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HYBRID METHOD FOR DESIGNING EXPLICIT HIERARCHICAL FIXED POINT APPROACH TO MONOTONE VARIATIONAL INEQUALITIES

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Abstract. Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $f : C \to H$ is L-Lipschitzian with constant $L \geq 0$ and $T, V : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Consider the hierarchical monotone variational inequality problem (in short, HMVIP):

VI (a): finding $z^* \in \mathrm{Fix}(T)$ such that $\langle (I-V)z^*, z-z^* \rangle \geq 0,\ \forall z \in \mathrm{Fix}(T)$;

VI (b): finding $x^* \in S$ such that $\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0,\ \forall z \in S$.

Here S denotes the nonempty solution set of the VI (a). This paper combines hybrid steepest-descent method, viscosity method and projection method to design an explicit algorithm, that can be used to find the unique solution of the HMVIP. Strong convergence of the algorithm is proved under very mild conditions. Applications in hierarchical minimization problems are also included.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and C be a nonempty closed convex subset of H. The so-called classical variational inequality (VI) means to find an element $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \leq 0, \quad \forall x \in C, \tag{1.1}$$

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where $A : C \to H$ is a nonlinear mapping. If A is a monotone operator, then the VI (1.1) is also known as a monotone variational inequality. It is well-known that the VI (1.1) is equivalent to the fixed point equation

$$(1.2) \qquad x^* = P_C(x^* - \nu Ax^*),$$

where $\nu > 0$ and P_C is the metric projection of H onto C; which assigns, to each $x \in H$, the unique point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

Therefore, fixed point algorithms can be applied to solve VIs. As a matter of fact, if A is Lipschitzian and strongly monotone (i.e., $\langle Ax-Ay, x-y\rangle \geq \alpha\|x-y\|^2,\ \forall x, y \in C$, for some $\alpha > 0$), the for small enough $\nu > 0$, the mapping $P_C(I - \nu A)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I-\nu A)x_{n-1}$ $(n \geq 1)$, converges strongly to the unique solution of the VI (1.1). Furthermore, whenever A is inverse-strongly monotone (i.e., there is a constant $\zeta > 0$, such that $\langle Ax-Ay, x-y\rangle \geq \zeta\|Ax-Ay\|^2,\ \forall x, y \in C$), the mapping $P_C(I-\nu A)$ is an averaged mapping (i.e., there are $\beta \in (0, 1)$ and a nonexpansive mapping T, such that $P_C(I-\nu A) = (1-\beta)I+\beta T$) and the sequence of Picard iterates, $\{(P_C(I-\nu A))^n x_0\}$, converges weakly to a solution of the VI (1.1) (if such solutions exist). Recall here that a mapping $T : C \to C$ is nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|,\ \forall x, y \in C$.

Very recently, Marino and Xu [10] investigated a special form of the VI (1.1), where the constraint set is the set of fixed points of a nonexpansive mapping T and A is the complement of another nonexpansive mapping V; that is, their VI is of the form

$$(1.3) \qquad x^* \in \mathrm{Fix}(T) : \quad \langle (I - V)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T),$$

where $T, V : C \to C$ are nonexpansive mappings, such that $\mathrm{Fix}(T) \neq \emptyset$ with C being a nonempty closed convex subset of a real Hilbert space H. It is not difficult to observe that x^* solves the VI (1.3) if and only if it is a fixed point of the nonexpansive mapping $P_{\mathrm{Fix}(T)}V$.

Throughout this paper, we use S to denote the solution set of the VI (1.3) (or $S = \mathrm{Fix}(P_{\mathrm{Fix}(T)}V)$) and always assume that $S \neq \emptyset$. Variational inequalities have extensively been investigated; see the monographs [1-5], and also the articles [23-31] (and the references therein). Extensions to vector variational inequalities have recently been made extensively; see [20] and the survey article [21].

Variational inequalities of form (1.3) cover several topics recently investigated in the literature, including monotone inclusions, convex optimization, quadratic minimization over fixed point sets; see [11, 14, 17-19, 22], and the references therein. The hierarchical fixed point approach was however introduced recently (see [7, 15]) and Marino and Xu [10] further studied hierarchical fixed point approach to monotone VIs of form (1.3).

A couple of particular cases of the VI (1.3) have been studied in the literature:

(a) V is a constant mapping on C : $Vx \equiv u$ for some $u \in C$ and all $x \in C$;

(b) V is a contraction with coefficient $\rho \in [0, 1)$; i.e., $\|Vx - Vy\| \leq \rho\|x - y\|$, $\forall x, y \in C$.

Case (a) is actually the VI:

$$\text{find } x^* \in \mathrm{Fix}(T) \text{ such that } \langle x^* - u, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T) \tag{1.4}$$

and is equivalent to finding the fixed point of T closest to u; that is,

$$x^* := P_{\mathrm{Fix}(T)}u = \mathrm{argmin}_{x\in\mathrm{Fix}(T)}\|u - x\|.$$

This problem has extensively been investigated; see [14, 32-42].

Case (b) corresponds to the VI

$$x^* \in \mathrm{Fix}(T) : \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.5}$$

where f is a contraction on H. This VI has been studied recently; see [17, 18].

There are two approaches to the VI (1.3): One implicit and one explicit. The implicit approach generates a net $\{x_{s,t}\}_{0<s,t<1}$ satisfying the fixed point equation:

$$x_{s,t} = sf(x_{s,t}) + (1 - s)[tVx_{s,t} + (1 - t)Tx_{s,t}], \tag{1.6}$$

where f is a given contraction on H. The behavior of the net $\{x_{s,t}\}$ has recently been studied in [15, 16]. Essentially these papers prove that, under appropriate conditions, $\{x_{s,t}\}$ converges in norm repeatedly as $s \to 0$ and $t \to 0$ (in this order), respectively.

Very recently, Marino and Xu [10] gave an explicit approach, which generates a sequence $\{x_n\}$ recursively by the iterative scheme:

$$x_{n+1} := \lambda_n f(x_n) + (1 - \lambda_n)(\alpha_n V x_n + (1 - \alpha_n)Tx_n), \quad \forall n \geq 0, \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in $(0, 1)$ satisfying certain conditions. They proved that $\{x_n\}$, under certain assumptions, converges in norm to a solution, which solves another variational inequality.

On the other hand, let $F : H \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, and let $T : H \to H$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. In 2001, Yamada [11] introduced the so-called hybrid steepest-descent method for solving the variational inequality problem: finding $\tilde{x} \in \mathrm{Fix}(T)$ such that

$$\langle F\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

This method generates a sequence $\{x_n\}$ via the following iterative scheme:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad \forall n \geq 0, \tag{1.8}$$

where $0 < \mu < 2\eta/\kappa^2$, the initial guess $x_0 \in H$ is arbitrary and the sequence $\{\lambda_n\}$ in $(0,1)$ satisfies the conditions:

$$\lambda_n \to 0, \quad \sum_{n=0}^{\infty} \lambda_n = \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

A key fact in Yamada's argument is that, for small enough $\lambda > 0$, the mapping

$$T^{\lambda}x := Tx - \lambda\mu F(Tx), \quad \forall x \in H$$

is a contraction, due to the κ-Lipschitz continuity and η-strong monotonicity of F.

In this paper, let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $f : C \to H$ is L-Lipschitzian with constant $L \geq 0$ and $T, V : C \to C$ are nonexpansive mappings with $\text{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$. Consider the hierarchical monotone variational inequality problem (in short, HMVIP):

VI (a) finding $z^* \in \text{Fix}(T)$ such that $\langle (I - V)z^*, z - z^* \rangle \geq 0, \ \forall z \in \text{Fix}(T)$;

VI (b) finding $x^* \in S$ such that $\langle (\mu F - \gamma f)x^*, x - x^* \rangle \geq 0, \ \forall z \in S$. Here S denotes the nonempty solution set of the VI (a).

Combining hybrid steepest-descent method, viscosity method and projection method, we design an explicit approach, which generates a sequence $\{x_n\}$ recursively by the formula:

$$(1.9) \quad x_{n+1} := P_C[\lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)(\alpha_n V x_n + (1 - \alpha_n)T x_n)], \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences in $(0,1)$ satisfying certain conditions to be made precisely in Sect. 3. We will prove that $\{x_n\}$, under certain assumptions, converges in norm to a solution of the HMVIP. In particular, if we put $\mu = 1$, $F = I$ and $\gamma = \tau = 1$ and let f be a contractive self-mapping on C with coefficient $\rho \in [0,1)$, then our results (including our algorithm (1.9) and its strong convergence) reduce to those in Marino and Xu [10]. There is no doubt that our results improve and extend the corresponding results in [10].

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a L-Lipschitzian mapping $f : C \to H$ is the mapping on C such that $\|f(x) - f(y)\| \leq L\|x - y\|, \ \forall x, y \in C$, where $L \in [0, \infty)$ is a constant. In particular, if $L \in [0,1)$ then f is called a contraction on C; if $L = 1$ then f is called a nonexpansive mapping on C.

Below we gather some basic facts that are needed in the argument of the subsequent section.

The following lemma plays a key role in proving strong convergence of our method.

Lemma 2.1. (see [9, Lemma 3.1] and also [11]). *Let λ be a number in $(0,1]$ and let $\mu > 0$. Let $F : C \to H$ be an operator on C such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. We sssociate with a nonexpansive mapping $T : C \to C$ the mapping $T^\lambda : C \to H$ defined by*

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < 2\eta/\kappa^2$, that is,

$$\|T^\lambda x - T^\lambda y\| \le (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Remark 2.1. *Put $F = I$, where I is the identity operator of H. Then $\kappa = \eta = 1$ and hence $\mu < 2\eta/\kappa^2 = 2$. Also, put $\mu = 1$. Then it is easy to see that*

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} = 1.$$

In particular, whenever $\lambda > 0$, we have $T^\lambda x := Tx - \lambda\mu F(Tx) = (1 - \lambda)Tx$.

It is not difficult to prove the following lemma.

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, $f : C \to H$ a L-Lipschitzian mapping with constant $L \in [0, \infty)$, and $F : C \to H$ a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$. Then for $0 \le \gamma L < \mu\eta$,*

$$\langle(\mu F - \gamma f)x - (\mu F - \gamma f)y, x - y\rangle \ge (\mu\eta - \gamma L)\|x - y\|^2, \quad \forall x, y \in C.$$

That is, $\mu F - \gamma f$ is strongly monotone with constant $\mu\eta - \gamma L > 0$.

Lemma 2.3. (see [12, Demiclosedness Principle]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Recall that the metric (or nearest point) projection from H onto K is the mapping $P_K : H \to K$ which assigns to each $x \in H$ the unique point $P_K x \in K$ satisfying the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K),$$

where K is a nonempty closed convex subset of a real Hilbert space H.

Lemma 2.4. (see [13]). *Let $x \in H$ and $z \in K$. Then:*

(i) $z = P_K x$ *if and only if there holds the relation*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in K.$$

(ii) $z = P_K x$ *if and only if there holds the relation*

$$\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \quad \forall y \in K.$$

(iii) *There holds the relation*

$$\langle P_K x - P_K y, x - y\rangle \geq \|P_K x - P_K y\|^2, \quad \forall x, y \in H.$$

Consequently, P_K is nonexpansive and monotone.

Lemma 2.5. (see [14]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \beta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$, such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) *either* $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \gamma_n|\delta_n| < \infty$;

(iii) $\sum_{n=0}^{\infty} \beta_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

It is easy to see that the following straightforward inequality holds.

Lemma 2.6. *There holds the following inequality in an inner product space X:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Notation. Let $\{x_n\}$ be a sequence and x be a point in a normed space X. Then we use $x_n \to x$ and $x_n \rightharpoonup x$ to denote strong and weak convergence to x of the sequence $\{x_n\}$, respectively.

3. Strong Convergence of Explicit Scheme

Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that $F : C \to H$ is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$, $f : C \to H$ is L-Lipschitzian with constant $L \geq 0$ and $T, V : C \to C$ are nonexpansive mappings with $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$.

Consider the variational inequality of finding a point x^* with the property

$$(3.1) \qquad x^* \in \mathrm{Fix}(T): \quad \langle (I-V)x^*, x-x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

The purpose of this section is to propose an iterative algorithm and prove the strong convergence of the suggested algorithm to a solution of the VI (3.1), i.e., the VI (a) in Sect 1. Meantime, it is also a solution of the VI (b). Thus, there is no doubt that we prove the strong convergence of the suggested algorithm to a solution of the HMVIP. Our algorithm generates a sequence $\{x_n\}$ through the recursive formula

$$(3.2) \quad x_{n+1} := P_C[\lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)(\alpha_n V x_n + (1-\alpha_n)Tx_n)], \quad \forall n \geq 0,$$

where the initial guess $x_0 \in C$ is arbitrary and $\{\lambda_n\}$ and $\{\alpha_n\}$ are sequences in $(0,1)$.

Introducing the mapping $W_n = \alpha_n V + (1-\alpha_n)T$, we can rewrite (3.2) as

$$(3.3) \qquad x_{n+1} = P_C[\lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)W_n x_n]. \quad \forall n \geq 0.$$

We will see that the strong convergence of the algorithm (3.2) depends on the choice of the sequences of parameters, $\{\lambda_n\}$ and $\{\alpha_n\}$. Therefore, for the sake of convenience, we list the following possible assumptions:

(A_1) $\alpha_n \leq \nu\lambda_n$ for all n and some constant ν.

(A_2) $\lim_{n\to\infty} \alpha_n/\lambda_n =: \sigma \in [0,\infty]$.

(A_3) $\lambda_n \to 0$ (as $n \to \infty$) and $\sum_{n=0}^{\infty} \lambda_n = \infty$.

(A_4) $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

(A_5) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.

(A_6) $|\lambda_{n+1} - \lambda_n|/\lambda_{n+1} \to 0$; i.e., $\lambda_n/\lambda_{n+1} \to 1$.

(A_7) $|\alpha_{n+1} - \alpha_n|/\alpha_{n+1} \to 0$; i.e., $\alpha_n/\alpha_{n+1} \to 1$.

(A_8) $(|\lambda_{n+1} - \lambda_n| + |\alpha_{n+1} - \alpha_n|)/(\lambda_{n+1}\alpha_{n+1}) \to 0$.

(A_9) There exists a constant $\mathcal{K}$ such that $\mathcal{K} \geq \frac{1}{\lambda_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}|$ for all $n \geq 1$.

Note that (A_8) implies both (A_6) and (A_7).

Throughout this section, $\{x_n\}$ always stands for the sequence generated by the algorithm (3.2). We first discuss some properties of $\{x_n\}$.

Lemma 3.1. Assume (A_1). Then $\{x_n\}$ is bounded.

Proof. Take a point $z \in \mathrm{Fix}(T)$ $(\subset C)$ arbitrarily. Utilizing Lemma 2.1 and (A_1), we deduce that

$$\begin{aligned}\|x_{n+1}-z\| &= \|P_C[\lambda_n\gamma f(x_n)+(I-\lambda_n\mu F)W_nx_n]-P_Cz\|\\ &\le \|\lambda_n\gamma f(x_n)+(I-\lambda_n\mu F)W_nx_n-z\|\\ &= \|\lambda_n\gamma(f(x_n)-f(z))+\lambda_n(\gamma f(z)-\mu FW_nz)\\ &\quad +(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nz+W_nz-z\|\\ &\le \lambda_n\gamma\|f(x_n)-f(z)\|+\lambda_n\|\gamma f(z)-\mu FW_nz\|\\ &\quad +\|(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nz\|+\|W_nz-z\|\\ &\le \lambda_n\gamma L\|x_n-z\|+\lambda_n(\gamma\|f(z)\|+\mu\|FW_nz\|)\\ &\quad +(1-\lambda_n\tau)\|x_n-z\|+\alpha_n\|Vz-z\|\\ &\le [1-\lambda_n(\tau-\gamma L)]\|x_n-z\|+\lambda_n(\gamma\|f(z)\|+\nu\|Vz-z\|+\mu\|FW_nz\|)\\ &= [1-\lambda_n(\tau-\gamma L)]\|x_n-z\|+\lambda_n(\tau-\gamma L)\\ &\quad \cdot\frac{\gamma\|f(z)\|+\nu\|Vz-z\|+\mu\|FW_nz\|}{\tau-\gamma L},\end{aligned}$$

where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$. Noticing $\alpha_n \in (0,1)$ and the κ-Lipschitzian property of F, we have

$$\|FW_nz\| \le \|FW_nz-Fz\|+\|Fz\| \le \kappa\|W_nz-z\|+\|Fz\| \le \kappa\|Vz-z\|+\|Fz\|,$$

and so

$$\|x_{n+1}-z\| \le [1-\lambda_n(\tau-\gamma L)]\|x_n-z\|+\lambda_n(\tau-\gamma L) \cdot\frac{\gamma\|f(z)\|+(\mu\kappa+\nu)\|Vz-z\|+\mu\|Fz\|}{\tau-\gamma L}.$$

Since $\lambda_n \in (0,1)$ and $0 \le \gamma L < \tau$, by induction we obtain

$$\|x_n-z\| \le \max\{\|x_0-z\|, \frac{\gamma\|f(z)\|+(\mu\kappa+\nu)\|Vz-z\|+\mu\|Fz\|}{\tau-\gamma L}\}, \quad \forall n \ge 0.$$

Hence, $\{x_n\}$ is bounded. ∎

Lemma 3.2. *Assume (A_1) and (A_3). Also, assume either "(A_4) and A_5" or "(A_6) and A_7". Then*

(a) $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$;

(b) $\lim_{n\to\infty}\|x_n-Tx_n\| = 0$;

(c) $\omega_w(x_n) \subset \mathrm{Fix}(T)$.

Proof. Utilizing (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \|P_C[\lambda_n\gamma f(x_n)+(I-\lambda_n\mu F)W_nx_n] \\
&\quad -P_C[\lambda_{n-1}\gamma f(x_{n-1})+(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}]\| \\
&\le \|[\lambda_n\gamma f(x_n)+(I-\lambda_n\mu F)W_nx_n] \\
&\quad -[\lambda_{n-1}\gamma f(x_{n-1})+(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}]\| \\
&= \|\lambda_n(\gamma f(x_n)-\gamma f(x_{n-1}))+(\lambda_n-\lambda_{n-1})\gamma f(x_{n-1}) \\
&\quad +(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nx_{n-1} \\
&\quad +(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&\le \|\lambda_n(\gamma f(x_n)-\gamma f(x_{n-1}))+(\lambda_n-\lambda_{n-1})\gamma f(x_{n-1})\| \\
&\quad +\|(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nx_{n-1}\| \\
&\quad +\|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&\le \lambda_n\gamma L\|x_n-x_{n-1}\|+\gamma|\lambda_n-\lambda_{n-1}|\|f(x_{n-1})\|+(1-\lambda_n\tau)\|x_n-x_{n-1}\| \\
&\quad +\|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&= [1-\lambda_n(\tau-\gamma L)]\|x_n-x_{n-1}\|+\gamma|\lambda_n-\lambda_{n-1}|\|f(x_{n-1})\| \\
&\quad +\|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\|.
\end{aligned} \tag{3.4}
$$

Again, utilizing Lemma 2.1, we get

$$
\begin{aligned}
&\|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&= \|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_n\mu F)W_{n-1}x_{n-1} \\
&\quad +(I-\lambda_n\mu F)W_{n-1}x_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&\le \|(I-\lambda_n\mu F)W_nx_{n-1}-(I-\lambda_n\mu F)W_{n-1}x_{n-1}\| \\
&\quad +\|(I-\lambda_n\mu F)W_{n-1}x_{n-1}-(I-\lambda_{n-1}\mu F)W_{n-1}x_{n-1}\| \\
&\le (1-\lambda_n\tau)\|W_nx_{n-1}-W_{n-1}x_{n-1}\|+\mu|\lambda_n-\lambda_{n-1}|\|FW_{n-1}x_{n-1}\| \\
&\le \|\alpha_nVx_{n-1}+(1-\alpha_n)Tx_{n-1}-[\alpha_{n-1}Vx_{n-1} \\
&\quad +(1-\alpha_{n-1})Tx_{n-1}]\|+\mu|\lambda_n-\lambda_{n-1}|\|FW_{n-1}x_{n-1}\| \\
&\le |\alpha_n-\alpha_{n-1}|(\|Vx_{n-1}\|+\|Tx_{n-1}\|)+\mu|\lambda_n-\lambda_{n-1}|\|FW_{n-1}x_{n-1}\|.
\end{aligned} \tag{3.5}
$$

Combining (3.4) and (3.5), we derive

$$
\begin{aligned}
\|x_{n+1}-x_n\| &\le [1-\lambda_n(\tau-\gamma L)]\|x_n-x_{n-1}\|+\gamma|\lambda_n-\lambda_{n-1}|\|f(x_{n-1})\| \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|Vx_{n-1}\|+\|Tx_{n-1}\|)+\mu|\lambda_n-\lambda_{n-1}|\|FW_{n-1}x_{n-1}\| \\
&= [1-\lambda_n(\tau-\gamma L)]\|x_n-x_{n-1}\| \\
&\quad +|\lambda_n-\lambda_{n-1}|(\gamma\|f(x_{n-1})\|+\mu\|FW_{n-1}x_{n-1}\|) \\
&\quad +|\alpha_n-\alpha_{n-1}|(\|Vx_{n-1}\|+\|Tx_{n-1}\|).
\end{aligned}
$$

Since V and T are nonexpansive mappings, and f and F are Lipschitzian mappings with constants $L \geq 0$ and $\kappa > 0$, respectively, and also since $\{x_n\}$ is bounded (Lemma 3.1), we can find a constant $M > 0$ big enough so that

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}|M + |\alpha_n - \alpha_{n-1}|M \\
&= [1 - \lambda_n(\tau - \gamma L)]\|x_n - x_{n-1}\| + (|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|)M \\
&= [1 - \lambda_n(\tau - \gamma L)]\|x_n - x_{n-1}\| + (\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\lambda_n})\lambda_n M \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\|x_n - x_{n-1}\| + (\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} + \nu\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n})\lambda_n M.
\end{aligned} \tag{3.6}
$$

Using the assumptions (A_4) and (A_5) or (A_6) and (A_7) and applying Lemma 2.5 to (3.6) we obtain that $\|x_{n+1} - x_n\| \to 0$, which together with the following estimate (noticing $\alpha_n \to 0$ and $\lambda_n \to 0$)

$$
\begin{aligned}
\|x_{n+1} - Tx_n\| &= \|P_C[\lambda_n\gamma f(x_n) + (I - \lambda_n\mu F)W_n x_n] - P_C T x_n\| \\
&\leq \|\lambda_n\gamma f(x_n) + (I - \lambda_n\mu F)W_n x_n - Tx_n\| \\
&\leq \lambda_n\gamma\|f(x_n)\| + \|W_n x_n - Tx_n\| + \lambda_n\mu\|FW_n x_n\| \\
&= \lambda_n\gamma\|f(x_n)\| + \alpha_n\|Vx_n - Tx_n\| + \lambda_n\mu\|FW_n x_n\| \to 0
\end{aligned}
$$

immediately implies part (b). Part (c) follows from part (b) and Lemma 2.3. ∎

We will see that the convergence of the sequence $\{x_n\}$ depends on the limit of the ratio in (A_2); i.e., on $\sigma = \lim_{n\to\infty} \alpha_n/\lambda_n$. Our first result treats the case where $\sigma = 0$.

Theorem 3.1. *Assume (A_2) with $\sigma = 0$ and (A_3). Also, assume either "(A_4) and (A_5)" or "(A_6) and (A_7)". Then $\{x_n\}$ converges in norm to the fixed point q of the contraction $P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)$, $q = P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)q$, i.e., the unique solution of the variational inequality*

$$
q \in \mathrm{Fix}(T): \quad \langle(\mu F - \gamma f)q, x - q\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{3.7}
$$

Proof. It is easy to obtain the uniqueness of a solution of the VI (3.7). Indeed, note that $0 \leq \gamma L < \tau$ and

$$
\begin{aligned}
\mu\eta \geq \tau &\Leftrightarrow \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\
&\Leftrightarrow \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \geq 1 - \mu\eta \\
&\Leftrightarrow 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\
&\Leftrightarrow \kappa^2 \geq \eta^2 \\
&\Leftrightarrow \kappa \geq \eta.
\end{aligned}
$$

It is clear that

$$\langle(\mu F-\gamma f)x-(\mu F-\gamma f)y, x-y\rangle \geq (\mu\eta-\gamma L)\|x-y\|^2, \quad \forall x, y \in C.$$

Hence it follows from $0 \leq \gamma L < \tau \leq \mu\eta$ that $\mu F-\gamma f$ is strongly monotone. Since $\mu F-\gamma f$ is Lipschitzian, the VI (3.7) has only one solution. Let q be the unique solution of the VI (3.7). Then the VI (3.7) can be rewritten as

$$q \in \mathrm{Fix}(T): \quad \langle q-(I-\mu F+\gamma f)q, x-q\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

According to Lemma 2.4 (i) we conclude that

$$q = P_{\mathrm{Fix}(T)}(I-\mu F+\gamma f)q.$$

By assumption (A_2) with $\sigma=0$, we can write $\alpha_n=\varepsilon_n\lambda_n$, where $\lim_{n\to\infty}\varepsilon_n=0$ (thus we may assume $0\leq\varepsilon_n\leq 1$ for all $n\geq 0$). Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)q, x_n-q\rangle = \lim_{j\to\infty}\langle(\gamma f-\mu F)q, x_{n_j}-q\rangle \tag{3.8}$$

Without loss of generality, we may assume $x_{n_j} \rightharpoonup x' \in \mathrm{Fix}(T)$ (Lemma 3.2 (c)). Now from (3.8) and (3.7) it follows that

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)q, x_n-q\rangle = \langle(\gamma f-\mu F)q, x'-q\rangle \leq 0. \tag{3.9}$$

Let us show that $x_n \to q$ as $n\to\infty$. Indeed, set

$$y_n = \lambda_n\gamma f(x_n)+(I-\lambda_n\mu F)W_nx_n, \quad \forall n \geq 0.$$

We then have $x_{n+1}=P_Cy_n$, and

$$\begin{aligned} x_{n+1}-q &= P_Cy_n-y_n+y_n-q \\ &= P_Cy_n-y_n+\lambda_n(\gamma f(x_n)-\mu Fq)+(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)q. \end{aligned} \tag{3.10}$$

Since P_C is the metric projection from H onto C, we have

$$\langle P_Cy_n-y_n, P_Cy_n-q\rangle \leq 0.$$

Utilizing Lemma 2.1, we deduce from (3.10) that

$$\begin{aligned} &\|x_{n+1}-q\|^2 \\ &= \langle P_Cy_n-y_n, P_Cy_n-q\rangle+\langle(I-\lambda_n\mu F)W_nx_n \\ &\quad -(I-\lambda_n\mu F)q, x_{n+1}-q\rangle+\lambda_n\langle\gamma f(x_n)-\mu Fq, x_{n+1}-q\rangle \\ &\leq \langle(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)q, x_{n+1}-q\rangle \\ &\quad +\lambda_n\langle\gamma f(x_n)-\mu Fq, x_{n+1}-q\rangle \\ &= \langle(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nq, x_{n+1}-q\rangle \\ &\quad +\langle(I-\lambda_n\mu F)W_nq-(I-\lambda_n\mu F)q, x_{n+1}-q\rangle \\ &\quad +\lambda_n\langle\gamma f(x_n)-\mu Fq, x_{n+1}-q\rangle \\ &\leq \|(I-\lambda_n\mu F)W_nx_n-(I-\lambda_n\mu F)W_nq\|\|x_{n+1}-q\| \\ &\quad +\langle(I-\lambda_n\mu F)W_nq-(I-\lambda_n\mu F)q, x_{n+1}-q\rangle \end{aligned} \tag{3.11}$$

$$
\begin{aligned}
&+\lambda_n\langle \gamma f(x_n) - \mu F q, x_{n+1} - q\rangle \\
&\leq (1-\lambda_n\tau)\|x_n - q\|\|x_{n+1} - q\| + \langle (I - \lambda_n \mu F)W_n q \\
&\quad -(I - \lambda_n\mu F)q, x_{n+1} - q\rangle \\
&\quad +\lambda_n\langle \gamma f(x_n) - \gamma f(q), x_{n+1} - q\rangle + \lambda_n\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\leq (1-\lambda_n\tau)\|x_n - q\|\|x_{n+1} - q\| + \lambda_n \gamma L\|x_n - q\|\|x_{n+1} - q\| \\
&\quad +\lambda_n\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle + \langle (I - \lambda_n\mu F)W_n q \\
&\quad -(I - \lambda_n\mu F)q, x_{n+1} - q\rangle \\
&= [1 - \lambda_n(\tau - \gamma L)]\|x_n - q\|\|x_{n+1} - q\| + \lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\quad +\frac{1}{\lambda_n}\langle (I - \lambda_n\mu F)W_n q - (I - \lambda_n\mu F)q, x_{n+1} - q\rangle] \\
&\leq [1 - \lambda_n(\tau - \gamma L)](\frac{1}{2}\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2) \\
&\quad +\lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\quad +\frac{1}{\lambda_n}\langle (I - \lambda_n\mu F)W_n q - (I - \lambda_n\mu F)q, x_{n+1} - q\rangle] \\
&\leq \frac{1}{2}[1 - \lambda_n(\tau - \gamma L)]\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 \\
&\quad +\lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\quad +\frac{1}{\lambda_n}(1 - \lambda_n\tau)\alpha_n\|Vq - q\|\|x_{n+1} - q\|] \\
&\leq \frac{1}{2}[1 - \lambda_n(\tau - \gamma L)]\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 \\
&\quad +\lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\quad +\varepsilon_n\|Vq - q\|\|x_{n+1} - q\|].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\|x_{n+1} - q\|^2 L)]\|x_n - q\|^2 + 2\lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\leq [1 - \lambda_n(\tau - \gamma + \varepsilon_n\|Vq - q\|\|x_{n+1} - q\|].
\end{aligned}
$$

Setting $a_n = \|x_n - q\|^2,\ \gamma_n = \lambda_n(\tau - \gamma L)$, and

$$
\delta_n = \frac{2}{\tau - \gamma L}[\varepsilon_n\|Vq - q\|\|x_{n+1} - q\| + \langle(\gamma f - \mu F)q, x_{n+1} - q\rangle],
$$

we then have

$$
a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n. \tag{3.12}
$$

In terms of (3.9) and the fact that $\varepsilon_n \to 0$, we have $\limsup_{n\to\infty}\delta_n \leq 0$. Therefore, by Lemma 2.5, we conclude that $a_n \to 0$; hence $x_n \to q$ in norm. This completes the proof. ∎

Remark 3.1. By Lemma 2.5, assumption ($A3$) and the estimate (3.11), we find that the following two conditions are sufficient to guarantee the strong convergence of $\{x_n\}$:

$$\limsup_{n\to\infty}\langle(\gamma f-\mu F)q, x_n-q\rangle\le 0 \tag{3.13}$$

and

$$\limsup_{n\to\infty}\frac{1}{\lambda_n}\langle(I-\lambda_n\mu F)W_nq-(I-\lambda_n\mu F)q, x_{n+1}-q\rangle\le 0. \tag{3.14}$$

Condition (3.13), proved in (3.9), is easily seen to satisfy due to the fact that q is the fixed point of the contraction $P_S(I-\mu F+\gamma f)$ together with the demiclosedness of nonexpansive mappings (Lemma 2.3). Condition (3.14), is however hard to verify unless an additional condition relating the growth of the displacement $\|x-Tx\|$ of the nonexpansive mapping T to its fixed point set $\mathrm{Fix}(T)$ is given (see Theorem 3.3 below).

We also see that under the assumptions of Theorem 3.1, the limit q of $\{x_n\}$ solves the VI (3.7), instead of (3.1). This is due to the fact that $\sigma=0$ makes $\{\alpha_n\}$ tend to 0 'much' faster than $\{\lambda_n\}$ do. Consequently, the term '$\lambda_n f(x_n)$' dominates while the term '$\alpha_n Vx_n$' looks 'negligible'. However, if we assume $\sigma\in(0,\infty]$, the situations differ, as shown below.

Theorem 3.2. *Assume (A_2) with $\sigma\in(0,\infty)$, (A_3), (A_8) and (A_9). Then $\{x_n\}$ converges strongly to the unique solution $\tilde{x}$ of the variational inequality*

$$\tilde{x}\in\mathrm{Fix}(T):\quad \langle\frac{1}{\sigma}(\mu F-\gamma f)\tilde{x}+(I-V)\tilde{x},\tilde{x}-x'\rangle\le 0,\quad \forall x'\in\mathrm{Fix}(T). \tag{3.15}$$

Proof. Since (A_2) with $\sigma\in(0,\infty)$ implies (A_1) and since (A_8) implies both (A_6) and (A_7), we still know that $\{x_n\}$ is bounded and the conclusions (a)-(c) in Lemma 3.2 hold.

As proven in Theorem 3.1, $\mu F-\gamma f$ is strongly monotone with constant $\mu\eta-\gamma L$ and is also Lipschitzian (because F and f are Lipschitzian). Since V is nonexpansive, $I-V$ is monotone and Lipschitzian. Thus, the mapping $\frac{1}{\sigma}(\mu F-\gamma f)+(I-V)$ is strongly monotone with constant $(\mu\eta-\gamma L)/\sigma$ due to the fact that

$$\begin{aligned}&\langle[\frac{1}{\sigma}(\mu F-\gamma f)+(I-V)]x-[\frac{1}{\sigma}(\mu F-\gamma f)+(I-V)]y, x-y\rangle\\ =&\frac{1}{\sigma}\langle(\mu F-\gamma f)x-(\mu F-\gamma f)y, x-y\rangle+\langle(I-V)x-(I-V)y, x-y\rangle\\ \ge&(\mu\eta-\gamma L)/\sigma.\end{aligned}$$

Meantime, it is clear that the mapping $\frac{1}{\sigma}(\mu F-\gamma f)+(I-V)$ is Lipschitzian. Consequently, the VI (3.15) has only one solution. Let $\tilde{x}$ be the unique solution of the VI (3.15).

Let us show that $x_n \to \tilde{x}$ as $n \to \infty$. Indeed, we still write $W_n = \alpha_n V + (1-\alpha_n)T$; then $x_{n+1} = P_C y_n$ where

$$y_n = \lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)W_n x_n, \quad \forall n \geq 0.$$

Now take a point $z \in \mathrm{Fix}(T)$ arbitrarily. Since P_C is the metric projection from H onto C, utilizing Lemma 2.4 (i) we have

$$\langle P_C y_n - y_n, P_C y_n - z \rangle \leq 0.$$

Also, observe that

$$\begin{aligned} y_n - z &= \lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)W_n x_n - z \\ &= \lambda_n(\gamma f(x_n) - \mu F W_n z) + (I - \lambda_n \mu F)W_n x_n \\ &\quad -(I - \lambda_n \mu F)W_n z + W_n z - z \\ &= \lambda_n \gamma (f(x_n) - f(z)) + \lambda_n(\gamma f(z) - \mu F W_n z) \\ &\quad +(I - \lambda_n \mu F)W_n x_n - (I - \lambda_n \mu F)W_n z + \alpha_n (V - I)z. \end{aligned} \tag{3.16}$$

Utilizing Lemma 2.1, we obtain from (3.2)

$$\begin{aligned} &\|x_{n+1} - z\|^2 \\ &= \langle P_C y_n - y_n, P_C y_n - z \rangle + \langle y_n - z, x_{n+1} - z \rangle \\ &\leq \langle y_n - z, x_{n+1} - z \rangle \\ &= \lambda_n \gamma \langle f(x_n) - f(z), x_{n+1} - z \rangle + \lambda_n \langle \gamma f(z) - \mu F W_n z, x_{n+1} - z \rangle \\ &\quad + \langle (I - \lambda_n \mu F)W_n x_n - (I - \lambda_n \mu F)W_n z, x_{n+1} - z \rangle + \alpha_n \langle (V-I)z, x_{n+1} - z \rangle \\ &\leq \lambda_n \gamma \|f(x_n) - f(z)\| \|x_{n+1} - z\| + \lambda_n \langle \gamma f(z) - \mu F W_n z, x_{n+1} - z \rangle \\ &\quad + \|(I - \lambda_n \mu F)W_n x_n - (I - \lambda_n \mu F)W_n z\| \|x_{n+1} - z\| + \alpha_n \langle (V-I)z, x_{n+1} - z \rangle \\ &\leq \lambda_n \gamma L \|x_n - z\| \|x_{n+1} - z\| + \lambda_n \langle \gamma f(z) - \mu F W_n z, x_{n+1} - z \rangle \\ &\quad + (1 - \lambda_n \tau) \|x_n - z\| \|x_{n+1} - z\| + \alpha_n \langle (V - I)z, x_{n+1} - z \rangle \\ &= [1 - \lambda_n(\tau - \gamma L)] \|x_n - z\| \|x_{n+1} - z\| + \lambda_n [\langle \gamma f(z) - \mu F W_n z, x_{n+1} - z \rangle \\ &\quad + \frac{\alpha_n}{\lambda_n} \langle (V - I)z, x_{n+1} - z \rangle] \\ &\leq \frac{1}{2}[1 - \lambda_n(\tau - \gamma L)] \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2 \\ &\quad + \lambda_n \langle \gamma f(z) - \mu F W_n z + \frac{\alpha_n}{\lambda_n}(V - I)z, x_{n+1} - z \rangle, \end{aligned}$$

which hence implies that

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq [1 - \lambda_n(\tau - \gamma L)] \|x_n - z\|^2 \\ &\quad + 2\lambda_n \langle \gamma f(z) - \mu F W_n z + \frac{\alpha_n}{\lambda_n}(V - I)z, x_{n+1} - z \rangle. \end{aligned} \tag{3.17}$$

In particular, whenever $z = \tilde{x}$, we have

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &\le [1 - \lambda_n(\tau - \gamma L)]\|x_n - \tilde{x}\|^2 + 2\lambda_n\langle \gamma f(\tilde{x}) - \mu F W_n \tilde{x} + \frac{\alpha_n}{\lambda_n}(V - I)\tilde{x}, x_{n+1} - \tilde{x}\rangle.\end{aligned}$$

Since $\{x_n\}$ is bounded, we can take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \mu F W_n \tilde{x} + \frac{\alpha_n}{\lambda_n}(V - I)\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &= \lim_{j\to\infty}\langle \gamma f(\tilde{x}) - \mu F W_{n_j} \tilde{x} + \frac{\alpha_{n_j}}{\lambda_{n_j}}(V - I)\tilde{x}, x_{n_j+1} - \tilde{x}\rangle.\end{aligned} \tag{3.18}$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup x' \in \mathrm{Fix}(T)$ (Lemma 3.2 (c)). Observe that

$$\begin{aligned}&|\langle \gamma f(\tilde{x}) - \mu F W_{n_j}\tilde{x}, x_{n_j+1} - \tilde{x}\rangle - \langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x' - \tilde{x}\rangle| \\ &\le |\langle \gamma f(\tilde{x}) - \mu F W_{n_j}\tilde{x} - (\gamma f(\tilde{x}) - \mu F\tilde{x}), x_{n_j+1} - \tilde{x}\rangle| \\ &\quad + |\langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_{n_j+1} - \tilde{x} - (x' - \tilde{x})\rangle| \\ &= \mu|\langle F W_{n_j}\tilde{x} - F\tilde{x}, x_{n_j+1} - \tilde{x}\rangle| + |\langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_{n_j+1} - x'\rangle| \\ &\le \mu\kappa\alpha_{n_j}\|V\tilde{x} - \tilde{x}\|\|x_{n_j+1} - \tilde{x}\| + \|\gamma f(\tilde{x}) - \mu F\tilde{x}\|\|x_{n_j+1} - x_{n_j}\| \\ &\quad + |\langle \gamma f(\tilde{x}) - \mu F\tilde{x}, x_{n_j} - x'\rangle| \to 0 \quad \text{as } j \to \infty,\end{aligned}$$

and

$$\begin{aligned}&|\tfrac{\alpha_{n_j}}{\lambda_{n_j}}\langle (V - I)\tilde{x}, x_{n_j+1} - \tilde{x}\rangle - \sigma\langle (V - I)\tilde{x}, x' - \tilde{x}\rangle| \\ &\le |\tfrac{\alpha_{n_j}}{\lambda_{n_j}} - \sigma||\langle (V - I)\tilde{x}, x_{n_j+1} - \tilde{x}\rangle| + \sigma|\langle (V - I)\tilde{x}, x_{n_j+1} - x'\rangle| \\ &\le |\tfrac{\alpha_{n_j}}{\lambda_{n_j}} - \sigma||\langle (V - I)\tilde{x}, x_{n_j+1} - \tilde{x}\rangle| + \sigma|\langle (V - I)\tilde{x}, x_{n_j+1} - x_{n_j}\rangle| \\ &\quad + \sigma|\langle (V - I)\tilde{x}, x_{n_j} - x'\rangle| \\ &\le |\tfrac{\alpha_{n_j}}{\lambda_{n_j}} - \sigma|\|(V - I)\tilde{x}\|\|x_{n_j+1} - \tilde{x}\| + \sigma\|(V - I)\tilde{x}\|\|x_{n_j+1} - x_{n_j}\| \\ &\quad + \sigma|\langle (V - I)\tilde{x}, x_{n_j} - x'\rangle| \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Therefore, by virtue of the fact that $\tilde{x}$ is the unique solution of the VI (3.15), we conclude from (3.18) that

$$\begin{aligned}&\limsup_{n\to\infty}\langle \gamma f(\tilde{x}) - \mu F W_n \tilde{x} + \frac{\alpha_n}{\lambda_n}(V - I)\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &= \langle (\gamma f - \mu F)\tilde{x} + \sigma(V - I)\tilde{x}, x' - \tilde{x}\rangle \le 0.\end{aligned} \tag{3.19}$$

Setting $a_n = \|x_n - \tilde{x}\|^2,\ \gamma_n = \lambda_n(\tau - \gamma L)$, and

$$\delta_n = \frac{2}{\tau - \gamma L}\langle \gamma f(\tilde{x}) - \mu F W_n \tilde{x} + \frac{\alpha_n}{\lambda_n}(V - I)\tilde{x}, x_{n+1} - \tilde{x}\rangle,$$

we then have

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n.$$

By means of (3.19) we have $\limsup_{n\to\infty} \delta_n \leq 0$. Therefore, by Lemma 2.5, we deduce that $a_n \to 0$; hence $x_n \to \tilde{x}$ in norm. This completes the proof. ■

Finally we consider the case where $\sigma = \infty$. This case is more delicate.

Theorem 3.3. *Assume (A_2) with $\sigma = \infty$ (i.e., $\lambda_n/\alpha_n \to 0$), (A_3), (A_8) and (A_9). Assume also that $\{x_n\}$ is bounded. Then every weak limit point of $\{x_n\}$ is a solution of the VI (3.1). If, in addition, there are constants $\xi > 0$ and $\theta > 0$ such that*

(A_{10}) $\|x - Tx\| \leq \xi[\mathrm{dist}(x, \mathrm{Fix}(T))]^\theta$ *for all* $x \in C$,

(A_{11}) $\frac{\alpha_n^{1+1/\theta}}{\lambda_n} \to 0$,

then $\{x_n\}$ converges strongly to a solution q of the VI (3.1), which is also the unique fixed point of the contraction $P_{\mathrm{Fix}(T)}(I - \mu F + \gamma f)$, or the unique solution of the VI (3.7).

Proof. Observe first that the boundedness of $\{x_n\}$ and (A_8) together ensure that the conclusions of Lemma 3.2 still hold. Now let us show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\|/\alpha_n = 0. \tag{3.20}$$

Indeed, utilizing (3.6) and (A_9) we have

$$\begin{aligned}
&\frac{\|x_{n+1} - x_n\|}{\alpha_n} \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\frac{\|x_n - x_{n-1}\|}{\alpha_n} + \frac{|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|}{\alpha_n}M \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + \frac{1}{\lambda_n}|\frac{1}{\alpha_n} - \frac{1}{\alpha_{n-1}}|(\lambda_n\|x_n - x_{n-1}\|) \\
&\quad + \frac{|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|}{\alpha_n}M \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} + \lambda_n\mathcal{K}\|x_n - x_{n-1}\| \\
&\quad + \frac{|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|}{\alpha_n}M \\
&= [1 - \lambda_n(\tau - \gamma L)]\frac{\|x_n - x_{n-1}\|}{\alpha_{n-1}} \\
&\quad + \lambda_n(\mathcal{K}\|x_n - x_{n-1}\| + \frac{|\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}|}{\lambda_n\alpha_n}M).
\end{aligned} \tag{3.21}$$

Using Lemma 2.5, Lemma 3.2 (a) and assumption (A_8), we deduce that (3.20) is valid. Furthermore, utilizing (3.17) we have for all $z \in \mathrm{Fix}(T)$

$$
\begin{aligned}
&\langle (I-V)z, x_{n+1}-z\rangle \\
&\le \frac{\|x_n-z\|^2-\|x_{n+1}-z\|^2}{2\alpha_n}+\frac{\lambda_n}{\alpha_n}\langle \gamma f(z)-\mu FW_n z, x_{n+1}-z\rangle \\
&\le \frac{\|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)}{2\alpha_n}+\frac{\lambda_n}{\alpha_n}\|\gamma f(z)-\mu FW_n z\|\|x_{n+1}-z\|.
\end{aligned}
$$

Since $\|x_{n+1}-x_n\|/\alpha_n \to 0$ and $\frac{\lambda_n}{\alpha_n}\to 0$ as $n\to\infty$, we arrive at

$$\limsup_{n\to\infty}\langle (I-V)z, x_n-z\rangle \le 0, \quad \forall z\in \mathrm{Fix}(T),$$

which implies that, for any $\hat{x}\in\omega_w(x_n)\subset \mathrm{Fix}(T)$,

$$\langle (I-V)z, \hat{x}-z\rangle \le 0, \quad \forall z\in \mathrm{Fix}(T)$$

which, in turns, implies that, for $t\in(0,1)$ and $x\in\mathrm{Fix}(T)$,

$$\langle (I-V)(\hat{x}+t(x-\hat{x})), \hat{x}-x\rangle \le 0, \quad \forall x\in \mathrm{Fix}(T).$$

Upon letting $t\to 0^+$, we immediately find that $\hat{x}$ solves the variational inequality

$$\hat{x}\in\mathrm{Fix}(T): \quad \langle (I-V)\hat{x}, \hat{x}-x\rangle \le 0, \quad \forall x\in \mathrm{Fix}(T).$$

This is the VI (3.1).

Finally, we prove that, under the additional conditions (A_{10}) and (A_{11}), the full sequence $\{x_n\}$ converges in norm to q, the unique fixed point of the contraction $P_S(I-\mu F+\gamma f)$. We have already seen that the relation (3.13) holds and so, according to Remark 3.1, we need to verify the relation (3.14). To see this, we use the fact that $q\in S$ solves the VI (3.1) to get

$$\langle Vq-q, P_{\mathrm{Fix}(T)}x_{n+1}-q\rangle \le 0.$$

This implies that

$$
\begin{aligned}
&\langle Vq-q, x_{n+1}-q\rangle \\
&=\langle Vq-q, x_{n+1}-P_{\mathrm{Fix}(T)}x_{n+1}\rangle+\langle Vq-q, P_{\mathrm{Fix}(T)}x_{n+1}-q\rangle \\
&\le \langle Vq-q, x_{n+1}-P_{\mathrm{Fix}(T)}x_{n+1}\rangle \\
&\le \|Vq-q\|\mathrm{dist}(x_{n+1},\mathrm{Fix}(T)) \\
&\le \|Vq-q\|(\frac{1}{\xi}\|x_{n+1}-Tx_{n+1}\|)^{1/\theta}.
\end{aligned} \tag{3.22}
$$

However, it is easy to see that there holds, for an appropriate constant $\xi_1>0$,

$$\|x_{n+1}-Tx_{n+1}\|\le\|x_{n+1}-Tx_n\|+\|x_{n+1}-x_n\|\le \xi_1(\alpha_n+\lambda_n)+\|x_{n+1}-x_n\|.$$

It follows from (3.22) that we have another appropriate constant $\xi_2 > 0$ such that

$$\langle Vq - q, x_{n+1} - q\rangle \leq \xi_2(\alpha_n + \lambda_n + \|x_{n+1} - x_n\|)^{1/\theta}.$$

Hence

$$\frac{\alpha_n}{\lambda_n}\langle Vq - q, x_{n+1} - q\rangle \leq \xi_2 \cdot \frac{\alpha_n^{1+1/\theta}}{\lambda_n}(1 + \frac{\lambda_n}{\alpha_n} + \frac{\|x_{n+1} - x_n\|}{\alpha_n})^{1/\theta}.$$

Now by conditions (A_2) with $\sigma = \infty$ and (A_{11}) and (3.20), we derive

$$\limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n}\langle Vq - q, x_{n+1} - q\rangle \leq 0.$$

Further, we get

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{\lambda_n}\langle (I - \lambda_n\mu F)W_n q - (I - \lambda_n\mu F)q, x_{n+1} - q\rangle \\
&= \limsup_{n\to\infty}[\frac{1}{\lambda_n}\langle W_n q - q, x_{n+1} - q\rangle - \mu\langle FW_n q - Fq, x_{n+1} - q\rangle] \\
&\leq \limsup_{n\to\infty}[\frac{\alpha_n}{\lambda_n}\langle Vq - q, x_{n+1} - q\rangle + \mu\|FW_n q - Fq\|\|x_{n+1} - q\|] \\
&\leq \limsup_{n\to\infty}[\frac{\alpha_n}{\lambda_n}\langle Vq - q, x_{n+1} - q\rangle + \mu\kappa\alpha_n\|Vq - q\|\|x_{n+1} - q\|] \\
&\leq \limsup_{n\to\infty} \frac{\alpha_n}{\lambda_n}\langle Vq - q, x_{n+1} - q\rangle + \mu\kappa\|Vq - q\| \cdot \limsup_{n\to\infty} \alpha_n\|x_{n+1} - q\| \\
&\leq 0.
\end{aligned}$$

That is, (3.14) is fulfilled.

Notice that we still have the inequality (3.11) and hence from (3.11) we obtain

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\leq [1 - \lambda_n(\tau - \gamma L)]\|x_n - q\|^2 + 2\lambda_n[\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle \\
&\quad + \frac{1}{\lambda_n}\langle (I - \lambda_n\mu F)W_n q - (I - \lambda_n\mu F)q, x_{n+1} - q\rangle].
\end{aligned} \tag{3.23}$$

So setting $\beta_n = \lambda_n(\tau - \gamma L)$ and

$$\gamma_n = \frac{2\langle(\gamma f - \mu F)q, x_{n+1} - q\rangle + \frac{2}{\lambda_n}\langle (I - \lambda_n\mu F)W_n q - (I - \lambda_n\mu F)q, x_{n+1} - q\rangle}{\tau - \gamma L},$$

we can rewrite (3.23) as

$$\|x_{n+1} - q\|^2 \leq (1 - \beta_n)\|x_n - q\|^2 + \beta_n\gamma_n. \tag{3.24}$$

Noticing (3.9) and (3.14), we can apply Lemma 2.5 to conclude that $\|x_n - q\|^2 \to 0$; namely, $x_n \to q$ in norm. This completes the proof. ∎

Remark 3.2. As given out in [10, Remark 3.2], the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ can be appropriately chosen such that they satisfy the requirements of Theorems 3.1-3.3, respectively.

Remark 3.3. Recently, Mainge and Moudafi [7] also obtained a strong convergence result of Marino and Xu [10, Theorem 3.3] type under some stronger assumptions than those in [10, Theorem 3.3]. For instance, the following assumption in [7]

$$(P3)\ |\alpha_n - \alpha_{n-1}|/(\alpha_n^2\lambda_n) \to 0 \text{ and } |\lambda_n - \lambda_{n-1}|/(\alpha_n\lambda_n) \to 0$$

is slightly strongly than (A_9).

Meantime, Marino and Xu [10, Theorem 3.3] completely removed the condition $\mathrm{Fix}(T) \cap \mathrm{int}C \neq \emptyset$ in [7] and they did not use any graph convergence (see [8]) argument.

4. Applications in Hierarchical Minimization

Let H be a real Hilbert space and let $\varphi_0, \varphi_1 : H \to \mathbf{R}$ be lower semicontinuous convex functions. Consider the following hierarchical minimization

$$\min_{x\in H} \varphi_0(x), \quad \min_{x\in S_0} \varphi_1(x), \tag{4.1}$$

where $S_0 := \mathrm{argmin}_{x\in H}\varphi_0(x) \neq \emptyset$.

Let $S_1 := \mathrm{argmin}_{x\in S_0}\varphi_1(x) \neq \emptyset$. Suppose that φ_0 and φ_1 are differentiable and their gradients are Lipschitz continuous:

$$\|\nabla\varphi_0(x)-\nabla\varphi_0(y)\| \le L_0\|x-y\| \text{ and } \|\nabla\varphi_1(x)-\nabla\varphi_1(y)\| \le L_1\|x-y\|, \ \forall x, y \in H,$$

where L_0 and L_1 are constants. Let

$$T = I - \gamma_0\nabla\varphi_0, \quad V = I - \gamma_1\nabla\varphi_1, \tag{4.2}$$

where $\gamma_0 > 0$ and $\gamma_1 > 0$.

It is easily seen that $S_0 = \mathrm{Fix}(T)$. It is also known that T and V are both nonexpansive, if $0 < \gamma_0 < 2/L_0$ and $0 < \gamma_1 < 2/L_1$ (we always restrict γ_0 and γ_1 to such ranges). To see this, we need a result of [6], which says that the Lipschitz continuity of $\nabla\varphi_0$ implies that it is inverse strongly monotone; that is, the following inequality holds:

$$\langle x-y, \nabla\varphi_0(x) - \nabla\varphi_0(y)\rangle \ge \frac{1}{L_0}\|\nabla\varphi_0(x) - \nabla\varphi_0(y)\|^2, \quad \forall x, y \in H.$$

Now it follows that

$$\begin{aligned}\|Tx-Ty\|^2 &= \|(x-y)-\gamma_0(\nabla\varphi_0(x)-\nabla\varphi_0(y))\|^2\\ &= \|x-y\|^2-2\gamma_0\langle x-y,\nabla\varphi_0(x)\\ &\quad -\nabla\varphi_0(y)\rangle+\gamma_0^2\|\nabla\varphi_0(x)-\nabla\varphi_0(y)\|^2\\ &\le \|x-y\|^2-\gamma_0(\frac{2}{L_0}-\gamma_0)\|\nabla\varphi_0(x)-\nabla\varphi_0(y)\|^2\\ &\le \|x-y\|^2.\end{aligned}$$

Hence, T is nonexpansive. Similarly, V is nonexpansive.

The optimality condition for $x^* \in S_0$ to be a solution of the hierarchical minimization (4.1) is

$$x^* \in S_0 : \quad \langle \nabla\varphi_1(x^*), x-x^*\rangle \ge 0, \quad \forall x \in S_0; \tag{4.3}$$

equivalently, the variational inequality

$$x^* \in \mathrm{Fix}(T) : \quad \langle (I-V)x^*, x-x^*\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T). \tag{4.4}$$

Therefore, our results obtained in Sect. 3 apply. In particular, for any fixed point $u \in H$, taking the Lipschitzian mapping f to be the constant mapping $f(x) \equiv u$ for all $x \in H$, we have the following results (see [10, Theorems 4.1-4.3] for $\gamma = 1,\ \mu = 1$ and $F = I$).

Theorem 4.1. *Initializing with $x_0 \in H$, we define a sequence $\{x_n\}$ by the recursive algorithm:*

$$\begin{aligned} x_{n+1} &= \lambda_n\gamma u + (I-\lambda_n\mu F)(x_n-\gamma_0(1-\alpha_n)\nabla\varphi_0(x_n)\\ &\quad -\gamma_1\alpha_n\nabla\varphi_1(x_n)), \quad \forall n \ge 0. \end{aligned} \tag{4.5}$$

Assume that the sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ satisfy the assumptions (A_2) with $\tau = 0$, (A_3), and either "(A_4) and A_5" or "(A_6) and A_7". Then $\{x_n\}$ converges in norm to the fixed point x^* of the contraction $P_{S_0}(I-\mu F+\gamma u),\ x^* = P_{S_0}(x^*-\mu F x^*+\gamma u)$, i.e., the unique solution of the variational inequality

$$x^* \in S_0 : \quad \langle \mu F(x^*)-\gamma u, \hat{x}-x^*\rangle \ge 0, \quad \forall \hat{x} \in S_0.$$

In particular, if we additionally restrict $\gamma = \mu = 1$ and $F = I$, then $\{x_n\}$ converges in norm to the u-minimal norm solution x^* of the hierarchical minimization (4.1); namely, $x^* \in S_1$ satisfies the property: $\|u-x^*\| \le \|u-\hat{x}\|$ for all $\hat{x} \in S_1$. In other words, x^* is the (nearest point) projection of u onto the solution set S_1 of the hierarchical minimization (4.1).

Proof. Let $C = H,\ f(x) \equiv u$ and T, V be given by (4.2). Then $L = 0$ and so $0 \le \gamma L < \tau$ for all $\gamma > 0$ where $\tau = 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)},\ 0 < \mu < 2\eta/\kappa^2$. Thus it is found that the algorithm (3.2) generates a sequence $\{x_n\}$ given by

$$\begin{aligned} & x_{n+1} \\ &= P_C[\lambda_n \gamma f(x_n) + (I - \lambda_n \mu F)(\alpha_n V x_n + (1-\alpha_n) T x_n)] \\ &= \lambda_n \gamma u + (I - \lambda_n \mu F)[\alpha_n (x_n - \gamma_1 \nabla \varphi_1(x_n)) + (1-\alpha_n)(x_n - \gamma_0 \nabla \varphi_0(x_n))] \\ &= \lambda_n \gamma u + (I - \lambda_n \mu F)(x_n - \gamma_0 (1-\alpha_n) \nabla \varphi_0(x_n) - \gamma_1 \alpha_n \nabla \varphi_1(x_n)). \end{aligned}$$

Consequently, we can apply Theorem 3.1 to conclude that the sequence $\{x_n\}$ converges in norm to the unique solution x^* of the variational inequality:

$$(4.6) \qquad x^* \in S_0: \quad \langle \mu F(x^*) - \gamma u, \hat{x} - x^* \rangle \geq 0, \quad \forall \hat{x} \in S_0$$

which is equivalent to the fact that $x^* = P_{S_0}(x^* - \mu F x^* + \gamma u)$. ∎

Theorem 4.2. *Initializing with $x_0 \in H$, we define a sequence $\{x_n\}$ by the recursive algorithm:*

$$(4.7) \qquad \begin{aligned} x_{n+1} &= \alpha_n \gamma u + (I - \alpha_n \mu F)(x_n - \gamma_0 (1-\alpha_n) \nabla \varphi_0(x_n) \\ &\quad - \gamma_1 \alpha_n \nabla \varphi_1(x_n)), \quad \forall n \geq 0. \end{aligned}$$

where the sequence $\{\alpha_n\}$ satisfies the assumptions:

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*ii*) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| / \alpha_{n+1}^2 = 0$*;*

(*iii*) *there exists a constant $\mathcal{K} > 0$ such that $\frac{1}{\alpha_{n+1}} \left| \frac{1}{\alpha_{n+1}} - \frac{1}{\alpha_n} \right| \leq \mathcal{K}$ for all $n \geq 0$. Then $\{x_n\}$ converges in norm to a solution x^* of the hierarchical minimization (4.1) which is also the solution of the variational inequality:*

$$(4.8) \qquad x^* \in S_0: \quad \langle \mu F(x^*) + \gamma_1 \nabla \varphi_1(x^*) - \gamma u, \hat{x} - x^* \rangle \geq 0, \quad \forall \hat{x} \in S_0.$$

In particular, if we additionally restrict $\gamma = \mu = 1$ and $F = I$, then $\{x_n\}$ converges in norm to a solution x^* of the hierarchical minimization (4.1) which is also the solution of the variational inequality:

$$x^* \in S_0: \quad \langle x^* + \gamma_1 \nabla \varphi_1(x^*) - u, \hat{x} - x^* \rangle \geq 0, \quad \forall \hat{x} \in S_0.$$

Proof. Taking $\lambda_n = \alpha_n$ for all $n \geq 0$, we find that algorithm (4.5) reduces to algorithm (4.7). By virtue of the assumptions (i)-(iii), we can apply Theorem 3.2 to conclude that the sequence $\{x_n\}$ converges in norm to a solution x^* of the hierarchical minimization (4.1) which is also the solution of the variational inequality (see (3.15) with $\sigma = 1$ and $f \equiv u$):

$$(4.9) \qquad x^* \in S_0: \quad \langle \mu F(x^*) - \gamma u + (I - V)x^*, \hat{x} - x^* \rangle \geq 0, \quad \forall \hat{x} \in S_0.$$

Since $V = I - \gamma_1 \nabla\varphi_1$, we see that the variational inequality (4.9) reduces to (4.8). ■

Theorem 4.3. *Let* $\{x_n\}$ *be generated by the algorithm* (*4.5*). *Assume that the sequences* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *satisfy the assumptions* (A_2) *with* $\sigma = \infty$, (A_3), (A_8), *and* (A_9). *Assume also* $\lim_{n\to\infty} \alpha_n^2/\lambda_n = 0$. *Then* $\{x_n\}$ *converges in norm to a solution* x^* *of the hierarchical minimization* (*4.1*) *which solves the VI* (*4.6*). *In particular, if we additionally restrict* $\gamma = \mu = 1$ *and* $F = I$, *then* $\{x_n\}$ *converges in norm to the* u*-minimal norm solution* x^* *of the hierarchical minimization* (*4.1*).

Proof. Observe that assumption (A_{10}) of Theorem 3.3 holds with $\xi = \gamma_0 L_0$ and $\theta = 1$. As a matter of fact, for any $x \in H$ and $\hat{x} \in S_0$ (hence $\nabla\varphi_0(\hat{x}) = 0$), we deduce that

$$\|\nabla\varphi_0(x)\| = \|\nabla\varphi_0(x) - \nabla\varphi_0(\hat{x})\| \le L_0\|x - \hat{x}\|,$$

and hence $\|\nabla\varphi_0(x)\| \le L_0 \mathrm{dist}(x, S_0)$. As $T = I - \gamma_0\nabla\varphi_0$, we have

$$\|(I-T)x\| = \gamma_0\|\nabla\varphi_0(x)\| \le \gamma_0 L_0 \mathrm{dist}(x, S_0).$$

Now we can apply Theorem 3.3 to conclude that $\{x_n\}$ converges in norm to a solution x^* of the hierarchical minimization (4.1) which solves the VI (4.6). ■

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