+1)}} f(x-\phi(|y|)y', x_{n+1}-\psi(|y|))\frac{\Omega_m(y')}{|y|^n}h(|y|)dy\\
&= \sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}\int_{2^{i-1}<|y|\le 2^i} f(x-\phi(|y|)y', x_{n+1}-\psi(|y|))\frac{\Omega_m(y')}{|y|^n}h(|y|)dy\\
&= \sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m} \sigma_{m,2^i,h} * f(x,x_{n+1}).
\end{aligned}$$

Since $\triangle_\gamma \subseteq \triangle_2$ for $\gamma \ge 2$, we may assume that $1 < \gamma \le 2$ and $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{\gamma'}$. For $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, let $\theta_{m,k} = \phi(2^{mk})$. From Lemma 2.1, we easily see that $\{\theta_{m,k}\}$ is a lacunary sequence with $\theta_{m,k+1}/\theta_{m,k} \ge 2^{m/b} > 1$, where $b = \|\phi(t)/(t\phi'(t))\|_\infty$. Let $\{\hat{\Psi}_{m,k}, k \in \mathbb{Z}\}$ be a smooth partition of unity in $(0,\infty)$, defined in Lemma 2.5, and set $(\widehat{T_{m,k}f})(\xi,\eta) = \widehat{\Psi}_{m,k}(|\xi|)\hat{f}(\xi,\eta)$, $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}$. Then

$$\begin{aligned}
&T_{m,\phi,\psi,h}f(x,x_{n+1})\\
&= \sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}\sum_{j\in\mathbb{Z}}(\Psi_{m,j+k}\otimes\delta_{\{0\}}) * \sigma_{m,2^i,h} * f(x,x_{n+1})\\
&= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}(\Psi_{m,j+k}\otimes\delta_{\{0\}}) * \sigma_{m,2^i,h} * f(x,x_{n+1})\\
&= \sum_{j\in\mathbb{Z}} Q_{m,j}f(x,x_{n+1}),
\end{aligned} \tag{4.5}$$

where $\delta_{\{0\}}$ is the Dirac's delta at the origin in the x_{n+1} space, and $Q_{m,j}f(x, x_{n+1}) = \sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}(\Psi_{m,j+k}\otimes\delta_{\{0\}}) * \sigma_{m,2^i,h} * f(x, x_{n+1})$.
From Lemma 3.1, we have the following estimates:

$$\left|\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right| \le Cm; \tag{4.6}$$

$$\left|\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right| \le Cm|\phi(2^{mk})\xi|^{-\alpha/m}; \tag{4.7}$$

$$\left|\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right| \le Cm|\phi(2^{m(k+1)})\xi|^{\alpha/m}. \tag{4.8}$$

First, we compute L^2 norm of $Q_{m,j}f$. By Plancherel's theorem, Fubini's theorem and (4.7), (4.8), we obtain

$$\begin{aligned}
\|Q_{m,j}f\|_2^2 &= \int_{\mathbb{R}^{n+1}}\left|\sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}(\Psi_{m,j+k}\otimes\delta_{\{0\}}) * \sigma_{m,2^i,h} * f(x,x_{n+1})\right|^2 dx\,dx_{n+1}\\
&= (2\pi)^n\int_{\mathbb{R}^{n+1}}\left|\sum_{k\in\mathbb{Z}}\sum_{i=km+1}^{(k+1)m}\widehat{\Psi}_{m,j+k}(\xi)\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\hat{f}(\xi,\eta)\right|^2 d\xi\,d\eta\\
&= (2\pi)^n\int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}\widehat{\Psi}_{m,j+k}(\xi)\left\{\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right\}\\
&\quad\times\left\{\sum_{t=-1}^{1}\overline{\widehat{\Psi}_{m,j+k+t}(\xi)}\sum_{i=(k+t)m+1}^{(k+t+1)m}\overline{\widehat{\sigma}_{m,2^i,h}(\xi,\eta)}\right\}|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le 3(2\pi)^n\int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}|\widehat{\Psi}_{m,j+k}(\xi)|^2\left|\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right|^2|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta\\
&\le 3(2\pi)^n\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\theta^{-1}_{m,k+j+1}\le|\xi|\le\theta^{-1}_{m,k+j-1}}\left|\sum_{i=km+1}^{(k+1)m}\widehat{\sigma}_{m,2^i,h}(\xi,\eta)\right|^2|\hat{f}(\xi,\eta)|^2 d\xi\,d\eta.
\end{aligned}$$

For $j \le -2$ and $\theta^{-1}_{m,k+j+1} \le |\xi| \le \theta^{-1}_{m,k+j-1}$ we get, using (4.7),

$$\|Q_{m,j}f\|_2 \le Cm(2^m)^{j\alpha/(bm)}\|f\|_2 \le Cm2^{j\alpha/b}\|f\|_2.$$

For $j \ge 2$ and $\theta^{-1}_{m,k+j+1} \le |\xi| \le \theta^{-1}_{m,k+j-1}$ we get, using (4.8),

$$\|Q_{m,j}f\|_2 \le Cm(2^m)^{-j\alpha/(bm)}\|f\|_2 \le Cm2^{-j\alpha/b}\|f\|_2.$$

For $-1 \le j \le 1$ and $\theta_{m,k+j+1}^{-1} \le |\xi| \le \theta_{m,k+j-1}^{-1}$ we get, using (4.6),

$$\|Q_{m,j}f\|_2 \le Cm\|f\|_2.$$

Hence, we obtain

$$\|Q_{m,j}f\|_2 \le Cm2^{-|j|\alpha/b}\|f\|_2. \tag{4.9}$$

Next, as for L^p estimate, by using Lemma 3.4, Lemma 3.2 and Lemma 2.5, we have

$$\|Q_{m,j}f\|_p \le Cm\|f\|_p, \qquad \text{for } \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{\gamma'}. \tag{4.10}$$

Interpolating between (4.9) and (4.10), we can find a number $0 < \theta < 1$ such that

$$\|Q_{m,j}f\|_p \le Cm2^{-|j|\theta\alpha/b}\|f\|_p, \qquad \text{for } \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{\gamma'}. \tag{4.11}$$

Hence, combining (4.5), (4.11), (4.4) and (4.2), we complete the proof of Theorem 1 (i).

Next, we show (ii). We first note that for any $\Omega \in L^1(S^{n-1})$, $\mathcal{M}_{\Omega,\phi,\psi,h}f(x,x_{n+1}) \le 2^n(2^n-1)^{-1}\sigma_h^* f(x,x_{n+1})$, and hence for any $m \in \mathbb{N}$

$$\mathcal{M}_{\Omega,\phi,\psi,h}f(x,x_{n+1}) \le \frac{2^{n+1}}{2^n-1}\tilde{\sigma}_{m,h}^* f(x,x_{n+1}), \tag{4.12}$$

where σ_h^* and $\tilde{\sigma}_{m,h}^*$ are defined by (3.6) and (3.8), respectively.

Now, by (4.3) and (4.12) we have

$$\begin{aligned} &\mathcal{M}_{\Omega,\phi,\psi,h}f(x,x_{n+1}) \\ &\le M_{\Omega_0,\phi,\psi,h}f(x,x_{n+1}) + \sum_{m\in\Lambda} A_m \mathcal{M}_{\Omega_m,\phi,\psi,h}f(x,x_{n+1}) \\ &\le \frac{2^{n+1}}{2^n-1}\left[\sigma_{0,\Omega_0,\phi,\psi,h}^* f(x,x_{n+1}) + \sum_{m\in\Lambda} A_m \sigma_{m,\Omega_m,\phi,\psi,h}^* f(x,x_{n+1})\right], \end{aligned} \tag{4.13}$$

where $\sigma_{m,\Omega_m,\phi,\psi,h}^* = \tilde{\sigma}_{m,h}^*$ which is defined by (3.8) for $\Omega = \Omega_m$. Then, using Lemma 3.2, Minkowski's inequality and (4.2), we obtain for $\gamma' < p \le \infty$

$$\|\mathcal{M}_{\Omega,\phi,\psi,h}f\|_p \le C\|f\|_p + \sum_{m\in\Lambda} CA_m m\|f\|_p \le C\|f\|_p.$$

This completes the proof of Theorem 1 (ii).

Finally we show (iii). It suffices to show that

$$\|T_{\Omega_m,\phi,\psi,h}^* f\|_{L^p(\mathbb{R}^{n+1})} \le C_p m\|f\|_{L^p(\mathbb{R}^{n+1})}. \tag{4.14}$$

To show this, since for every $0 < \varepsilon < A < \infty$ there exist $k_0, k_1 \in \mathbb{Z}$ with $2^{mk_0} \le \varepsilon < 2^{m(k_0+1)}$ and $2^{mk_1} \le A < 2^{m(k_1+1)}$, we see by Lemma 3.1 that it suffices to show

$$\|T^{**}_{\Omega_m,\phi,\psi,h} f\|_{L^p(\mathbb{R}^{n+1})} \le C_p m \|f\|_{L^p(\mathbb{R}^{n+1})}, \tag{4.15}$$

where

$$\begin{aligned}
&T^{**}_{\Omega_m,\phi,\psi,h} f(x, x_{n+1}) \\
&= \sup_{k\in\mathbb{Z}} \left| \int_{|y|>2^{mk}} f(x-\phi(|y|)y', x_{n+1}-\psi(|y|)) \frac{\Omega_m(y')}{|y|^n} h(|y|)dy \right| \\
&= \sup_{k\in\mathbb{Z}} \left| \sum_{j=k}^{\infty} \int_{2^{mj}<|y|\le 2^{m(j+1)}} f(x-\phi(|y|)y', x_{n+1}-\psi(|y|)) \frac{\Omega_m(y')}{|y|^n} h(|y|)dy \right| \\
&= \sup_{k\in\mathbb{Z}} \left| \sum_{j=k}^{\infty} \tilde{\sigma}_{m,j,h} * f(x, x_{n+1}) \right| =: \sup_{k\in\mathbb{Z}} |I_k(f)(x, x_{n+1})|.
\end{aligned} \tag{4.16}$$

We take a radial function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ when $|\xi| \le 2^{-m/b}$ and $\varphi(\xi) = 0$ when $|\xi| \ge 2^{m/b}$, where $b = \|\phi(t)/(t\phi'(t)\|_\infty$. We also assume $\|\varphi\|_\infty = 1$. Let $\varphi_k(\xi) = \varphi\big(\phi(2^m k)\xi\big)$ and $\widehat{\Phi}_k(\xi) = \varphi_k(\xi)$. Then it is easily seen that

$$|\Phi_k(x)| \le \frac{C_0}{\phi(2^{mk})^n \big(1 + |x|/\phi(2^{mk})\big)^{n+1}} \tag{4.17}$$

for some $C_0 > 0$. Let δ be the Dirac delta function in $\mathbb{R}^{n+1}$ and as before, $\delta_{(0)}$ be the 1-dimensional Dirac delta function in x_{n+1}. Now

$$\begin{aligned}
I_k(f) = \big(\delta - \Phi_k \otimes \delta_{(0)}\big) * \sum_{j=k}^{\infty} \tilde{\sigma}_{m,j} * f + \Phi_k \otimes \delta_{(0)} * T_{\Omega_m,\phi,\psi,h} f \\
- \Phi_k \otimes \delta_{(0)} * \sum_{j=-\infty}^{k-1} \tilde{\sigma}_{m,j} * f =: I_{k,1}(f) + I_{k,2}(f) + I_{k,3}(f).
\end{aligned}$$

Clearly, by using (4.17) we see that

$$|I_{k,2}(f)| \le C M_{\mathbb{R}^n} \otimes \mathrm{id}_{\mathbb{R}}(T_{\Omega_m,\phi,\psi,h} f).$$

Hence, by the L^p boundedness of the Hardy-Littlewood maximal function and the fact $\|T_{\Omega_m,\phi,\psi,h} f\|_p \le C_p m \|f\|_p$ for p with $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ in (i), we obtain

$$\|\sup_{k\in\mathbb{Z}} |I_{k,2}(f)|\|_p \le C_p m \|f\|_p \tag{4.18}$$

when $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Next

$$\begin{aligned}\sup_{k\in\mathbb{Z}}|I_{k,3}(f)| &= \sup_{k\in\mathbb{Z}}\Big|\sum_{j=1}^{\infty}\tilde{\sigma}_{m,k-j,h}*(\Phi_k\otimes\delta_{(0)})*f\Big| \\ &\le \sum_{j=1}^{\infty}\Big(\sup_{k\in\mathbb{Z}}\big|\tilde{\sigma}_{m,k-j,h}*(\Phi_k\otimes\delta_{(0)})*f\big|\Big) =: \sum_{j=1}^{\infty}G_j(f).\end{aligned}$$

Since we see by using (4.17)

$$G_j(f) \le \sup_{k\in\mathbb{Z}}|\tilde{\sigma}_{m,k-j,h}|*(M_{\mathbb{R}^n}\otimes \mathrm{id}_{\mathbb{R}})f,$$

we have by Lemma 3.3

$$\|G_j(f)\|_p \le Cm\|(M_{\mathbb{R}^n}\otimes \mathrm{id}_{\mathbb{R}})f\|_p \le Cm\|f\|_p \tag{4.19}$$

for $p>\gamma'$. On the other hand, it holds

$$G_j(f) \le \Big(\sum_{k\in\mathbb{Z}}|\tilde{\sigma}_{m,k-j,h}*(\Phi_k\otimes\delta_{(0)})*f|^2\Big)^{1/2}.$$

So, by Plancherel's theorem, Lemma 3.1, the support property of Φ_k and Lemma 2.1, we see that

$$\begin{aligned}\|G_j(f)\|_2 &= C\Big\|\Big(\sum_{k\in\mathbb{Z}}|\widehat{\tilde{\sigma}}_{m,k-j,h}(\xi,\eta)\widehat{\Phi}_k(\xi)\hat{f}(\xi,\eta)|^2\Big)^{1/2}\Big\|_2 \\ &\le Cm\Big\|\Big(\sum_{\phi(2^{mk})|\xi|\le 2^{m/b}}|\phi(2^{m(k-j+1)})\xi|^{2\alpha/m}\Big)^{1/2}\hat{f}(\xi,\eta)\Big\|_2 \\ &\le Cm2^{-\alpha(j-1)/b}\Big\|\Big(\sum_{\phi(2^{mk})|\xi|\le 2^{m/b}}|\phi(2^{mk})\xi|^{2\alpha/m}\Big)^{1/2}\hat{f}(\xi,\eta)\Big\|_2 \\ &\le Cm2^{-\alpha(j-1)/b}\|f\|_2.\end{aligned} \tag{4.20}$$

For any fixed $p>\gamma'$, we take $p_0>p$ when $p>2$ and $\gamma'<p_0<p$ when $p<2$, and interpolate between (4.19) for p_0 and (4.20), and get for some $\beta>0$

$$\|G_j(f)\|_p \le Cm2^{-\beta j}\|f\|_p, \tag{4.21}$$

which leads to

$$\Big\|\sup_{k\in\mathbb{Z}}I_{k,3}(f)\Big\|_p \le Cm\|f\|_p \tag{4.22}$$

for $p>\gamma'$.

As for $I_{k,1}(f)$, we have

$$\begin{aligned}\sup_{k\in\mathbb{Z}}|I_{k,1}(f)| &= \sup_{k\in\mathbb{Z}}\Big|(\delta-\Phi_k\otimes\delta_{(0)})*\sum_{j=0}^{\infty}\tilde{\sigma}_{m,k+j,h}*f\Big|\\ &\le \sum_{j=0}^{\infty}\sup_{k\in\mathbb{Z}}\big|(\delta-\Phi_k\otimes\delta_{(0)})*\tilde{\sigma}_{m,k+j,h}*f\big| =: H_j(f).\end{aligned}$$

As before, we get

$$\|H_j(f)\|_p \le Cm\|f\|_p \tag{4.23}$$

for $p>\gamma'$. On the other hand, it holds

$$H_j(f) \le \Big(\sum_{k\in\mathbb{Z}}|\tilde{\sigma}_{m,k+j,h}*(\delta-\Phi_k\otimes\delta_{(0)})*f|^2\Big)^{1/2}.$$

So, by Plancherel's theorem, Lemma 3.1, the support property of Φ_k and Lemma 2.1, we see that

$$\begin{aligned}\|H_j(f)\|_2 &= C\Big\|\Big(\sum_{k\in\mathbb{Z}}|\widehat{\tilde{\sigma}}_{m,k+j,h}(\xi,\eta)\big(1-\widehat{\Phi}_k(\xi)\big)\hat{f}(\xi,\eta)|^2\Big)^{1/2}\Big\|_2\\ &\le Cm\Big\|\Big(\sum_{\phi(2^{mk})|\xi|\ge 2^{-m/b}}\frac{1}{|\phi(2^{m(k+j)})\xi|^{2\alpha/m}}\Big)^{1/2}\hat{f}(\xi,\eta)\Big\|_2\\ &\le Cm2^{-\alpha j/b}\Big\|\Big(\sum_{\phi(2^{mk})|\xi|\ge 2^{-m/b}}\frac{1}{|\phi(2^{mk})\xi|^{2\alpha/m}}\Big)^{1/2}\hat{f}(\xi,\eta)\Big\|_2\\ &\le Cm2^{-\alpha j/b}\|f\|_2.\end{aligned} \tag{4.24}$$

Thus, as in the estimate for $I_{k,3}$, we obtain

$$\Big\|\sup_{k\in\mathbb{Z}}I_{k,1}(f)\Big\|_p \le Cm\|f\|_p \tag{4.25}$$

for $p>\gamma'$. Combining (4.18), (4.22), (4.25) and (4.16), we obtain the desired estimate (4.15), i.e.,

$$\|T^{**}_{\Omega_m,\phi,\psi,h}f\|_{L^p(\mathbb{R}^{n+1})} \le C_p m\|f\|_{L^p(\mathbb{R}^{n+1})},$$

for p satisfying $p>\gamma'$ and $|1/2-1/p|<\min\{1/2,1/\gamma'\}$, i.e., $\gamma'<p<1/(1/2-\min\{1/2,1/\gamma'\})$. This completes the proof of Theorem 1 (iii). ∎

5. Proof of Theorem 1 in the Case of $\Omega\in\cup_{1<q<\infty}B_q^{(0,0)}(S^{n-1})$

Let $q>1$. Then if $\Omega\in B_q^{(0,0)}(S^{n-1})$ and satisfies the cancelation condition, it can be written as $\Omega=\sum_{\ell=1}^{\infty}\lambda_\ell\breve{\Omega}_\ell$, where $\lambda_\ell\in\mathbb{C}$ and $\breve{\Omega}_\ell$ is a q-block supported on a cap $B_\ell=B(x_\ell,\tau_\ell)\cap S^{n-1}$ on S^{n-1} and

$$\sum_{\ell=1}^{\infty}|\lambda_\ell|\big\{1+\log\big(|B_\ell|^{-1}\big)\big\} < 2\|\Omega\|_{B_q^{(0,0)}(S^{n-1})} < \infty. \tag{5.1}$$

To each block $\breve{\Omega}_\ell$, we define

$$\Omega_\ell(y') = \breve{\Omega}_\ell(y') - \frac{1}{|S^{n-1}|}\int_{S^{n-1}} \breve{\Omega}_\ell(x')\, d\sigma(x').$$

Let $\Lambda = \{\ell \in \mathbb{N};\ |B_\ell| \le 1/2\}$ and set

$$\Omega_0 = \Omega - \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{5.2}$$

Then there exists a positive constant C such that the followings hold for all $\ell \in \Lambda$:

$$\int_{S^{n-1}} \Omega_\ell(x')\, d\sigma(x') = 0, \tag{5.3}$$

$$\|\Omega_\ell\|_{L^q(S^{n-1})} \le C|B_\ell|^{-1/q'}, \tag{5.4}$$

$$\|\Omega_\ell\|_{L^1(S^{n-1})} \le 2, \tag{5.5}$$

$$\Omega = \Omega_0 + \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{5.6}$$

Moreover, from (5.1) and the definition of Ω_ℓ it follows that

$$\|\Omega_0\|_{L^q(S^{n-1})} \le C \sum_{\ell\in\mathbb{N}\setminus\Lambda} 2^{-1/q'}|\lambda_\ell| \le C\|\Omega\|_{B_q^{(0,0)}(S^{n-1})}, \tag{5.7}$$

$$\int_{S^{n-1}} \Omega_0(x')\, d\sigma(x') = 0. \tag{5.8}$$

For $\ell \in \Lambda$, define a family of measures $\sigma^{(\ell)} = \{\sigma_{\ell,t,h}; 0 < t < \infty\}$ on $\mathbb{R}^{n+1}$, as in Definition 3, by

$$\int_{\mathbb{R}^{n+1}} f\, d\sigma_{\ell,t,h} = \int_{t/2<|y|<t} f\big(\phi(|y|)y', \psi(|y|)\big)h(|y|)\frac{\Omega_\ell(y')}{|y|^n}\, dy.$$

We only discuss the case of increasing ϕ in the proof of Theorem 1 in the case of $\Omega \in \cup_{1<q<\infty} B_q^{(0,0)}(S^{n-1})$, since decreasing case can be proved in the same way.

For $k \in \mathbb{Z}$ and $\ell \in \Lambda \cup 0$, we set $\omega_\ell = 2^{\theta_\ell}$, $\theta_\ell = [\log_2 |B_\ell|^{-1/q'}] + 1$, where $[\cdot]$ denotes the greatest integer function.

From Lemma 3.1, we have the following estimates:

$$\left|\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell} \widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right| \le C\theta_\ell; \tag{5.9}$$

$$\left|\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right|\le C\theta_\ell|\phi(\omega_\ell^k)\xi|^{-\alpha/\theta_\ell};\tag{5.10}$$

$$\left|\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right|\le C\theta_\ell|\phi(\omega_\ell^{k+1})\xi|^{\alpha/\theta_\ell}.\tag{5.11}$$

Moreover, we can use Lemmas 3.3 and 3.4, taking $m=\theta_\ell$. Now, we begin to prove Theorem 1 in the case of $\Omega\in\cup_{1<q<\infty}B_q^{(0,0)}(S^{n-1})$. By (5.6), we have

$$|T_{\Omega,\phi,\psi,h}f(x,x_{n+1})|\le\sum_{\ell\in\Lambda\cup 0}|\lambda_\ell|\,|T_{\ell,\phi,\psi,h}f(x,x_{n+1})|,\tag{5.12}$$

where

$$T_{\ell,\phi,\psi,h}f(x,x_{n+1})=\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k\le|y|\le\omega_\ell^{k+1}}\frac{\Omega_\ell(u')}{|u|^n}h(|u|)f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du.$$

So, we have only to show the boundedness of $T_{\ell,\phi,\psi,h}f$.

$$\begin{aligned}
&T_{\ell,\phi,\psi,h}f(x,x_{n+1})\\
&=\sum_{k\in\mathbb{Z}}\int_{\omega_\ell^k\le|y|\le\omega_\ell^{k+1}}\frac{\Omega_\ell(u')}{|u|^n}h(|u|)f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du\\
&=\sum_{k\in\mathbb{Z}}\sum_{j=k\theta_\ell+1}^{(k+1)\theta_\ell}\int_{2^{j-1}\le|y|\le 2^j}\frac{\Omega_\ell(u')}{|u|^n}h(|u|)f(x-\phi(|u|)u',x_{n+1}-\psi(|u|))du\\
&=\sum_{k\in\mathbb{Z}}\sum_{j=k\theta_\ell+1}^{(k+1)\theta_\ell}\sigma_{\ell,2^j,h}*f(x,x_{n+1}).
\end{aligned}$$

Since $\Delta_\gamma\subseteq\Delta_2$ for $\gamma\ge 2$, we may assume that $1<\gamma\le 2$ and $|1/p-1/2|<\frac{1}{\gamma'}$. For $\ell\in\mathbb{Z}$, let $\theta_{\ell,j}=\phi(\omega_\ell^j)$. From Lemma 2.1, we easily see that $\{\theta_{\ell,j},j\in\mathbb{Z}\}$ is a lacunary sequence with $\theta_{\ell,j+1}/\theta_{\ell,j}\ge\omega_\ell^{1/b}>1$. Let $\{\widehat{\Psi}_{\ell,j},j\in\mathbb{Z}\}$ be a smooth partition of unity in $(0,\infty)$, defined in Lemma 2.5, and set $\widehat{(T_{\ell,j}f)}(\xi,\eta)=\widehat{\Psi}_{\ell,j}(|\xi|)\hat{f}(\xi,\eta)$, $(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}$. Then

$$\begin{aligned}
T_{\ell,\phi,\psi,h}f(x,x_{n+1})&=\sum_{k\in\mathbb{Z}}\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\sum_{j\in\mathbb{Z}}(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,2^i,h}*f(x,x_{n+1})\\
&=\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}(\Psi_{\ell,j+k}\otimes\delta_{\{0\}})*\sigma_{\ell,2^i,h}*f(x,x_{n+1})\\
&=\sum_{j\in\mathbb{Z}}Q_{\ell,j}f(x,x_{n+1}),
\end{aligned}\tag{5.13}$$

where $\delta_{\{0\}}$ is the Dirac's delta at the origin in the x_{n+1} space, and $Q_{\ell,j}f(x,x_{n+1}) = \sum_{k\in\mathbb{Z}}\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}(\Psi_{\ell,j+k}\otimes\delta_{\{0\}}) * \sigma_{\ell,2^i,h} * f(x,x_{n+1})$.

First, we compute L^2 norm of $Q_{\ell,j}f$. By Plancherel's theorem, Fubini's theorem and (5.10), (5.11), we obtain

$$\begin{aligned}
\|Q_{\ell,j}f\|_2^2 &= \int_{\mathbb{R}^{n+1}}\left|\sum_{k\in\mathbb{Z}}\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}(\Psi_{\ell,j+k}\otimes\delta_{\{0\}}) * \sigma_{\ell,2^i,h} * f(x,x_{n+1})\right|^2 dx\,dx_{n+1} \\
&= (2\pi)^n\int_{\mathbb{R}^{n+1}}\left|\sum_{k\in\mathbb{Z}}\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\Psi}_{\ell,j+k}(\xi)\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\hat{f}(\xi,\eta)\right|^2 d\xi\,d\eta \\
&= (2\pi)^n\int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}\widehat{\Psi}_{\ell,j+k}(\xi)\left\{\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right\} \\
&\qquad\times\left\{\sum_{m=-1}^{1}\overline{\widehat{\Psi}_{\ell,j+k+m}(\xi)}\sum_{i=(k+m)\theta_\ell+1}^{(k+m+1)\theta_\ell}\overline{\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)}\right\}|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta \\
&\le 3(2\pi)^n\int_{\mathbb{R}^{n+1}}\sum_{k\in\mathbb{Z}}|\widehat{\Psi}_{\ell,j+k}(\xi)|^2\left|\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right|^2|\hat{f}(\xi,\eta)|^2\,d\xi\,d\eta \\
&\le 3(2\pi)^n\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\int_{\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}}\left|\sum_{i=k\theta_\ell+1}^{(k+1)\theta_\ell}\widehat{\sigma}_{\ell,2^i,h}(\xi,\eta)\right|^2|\hat{f}(\xi,\eta)|^2 d\xi\,d\eta.
\end{aligned}$$

For $j\le -2$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.10),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell\omega_\ell^{j\alpha/(b\theta_\ell)}\|f\|_2 \le C(\log|B_\ell|^{-1})2^{j\alpha/b}\|f\|_2,$$

For $j\ge 2$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.11),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell\omega_\ell^{-j\alpha/(b\theta_\ell)}\|f\|_2 \le C(\log|B_\ell|^{-1})2^{-j\alpha/b}\|f\|_2.$$

For $-1\le j\le 1$ and $\theta_{\ell,k+j+1}^{-1}\le|\xi|\le\theta_{\ell,k+j-1}^{-1}$ we get, using (5.9),

$$\|Q_{\ell,j}f\|_2 \le C\theta_\ell \le C(\log|B_\ell|^{-1})\|f\|_2.$$

Hence, we obtain

$$\|Q_{\ell,j}f\|_2 \le C(\log|B_\ell|^{-1})2^{-|j|\alpha/b}\|f\|_2. \tag{5.14}$$

Next, as for L^p estimate, by using Lemma 3.4, Lemma 3.2 and Lemma 2.5, we have

$$\|Q_{\ell,j}f\|_p \le C(\log|B_\ell|^{-1})\|f\|_p, \qquad \text{for } \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{1}{\gamma'}. \tag{5.15}$$

Interpolating between (5.14) and (5.15), we can find a number $0<\theta<1$ such that

$$\|Q_{\ell,j}f\|_p \le C(\log|B_\ell|^{-1})2^{-|j|\theta\alpha/b}\|f\|_p, \qquad \text{for } \left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{\gamma'}. \tag{5.16}$$

Hence, combining (5.12), (5.13) and (5.16), we complete the proof of Theorem 1 (i) in the case of $\Omega\in\cup_{1<q<\infty}B_q^{(0,0)}(S^{n-1})$.

We can prove Theorem 1 (ii) and (iii) in the case of $\Omega\in\cup_{1<q<\infty}B_q^{(0,0)}(S^{n-1})$ in the same way as those of the proofs of Theorem 1 (ii) and (iii) in the case of $\Omega\in L(\log L)(S^{n-1})$, respectively. So, we omit the details. ∎

6. Appendix

We state the claim in Remark 2 as a lemma.

Lemma 6.1. *Let* $\Phi(t)=\log^\alpha(1+t)$ $(0<\alpha\le 1/n)$ *and*

$$Tf(x):=\mathrm{p.v.}\int\frac{y_1}{|y|^{n+1}}f\big(x-\Phi(|y|)y'\big)\,dy. \tag{6.1}$$

Then T is non-bounded on any $L^p(\mathbb{R}^n)$ for $1<p<\infty$.

Proof. Set

$$f(x)=\frac{\chi_{\{x\in\mathbb{R}^n;\,x_1<0,\,|\tilde{x}|<-x_1\}}}{(1+|x|)^n},$$

where $\tilde{x}=(x_2,\dots,x_n)$. Then $f\in L^p(\mathbb{R}^n)$ for any $p>1$. Let $x\in\mathbb{R}^n$ with $x_1>0$ and $|\tilde{x}|<x_1/3$. Note first that, if $y_1\le 0$, we have $x_1-y_1|y|^{-1}\log^\alpha(1+|y|)>0$, and so $f\big(x-y|y|^{-1}\log^\alpha(1+|y|)\big)=0$. Also, if $|y|<e^{x_1^{1/\alpha}}-1$, $f\big(x-y|y|^{-1}\log^\alpha(1+|y|)\big)=0$. Next, for $y\in\mathbb{R}^n$ with $y_1>1$, $|\tilde{y}|<y_1/3$ and $|y|>e^{(3x_1)^{1/\alpha}}$, we have

$$\frac{y_1}{|y|}=\frac{y_1}{\sqrt{y_1^2+|\tilde{y}|^2}}>\frac{3}{\sqrt{10}},$$
$$2x_1<\frac{2}{3}\log^\alpha(1+|y|)<\frac{3}{\sqrt{10}}\log^\alpha(1+|y|)<\frac{y_1}{|y|}\log^\alpha(1+|y|),$$

and hence

$$\begin{aligned}&x_1-\frac{y_1}{|y|}\log^\alpha(1+|y|)<0,\\ &\frac{y_1}{|y|}\log^\alpha(1+|y|)-x_1>\frac{1}{2}\frac{y_1}{|y|}\log^\alpha(1+|y|).\end{aligned} \tag{6.2}$$

From these we obtain

$$\begin{aligned}&\left|\tilde{x}-\frac{\tilde{y}}{|y|}\log^\alpha(1+|y|)\right|\le|\tilde{x}|+\frac{|\tilde{y}|}{|y|}\log^\alpha(1+|y|)\le\frac{x_1}{3}+\frac{1}{3}\frac{y_1}{|y|}\log^\alpha(1+|y|)\\ &<\frac{1}{2}\frac{y_1}{|y|}\log^\alpha(1+|y|)<\frac{y_1}{|y|}\log^\alpha(1+|y|)-x_1.\end{aligned} \tag{6.3}$$

We also have

$$\begin{aligned}\left|x-\frac{y}{|y|}\log^{\alpha}(1+|y|)\right| &\le |x|+\log^{\alpha}(1+|y|)\\ \le \frac{\sqrt{10}}{3}x_1+\log^{\alpha}(1+|y|) &< 2\log^{\alpha}(1+|y|).\end{aligned} \tag{6.4}$$

Thus, using (6.2), (6.3), (6.4) and the definition of f, we have for $0<\varepsilon<1$

$$\int_{|y|>\varepsilon}\frac{y_1}{|y|^{n+1}}f\big(x-\Phi(|y|)y'\big)\,dy=\int_{\left\{|y|>\max\{\varepsilon,\,e^{x_1^{1/\alpha}}-1\}\right\}}\frac{y_1}{|y|^{n+1}}f\big(x-\Phi(|y|)y'\big)\,dy$$
$$>\int_{\{y_1>1,|y|>e^{(3x_1)^{1/\alpha}},\,|\tilde{y}|<y_1/3\}}\frac{3}{\sqrt{10}}\frac{dy}{|y|^n\big(1+2\log^{\alpha}(1+|y|)\big)^n}=+\infty.$$

This implies that T is not bounded on any $L^p(\mathbb{R}^n)$. ∎

Remark 5. For $\Phi(t)=\log^{\alpha}(1+t)$ $(\alpha>0)$ we have $\Phi(t)/\big(t\Phi'(t)\big)=\frac{(1+t)\log(1+t)}{\alpha t}\notin L^{\infty}(0,\infty)$, and $\Phi(2t)\le 2^{\alpha}\Phi(t)$, $t>0$, since $\log(1+2t)<\log(1+t)^2=2\log(1+t)$.

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