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AN IMPROVED CHEN-RICCI INEQUALITY FOR KAEHLERIAN SLANT SUBMANIFOLDS IN COMPLEX SPACE FORMS

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Abstract. B. Y. Chen proved in [4] an optimal inequality for Lagrangian submanifolds in complex space forms in terms of the Ricci curvature and the squared mean curvature, well-known as the Chen-Ricci inequality. Recently, the Chen-Ricci inequality was improved in [7, 11] for Lagrangian submanifolds in complex space forms. In this article we extend the improved Chen-Ricci inequality to Kaehlerian slant submanifolds in complex space forms. We also investigate the equality case of the inequality.

1. PRELIMINARIES

Let $\widetilde{M}$ be a complex m-dimensional Kaehler manifold, i.e., $\widetilde{M}$ is endowed with an almost complex structure J and with a J-Hermitian metric $\widetilde{g}$. By a *complex space form* $\widetilde{M}(4c)$ we mean an m-dimensional Kaehler manifold with constant holomorphic sectional curvature $4c$. A complete simply-connected complex space form $\widetilde{M}(4c)$ is holomorphically isometric to the complex Euclidean n-space $\mathbf{C}^m$, the complex projective m-space $CP^m(4c)$, or the complex hyperbolic m-space $CH^m(4c)$, according to $c = 0$, $c > 0$ or $c < 0$, respectively.

Let $f : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional Riemannian manifold *M* into a Kaehler m-manifold $\widetilde{M}$. Then *M* is called a *totally real submanifold* if $J(T_pM) \subset T_p^{\perp}M$, $\forall p \in M$ (cf. [6]). A *Lagrangian submanifold* is a totally real submanifold of maximum dimension.

We denote by $K(\pi)$ the sectional curvature of *M* associated with a plane section $\pi \subset T_pM$, $p \in M$, by *h* the second fundamental form and by *R* the Riemann curvature tensor of *M*. Then the Gauss equation is given by:

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) \\ & + g(h(X,W),h(Y,Z)),\end{aligned}$$

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for any vectors *X,Y,Z,W* tangent to *M*.

Let $p \in M$ and $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space T_pM. We denote by H the *mean curvature vector*, i.e.,

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i)$$

and by

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$$

the squared norm of the second fundamental form.

For any tangent vector X to *M*, one decomposes $JX = PX + FX$, where *PX* and *FX* are the tangential and normal components of *JX*, respectively.

The submanifold M is said to be a *slant submanifold* if the angle between JX and the tangent space T_pM, for any nonzero vector X tangent to M, called the *Wirtinger angle* $\theta(X)$ of X, is constant, i.e., is independent of the choice of the point p and of the vector X (cf. [1]).

Slant submanifolds are characterized by the condition $P^2 = \lambda I$, for some $\lambda \in [-1, 0]$, where I is the identity transformation of TM. If $\lambda = -1$, then $\theta = 0$ and f is an *invariant immersion*; if $\lambda = 0$, then $\theta = \frac{\pi}{2}$ and f is an *totally real immersion*; if $\lambda = -\cos^2\theta$, with $\theta \neq 0, \frac{\pi}{2}$, then f is a *proper slant immersion*.

A proper slant submanifold is said to be *Kaehlerian slant* if $\nabla P = 0$ (the canonical endomorphism P is parallel), where ∇ is the Levi-Civita connection on M. A Kaehlerian slant submanifold is a Kaehler manifold with respect to the induced metric and the almost complex structure $\widetilde{J} = (\sec\theta)J$, where θ is the slant angle.

Let M be a proper slant submanifold $p \in M$, $\pi \subset T_pM$ a 2-plane section and $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of tangent space T_pM such that $e_1, e_2 \in \pi$. If $m = n$, an orthonormal basis $\{e_1^*, e_2^*, ..., e_n^*\}$ of the normal space $T_p^{\perp}M$ is defined by

$$e_k^* = \frac{1}{\sin\theta} Fe_k, \quad k = 1, \dots, n. \tag{1.1}$$

For a Kaehlerian slant submanifold one has (cf. [1])

$$A_{FX}Y = A_{FY}X, \quad \forall X, Y \in T_pM,$$

or equivalently,

$$h_{ij}^k = h_{ik}^j = h_{jk}^i, \tag{1.2}$$

where A is the shape operator and

$$h_{ij}^k = g(h(e_i, e_j), e_k^*), \quad i, j, k = 1, ..., n. \tag{1.3}$$

The following propositions give characterizations of submanifolds with $\nabla P = 0$.

Proposition 1.1. [1]. *Let M be a submanifold of an almost Hermitian manifold $\widetilde{M}$. Then $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times ... \times M_k$, where each M_i is either a complex submanifold, a totally real submanifold or a Kaehlerian slant submanifold of $\widetilde{M}$.*

Proposition 1.2. [1]. *Let M be an irreducible submanifold of an almost Hermitian manifold $\widetilde{M}$. If M is neither invariant nor totally real, then M is a Kaehlerian slant submanifold if and only if the endomorphism P is parallel, i.e., $\nabla P = 0$.*

Definition 1.3. A slant H-*umbilical submanifold* of a Kaehler manifold $\widetilde{M}^n$ is a slant submanifold for which the second fundamental form takes the following forms:

$$h(e_1, e_1) = \lambda e_1^*, \quad h(e_2, e_2) = \cdots = h(e_n, e_n) = \mu e_1^*,$$
$$h(e_1, e_j) = \mu e_j^*, \quad h(e_j, e_k) = 0, \;\; 2 \leq j \neq k \leq n,$$

where $e_1^*, \ldots, e_n^*$ are defined by (1.1).

2. Ricci Curvature of Submanifolds

In [3], B. Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any n-dimensional Riemannian submanifold of a real space form $\widetilde{M}(c)$ of constant sectional curvature c; namely,

$$Ric(X) \leq (n-1)c + \frac{n^2}{4} \|H\|^2 ,$$

which is well-known as the Chen-Ricci inequality. The same inequality holds for Lagrangian submanifolds in a complex space form $\widetilde{M}(4c)$ as well (see [4]).

I. Mihai proved a similar inequality in [9] for certain submanifolds of Sasakian space forms.

In [8], Matsumoto, Mihai and Oiaga extended the Chen-Ricci equality to the following inequality for submanifolds in complex space forms.

Theorem 2.1. [8]. *Let M be an n-dimensional submanifold of a complex m-dimensional complex space form $\widetilde{M}(4c)$. Then*:

(i) *For each vector $X \in T_pM$ we have*

$$Ric(X) \leq (n-1)c + \frac{n^2}{4} \|H\|^2 + 3c \|PX\|^2 .$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case if and only if $X \in \ker h_p$;*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if p is a totally geodesic point or $n = 2$ and p is* a *totally umbilical point*.

In particular, for θ-slant submanifolds, the following result holds.

Corollary 2.2. [8]. *Let M be an n-dimensional θ-slant submanifold of a complex space form $\widetilde{M}(4c)$. Then*:
(i) *For each vector $X \in T_pM$ we have*

$$Ric(X) \leq (n-1)c + \frac{n^2}{4}\|H\|^2 + 3c\cos^2\theta.$$

(ii) *If $H(p) = 0$, then a unit tangent vector X at p satisfies the equality case if and only if $X \in \ker h_p$*;
(iii) *The equality case holds identically for all unit tangent vectors at p if and only if p is a totally geodesic point or $n = 2$ and p is* a *totally umbilical point*.

The Chen-Ricci inequality was further improved to the following for Lagrangian submanifolds (cf. [7, 11]).

Theorem 2.3. *Let M be a Lagrangian submanifold of dimension $n \geq 2$ in a complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$ and X a unit tangent vector in T_pM, $p \in M$. Then, we have*

$$Ric(X) \leq (n-1)\left(c + \frac{n}{4}\|H\|^2\right).$$

The equality sign holds for any unit tangent vector at p if and only if either:
(i) *p is a totally geodesic point, or*
(ii) *$n = 2$ and p is an H-umbilical point with $\lambda = 3\mu$.*

Lagrangian submanifolds in complex space forms satisfying the equality case of the inequality were determined by Deng in [7]. More precisely, he proved the following.

Corollary 2.4. *Let M be a Lagrangian submanifold of real dimension $n \geq 2$ in a complex space form $\widetilde{M}(4c)$. If*

$$Ric(X) = (n-1)\left(c + \frac{n}{4}\|H\|^2\right),$$

for any unit tangent vector X of M, then either
(i) *M is a totally geodesic submanifold in $\widetilde{M}(4c)$ or,*
(ii) *$n = 2$ and M is a Lagrangian H-umbilical submanifold of $\widetilde{M}(4c)$ with $\lambda = 3\mu$.*

3. Ricci Curvature of Kaehlerian Slant Submanifolds

In this section, we extend Theorem 2.3 to Kaehlerian slant submanifolds in complex space forms. We shall apply the following two Lemmas from [7].

Lemma 3.1. *Let $f_1(x_1, x_2, ..., x_n)$ be a function in $\mathbf{R}^n$ defined by:*

$$f_1(x_1, x_2, ..., x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + ... + x_n = 2na$, then we have

$$f_1(x_1, x_2, ..., x_n) \leq \frac{n-1}{4n} (x_1 + x_2 + ... + x_n)^2 ,$$

with the equality sign holding if and only if $\frac{1}{n+1} x_1 = x_2 = ... = x_n = a$.

Lemma 3.2. *Let $f_2(x_1, x_2, ..., x_n)$ be a function in $\mathbf{R}^n$ defined by:*

$$f_2(x_1, x_2, ..., x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + ... + x_n = 4a$, then we have

$$f_2(x_1, x_2, ..., x_n) \leq \frac{1}{8} (x_1 + x_2 + ... + x_n)^2 ,$$

with the equality sign holding if and only if $x_1 = a$ and *$x_2 + ... + x_n = 3a$.*

The main result of this section is the following theorem.

Theorem 3.3. *Let M be an n-dimensional Kaehlerian proper θ-slant submanifold in a complex n-dimensional complex space form $\widetilde{M}(4c)$ of constant holomorphic sectional curvature $4c$. Then for any unit tangent vector X to M we have*

$$Ric(X) \leq (n-1)\left(c + \frac{n}{4} \|H\|^2\right) + 3c \cos^2 \theta. \tag{3.1}$$

The equality sign of (3.1) *holds identically if and only if either*
(i) *$c = 0$ and M is totally geodesic, or*
(ii) *$n = 2$, $c < 0$ and M is a slant H-umbilical surface with $\lambda = 3\mu$.*

Proof. For a given point $p \in M$ and a given unit vector $X \in T_pM$, we choose an orthonormal basis $\{e_1 = X, e_2, \dots, e_n\} \subset T_pM$ and

$$\left\{e_1^* = \frac{Fe_1}{\sin\theta}, \dots, e_n^* = \frac{Fe_n}{\sin\theta}\right\} \subset T_p^{\perp}M.$$

For that $X = Z = e_1$ and $Y = W = e_j$, $j = 2, .., n$, Gauss' equation gives

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - g(h(e_1, e_1), h(e_j, e_j)) + g(h(e_1, e_j), h(e_1, e_j)),$$

or equivalently,

$$\widetilde{R}(e_1, e_j, e_1, e_j) = R(e_1, e_j, e_1, e_j) - \sum_{r=1}^{n} (h_{11}^r h_{jj}^r - (h_{1j}^r)^2), \quad \forall j \in \overline{2, n}.$$

Since the Riemannian curvature tensor of $\widetilde{M}(4c)$ is given by

$$\widetilde{R}(X,Y,Z,W) = c\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)$$
$$+g(JX,Z)g(JY,W) - g(JX,W)g(JY,Z) + 2g(JX,Y)g(JZ,W)\},$$

we find

$$\widetilde{R}(e_1,e_j,e_1,e_j) = c[1+3g^2(Je_1,e_j)]. \tag{3.2}$$

By summing after $j = \overline{2,n}$, we get

$$(n-1+3\|PX\|^2)c = Ric(X) - \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right],$$

or,

$$(n-1+3\cos^2\theta)c = Ric(X) - \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right].$$

It follows that

$$Ric(X) - (n-1+3\cos^2\theta)c = \sum_{r=1}^{n}\sum_{j=2}^{n}\left[h_{11}^r h_{jj}^r - (h_{1j}^r)^2\right] \tag{3.3}$$
$$\le \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{1j}^1)^2 - \sum_{j=2}^{n}(h_{1j}^j)^2.$$

Since M is a Kaehlerian slant submanifold, we have the relations (1.2) and

$$Ric(X) - (n-1+3\cos^2\theta)c \le \sum_{r=1}^{n}\sum_{j=2}^{n} h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{11}^j)^2 - \sum_{j=2}^{n}(h_{jj}^1)^2. \tag{3.4}$$

Now we put

$$f_1(h_{11}^1, h_{22}^1, ..., h_{nn}^1) = h_{11}^1\sum_{j=2}^{n} h_{jj}^1 - \sum_{j=2}^{n}(h_{jj}^1)^2$$

and

$$f_r(h_{11}^r, h_{22}^r, ..., h_{nn}^r) = h_{11}^r\sum_{j=2}^{n} h_{jj}^r - (h_{11}^r)^2, \quad \forall r \in \overline{2,n}.$$

Since $nH^1 = h_{11}^1 + h_{22}^1 + ... + h_{nn}^1$, we obtain by using Lemma 3.1 that

$$f_1(h_{11}^1, h_{22}^1, ..., h_{nn}^1) \le \frac{n-1}{4n}(nH^1)^2 = \frac{n(n-1)}{4}(H^1)^2. \tag{3.5}$$

By applying Lemma 3.2 for $2 \le r \le n$, we get

$$f_r(h_{11}^r, h_{22}^r, ..., h_{nn}^r) \le \frac{1}{8}(nH^r)^2 = \frac{n^2}{8}(H^r)^2 \le \frac{n(n-1)}{4}(H^r)^2. \tag{3.6}$$

From (3.4), (3.5) and (3.6), we obtain

$$Ric(X) - (n-1+3\cos^2\theta)c \leq \frac{n(n-1)}{4}\sum_{r=1}^{n}(H^r)^2 = \frac{n(n-1)}{4}\|H\|^2.$$

Thus we have

$$Ric(X) \leq (n-1+3\cos^2\theta)c + \frac{n(n-1)}{4}\|H\|^2,$$

which implies (3.1).

Next, we shall study the equality case. For $n \geq 3$, we choose Fe_1 parallel to H. Then we have $H^r = 0$, for $r \geq 2$. Thus, by Lemma 3.2, we get

$$h^1_{1j} = h^j_{11} = \frac{nH^j}{4} = 0, \quad \forall j \geq 2,$$

and

$$h^1_{jk} = 0, \quad \forall j,k \geq 2, \quad j \neq k.$$

From Lemma 3.1, we have $h^1_{11} = (n+1)a$ and $h^1_{jj} = a$, $\forall j \geq 2$, with $a = \frac{H^1}{2}$.

In (3.3) we compute $Ric(X) = Ric(e_1)$. Similarly, by computing $Ric(e_2)$ and using the equality, we get

$$h^r_{2j} = h^2_{jr} = 0, \ \ \forall r \neq 2, \ \ j \neq 2, \ \ r \neq j.$$

Then we obtain

$$\frac{h^2_{11}}{n+1} = h^2_{22} = ... = h^2_{nn} = \frac{H^2}{2} = 0.$$

The argument is also true for matrices $\left(h^r_{jk}\right)$ because the equality holds for all unit tangent vectors; so, $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0, \ \forall j \geq 3$.

The matrix $\left(h^2_{jk}\right)$ (respectively the matrix $\left(h^r_{jk}\right)$) has only two possible nonzero entries $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$ (respectively $h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}$, $\forall r \geq 3$). Now, after putting $X = Z = e_2$ and $Y = W = e_j$, $j = 3, \ldots, n$, in Gauss' eqution, we obtain

$$\widetilde{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \quad \forall j \geq 3.$$

If we put $X = Z = e_2$ and $Y = W = e_1$ in Gauss' eqution, we get

$$\widetilde{R}(e_2, e_1, e_2, e_1) = R(e_2, e_1, e_2, e_1) - (n+1)\left(\frac{H^1}{2}\right)^2 + \left(\frac{H^1}{2}\right)^2.$$

After combining the last two relations, we find

$$Ric(e_2) - (n-1+3\cos^2\theta)c = 2(n-1)\left(\frac{H^1}{2}\right)^2.$$

On the other hand, the equality case of (3.1) implies that

$$Ric(e_2) - (n-1+3\cos^2\theta)c = \frac{n(n-1)}{4}\|H\|^2 = n(n-1)\left(\frac{H^1}{2}\right)^2.$$

Since $n \neq 1, 2$, by equating the last 2 equations we find $H^1 = 0$. Thus, $\left(h_{jk}^r\right)$ are all zero, i.e., M is a totally geodesic submanifold in $\widetilde{M}(4c)$. In particular, M is a curvature-invariant submanifold of $\widetilde{M}(4c)$. Therefore, when $c \neq 0$, it follows from a result of Chen and Ogiue [6] that M is either a complex submanifold or a Lagrangian submanifold of $\widetilde{M}(4c)$. Hence, M is a non-proper θ-slant submanifold, which is a contradiction. Consequently, we have either

(1) $c = 0$ and M is totally geodesic, or,

(2) $n = 2$.

If (1) occurs, we obtain (i) of the theorem. Now, let us assume that $n = 2$. Let us recall a result of Chen from [2] states that if M is a proper slant surface in a complex 2-dimensional complex space form $\widetilde{M}^2(4c)$ satisfying the equality case of (3.1) identically, then M is either totally geodesic or $c < 0$. In particular, when M is not totally geodesic, one has

$$h(e_1, e_1) = \lambda e_1^*, \quad h(e_2, e_2) = \mu e_1^*, \quad h(e_1, e_2) = \mu e_2^*,$$

with $\lambda = 3\mu = \frac{3H^1}{2}$, i.e., M is H-umbilical. This gives case (ii) of the theorem. ∎

Since a proper slant surface is Kaehlerian slant automatically (cf. [1]), we rediscover the following result of [2] from Theorem 3.3.

Theorem 3.4. *If M is a proper slant surface in a complex space form $\widetilde{M}(4c)$ of complex dimension 2, then the squared mean curvature and the Gaussian curvature of M satisfy:*

$$\|H\|^2 \geq 2[G - (1 + 3\cos^2\theta)c]$$

at each point $p \in M$, where θ is the slant angle of the slant surface.

Example 3.5. The explicit representation of the slant surface in $CH^2(-4)$ satisfying the equality case of inequality (3.1) was determined by Chen and Tazawa in [5, Theorem 5.2] as follows:

Let z be the immersion $z : \mathbf{R}^3 \to \mathbf{C}_1^3$ defined by

$$\begin{aligned} &z(u, v, t) \\ &= e^{it}\Bigg(1 + \frac{3}{2}\Big(\cosh\Big(\tfrac{\sqrt{2}}{\sqrt{3}}v\Big) - 1\Big) + \frac{u^2}{6}e^{-\sqrt{\frac{2}{3}}v} - i\frac{u}{\sqrt{6}}(1 + e^{-\sqrt{\frac{2}{3}}v}), \\ &\frac{u}{3}\Big(1 + 2e^{-\sqrt{\frac{2}{3}}v}\Big) + \frac{i}{6\sqrt{6}}e^{-\sqrt{\frac{2}{3}}v}\Big(\Big(e^{\sqrt{\frac{2}{3}}v} - 1\Big)\Big(9e^{\sqrt{\frac{2}{3}}v} - 3\Big) + 2u^2\Big), \\ &\frac{u}{3\sqrt{2}}\Big(1 - e^{-\sqrt{\frac{2}{3}}v}\Big) + \frac{i}{12\sqrt{3}}\Big(6 - 15e^{-\sqrt{\frac{2}{3}}v} + 9e^{\sqrt{\frac{2}{3}}v} + 2e^{-\sqrt{\frac{2}{3}}v}u^2\Big)\Bigg). \end{aligned} \tag{3.7}$$

It was proved in [5] that $\langle z, z\rangle = -1$. Hence, z defines an immersion from $\mathbf{R}^3$ into the anti-de Sitter spacetime $H_1^5(-1)$. Moreover, it was proved in [5] that the image $z(\mathbf{R}^3)$ in $H_1^5(-1)$ is invariant under the action of $\mathbf{C}^* = \mathbf{C} - \{0\}$. Let $\pi : H_1^4(-1) \to CH^2(-4)$ denote the Hopf fibration. It was shown in [5] that the composition

$$\pi \circ z : \mathbf{R}^3 \to CH_1^2(-4)$$

defines a slant surface with slant angle $\theta = \cos^{-1}(\frac{1}{3})$. Also, it was proved in [5] that $\pi \circ z$ is a H-umbilical immersion satisfying $\lambda = 3\mu$. Consequently, this example of slant H-umbilical surface satisfies the equality case of inequality (3.1) identically.

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