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CONVERGENCE BEHAVIOR FOR NEWTON-STEFFENSEN'S METHOD UNDER LIPSCHITZ CONDITION OF SECOND DERIVATIVE

Shaohua Yu, Xiubin Xu*, Jianqiu Li and Yonghui Ling

Abstract. The present paper is concerned with the semilocal as well as the local convergence problems of Newton-Steffensen's method to solve nonlinear operator equations in Banach spaces. Under the assumption that the second derivative of the operator satisfies Lipschitz condition, the convergence criterion and convergence ball for Newton-Steffensen's method are established.

1. Introduction

Let X and Y be real or complex Banach spaces, $D \subset X$ be an open subset and let $F : D \subset X \to Y$ be a nonlinear operator. Finding solutions of the nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

in Banach spaces is a very general subject which is widely studied in both theoretical and applied areas of mathematics.

When F is Fréchet differentiable, the most important method to find an approximation of a solution of (1.1) is Newton's method which takes the following form:

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

Usually one's interest is focused on two types of convergence issues about Newton's method: local and semi-local convergence analyses. The first is to determine the convergence ball based on the information in a neighborhood of the solution of (1.1), see for example, [16, 19, 17]; The second is the convergence criterion based on the

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*Corresponding author.

information only in a neighborhood of the initial point x_0. Among the semilocal convergence results on Newton's method, one of the famous results is the well-known Kantorovich theorem [9, 10] which provides the convergence criterion of Newton's method under the very mild condition.

As it is well known, in the case when F has the second continuous Fréchet derivative on D, there are several kinds of cubic generalizations for Newton's method. The most important two are Chebyshev's method and Halley's method, see e.g., [2, 3, 6, 4]. Another more general family of cubic extensions is the family of Chebyshev–Halley–type methods, which includes Chebyshev's method and Halley's method as well as the convex acceleration of Newton's method. However, the disadvantage of this family is that evaluation of the second derivative of the operator F is required at every step, the operation cost of which may be very large in fact. To reduce the operation cost but also retain the cubical convergence, the variants of the above methods have been studied extensively in [5, 7, 21, 22] and references therein.

Recently, Sharma in [13] proposed the following Newton-Steffensen's method which avoids the computation of the second Fréchet derivative. Let $f:\mathbb{R}\to\mathbb{R}$. The method is defined as follows:

$$\left\{\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{g(x_n)}, \quad n = 0, 1, \ldots, \end{aligned}\right. \tag{1.3}$$

where $g(x_n) = \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$. The author gave the cubic convergence result for (1.3) under the assumption that f is sufficiently smooth in the neighborhood of the solution. Motivated by the works mentioned above, we extend this method to Banach spaces which is described as follows:

$$\left\{\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - [y_n, x_n; F]^{-1}F(x_n), \quad n = 0, 1, \ldots, \end{aligned}\right. \tag{1.4}$$

where the divided difference operator is defined by

$$[y_n, x_n; F] = \int_0^1 F'(x_n + t(y_n - x_n))\,\mathrm{d}t. \tag{1.5}$$

Our goal in the present paper is to establish the semilocal convergence as well as the local convergence of Newton-Steffensen's method (1.4) under the assumption that F'' satisfies Lipschitz condition on some ball. Furthermore, the uniqueness ball is obtained by using the unified convergence theorem established in [18] for the class of operators whose second derivatives satisfy Lipschitz condition. In Section

2, we introduce some preliminary notations and an important majorizing function with some useful properties. The Kantorovich-type semilocal convergence criterion is established under Lipschitz condition in Section 3. In Section 4, we analyze the local convergence of Newton-Steffensen's method. Finally in Section 5, we apply the obtained semilocal convergence result to a nonlinear boundary value problem. From this, we can see that Newton-Steffensen's method is applicable and converges rapidly.

2. Notations and Preliminary Results

Throughout this paper, we assume that X and Y are two Banach spaces. Let $D \subset X$ be an open subset and let $F : D \subset X \to Y$ be a nonlinear operator with continuous twice Fréchet derivative. For $x \in X$ and $r > 0$, we use $\mathbf{B}(x, r)$ and $\overline{\mathbf{B}(x, r)}$ to denote the open ball with radius r and center x and its closure, respectively. Let $\overline{x} \in D$ be such that $F'(\overline{x})^{-1}$ exists and $\overline{\mathbf{B}(\overline{x}, r)} \subset D$.

We say that $F'(\overline{x})^{-1}F''$ satisfies Lipschitz condition on $\boldsymbol{B}(\overline{x}, r)$ with the positive constant L if

$$\|F'(\overline{x})^{-1}[F''(x) - F''(y)]\| \leq L\|x - y\|, \quad x, y \in \boldsymbol{B}(\overline{x}, r). \tag{2.1}$$

The lemma below is taken from [20], which is useful in the next two sections.

Lemma 2.1. *Suppose that $\|F'(\overline{x})^{-1}F''(\overline{x})\| \leq \eta$ and $F'(\overline{x})^{-1}F''$ satisfies Lipschitz condition (2.1) on $\boldsymbol{B}(\overline{x}, r)$, where η is some positive constant. Then for each $x \in \boldsymbol{B}(\overline{x}, r)$, $F'(x)^{-1}$ exists and the following inequalities hold:*

$$\|F'(\overline{x})^{-1}F''(x)\| \leq L\|x - \overline{x}\| + \eta, \tag{2.2}$$

$$\|F'(x)^{-1}F'(\overline{x})\| \leq \frac{1}{1 - \eta\|x - \overline{x}\| - \frac{L}{2}\|x - \overline{x}\|^2}. \tag{2.3}$$

Let β, η and L be some fixed positive constants. The majorizing function h defined below was introduced by Wang in [15]:

$$h(t) = \beta - t + \frac{\eta}{2}t^2 + \frac{L}{6}t^3, \quad t \geq 0. \tag{2.4}$$

Let $\{s_n\}$ and $\{t_n\}$ denote the corresponding sequences generated by Newton-Steffensen's method for the majorizing function h with the initial point $t_0 = 0$, that is,

$$\begin{cases} s_n = t_n - \dfrac{h(t_n)}{h'(t_n)}, \\ t_{n+1} = t_n - \left(\dfrac{h(s_n) - h(t_n)}{s_n - t_n}\right)^{-1} h(t_n), \quad n = 0, 1, \ldots. \end{cases} \tag{2.5}$$

For simplicity, set

$$\triangle(L;\eta)=\frac{2(\eta+2\sqrt{\eta^2+2L})}{3(\eta+\sqrt{\eta^2+2L})^2} \tag{2.6}$$

and

$$r_0=\frac{-\eta+\sqrt{\eta^2+2L}}{L}. \tag{2.7}$$

The following two lemmas respectively describe some properties about h and the convergence properties of the sequences $\{s_n\}$ and $\{t_n\}$. The first one is taken from [18].

Lemma 2.2. *Suppose that*

$$\beta\le\triangle(L;\eta). \tag{2.8}$$

Then h is decreasing in $[0,r_0]$, while it is increasing in $[r_0,+\infty)$. Moreover, h has a unique zero in each interval, denoted by t^* and t^{**}. They satisfy

$$\beta<t^*<r_0<t^{**} \tag{2.9}$$

when $\beta<\triangle(L;\eta)$ and $t^*=t^{**}$ when $\beta=\triangle(L;\eta)$.

Lemma 2.3. *Suppose that (2.8) holds. Let $\{s_n\}$ and $\{t_n\}$ be the sequences generated by (2.5). Then*

$$0\le t_n<s_n<t_{n+1}<t^* \quad \text{for all } n\ge 0. \tag{2.10}$$

Moreover, $\{s_n\}$ and $\{t_n\}$ converge increasingly to the same point t^*.

Proof. To show (2.10) holds for $n=0$, we note that $0=t_0<s_0=\beta$ and $t_1=-\beta/(\frac{\eta}{2}\beta+\frac{L}{6}\beta^2-1)$. Since $h'(t)=-1+\eta t+\frac{L}{2}t^2<0$ for all $t\in[0,r_0)$ and $0<\beta<r_0$ due to (2.9), we have

$$0<\frac{\eta}{2}\beta+\frac{L}{6}\beta^2<\eta\beta+\frac{L}{2}\beta^2<1.$$

This implies $t_1>\beta=s_0$. It remains to show $t_1<t^*$ for the case $n=0$. To this end, we define a real function as

$$\Phi(t)=1-\frac{\eta}{2}t-\frac{L}{6}t^2, \quad t\in[0,+\infty).$$

It is clear that $\Phi(t)=-(h(t)-\beta)/t$ and that $\Phi(t)$ is decreasing in $[0,+\infty)$. It follows from (2.9) that $\Phi(\beta)>\Phi(t^*)$. In view of the fact that t^* is the unique zero of h in $[0,r_0]$, i.e., $h(t^*)=0$, from which we obtain $\beta/t^*=\Phi(t^*)<\Phi(\beta)$. This is equivalent to

$$t^* > \frac{\beta}{1 - \frac{\eta}{2}\beta - \frac{L}{6}\beta^2} = t_1.$$

Hence (2.10) holds for $n = 0$.

Now we assume that

$$0 \le t_{n-1} < s_{n-1} < t_n < t^* \quad \text{for some } n \ge 1.$$

From Lemma 2.2, we have $h(t) \ge 0$ for each $t \in [0, t^*]$ and $h(t_n)/h'(t_n) < 0$. The later implies that $s_n > t_n$. Define function

$$N(t) = t - \frac{h(t)}{h'(t)}, \quad t \in [0, t^*].$$

Then, $N'(t) = h(t)h''(t)/h'(t)^2 > 0$, which implies that $N(t)$ is increasing in $[0, t^*]$. Therefore we have

$$s_n = t_n - \frac{h(t_n)}{h'(t_n)} < t^* - \frac{h(t^*)}{h'(t^*)} = t^*.$$

Since h is convex in $[0, t^*]$, we get $h'(t_n) < (h(s_n) - h(t_n))/(s_n - t_n)$ and so $s_n < t_{n+1}$.

Furthermore, it also follows from the convexity of h that

$$t' - \left(\frac{h(t) - h(t')}{t - t'}\right)^{-1} h(t') < t'' - \left(\frac{h(t'') - h(t)}{t'' - t}\right)^{-1} h(t'') \tag{2.11}$$

for all $t', t, t'' \in [0, t^*]$ and $t' < t < t''$. Indeed, we can obtain that

$$\begin{aligned} -\frac{h(t'')}{h'(t)} &\le -\left(\frac{h(t'') - h(t)}{t'' - t}\right)^{-1} h(t'') \le -\frac{h(t'')}{h'(t'')}, \\ &\le -\left(\frac{h(t) - h(t')}{t - t'}\right)^{-1} h(t') \le -\frac{h(t')}{h'(t)}, \end{aligned}$$

from which we have

$$(t'' - t') + \frac{h(t') - h(t'')}{h'(t)} \le (t'' - t') + (T'' - T') \le (t'' - t') - \frac{h(t'')}{h'(t'')} + \frac{h(t')}{h'(t')},$$

where

$$T' = -\left(\frac{h(t) - h(t')}{t - t'}\right)^{-1} h(t') \text{ and } T'' = -\left(\frac{h(t'') - h(t)}{t'' - t}\right)^{-1} h(t'').$$

Noting that $-1 < h'(t) < 0$ for all $t \in [0, t^*]$, we have

$$(t'' - t') + \frac{h(t') - h(t'')}{h'(t)} > (t'' - t') + [h(t'') - h(t')] \ge 0.$$

Then (2.11) follows. By (2.11), we conclude that

$$t_{n+1} = t_n - \left(\frac{h(s_n) - h(t_n)}{s_n - t_n}\right)^{-1} h(t_n) < t^* - \left(\frac{h(t^*) - h(s_n)}{t^* - s_n}\right)^{-1} h(t^*) = t^*.$$

Therefore, (2.10) holds for all $n \geq 0$. The inequalities in (2.10) imply that $\{s_n\}$ and $\{t_n\}$ converge increasingly to some same point, say τ. Clearly $\tau \in [0, t^*]$ and τ is a zero of h in $[0, t^*]$. Noting that t^* is the unique zero of h in $[0, r_0]$, one has that $\tau = t^*$. The proof is complete. ∎

3. Convergence Criterion

Throughout this section, let $x_0 \in D$ be the initial point such that the inverse $F'(x_0)^{-1}$ exists and let $\mathbf{B}(x_0, r_0) \subset D$, where r_0 is defined by (2.7). Moreover, we assume that $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$, $\|F'(x_0)^{-1}F''(x_0)\| \leq \eta$ and $F'(x_0)^{-1}F''(x)$ satisfies Lipschitz condition on $\mathbf{B}(x_0, r_0)$. For any $x \in \mathbf{B}(x_0, r_0)$, it follows from Lemma 2.1 that $F'(x)^{-1}$ exists and

$$\|F'(x_0)^{-1}F''(x)\| \leq L\|x - x_0\| + \eta, \tag{3.1}$$

$$\|F'(x)^{-1}F'(x_0)\| \leq \frac{1}{1 - \eta\|x - x_0\| - \frac{L}{2}\|x - x_0\|^2}. \tag{3.2}$$

Below we list a series of useful lemmas.

Recall that the divided difference operator $[y, x; F]$ and r_0 are respectively defined by (1.5) and (2.7), and $F'(x)^{-1}$ exists due to the above assumption. The following lemma gives the expressions of some desired estimates in the proof of Lemma 3.5.

Lemma 3.4. *Let $x \in \boldsymbol{B}(x_0, r_0)$. Define*

$$y := x - F'(x)^{-1}F(x) \quad \textit{and} \quad \overline{x} := x - [y, x; F]^{-1}F(x).$$

Then the following formulas hold:

(i) $[y, x; F] - F'(x_0) = \int_0^1 \int_0^1 F''(x_0 + s[(x - x_0) + t(y - x)])[(x - x_0) + t(y - x)]\, \mathrm{d}s\mathrm{d}t.$

(ii) $\overline{x} - y = -F'(x)^{-1} \int_0^1 \int_0^1 F''(x + st(y - x))(y - x)(\overline{x} - x)t\, \mathrm{d}s\mathrm{d}t.$

(iii) $F(\overline{x}) = \int_0^1 \int_0^1 F''(x + t(y - x) + st(\overline{x} - y))(\overline{x} - x)(\overline{x} - y)t\, \mathrm{d}s\mathrm{d}t.$

Proof. For (i), we notice that

$$
\begin{aligned}
&[y, x; F] - F'(x_0) \\
&= \int_0^1 [F'(x + t(y - x)) - F'(x_0)]\, \mathrm{d}t \\
&= \int_0^1 \int_0^1 F''(x_0 + s[(x - x_0) + t(y - x)])[(x - x_0) + t(y - x)]\, \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

As for (ii), one has that

$$
\begin{aligned}
\overline{x} - y &= F'(x)^{-1}F(x) - [y, x; F]^{-1}F(x) \\
&= F'(x)^{-1}\left[F'(x) - \int_0^1 F'(x + t(y - x))\, \mathrm{d}t\right](\overline{x} - x) \\
&= -F'(x)^{-1}\int_0^1 \int_0^1 F''(x + st(y - x))(y - x)(\overline{x} - x)t\, \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
F(\overline{x}) &= F(\overline{x}) - F(x) - [y, x; F](\overline{x} - x) \\
&= \int_0^1 F'(x + t(\overline{x} - x))(\overline{x} - x)\, \mathrm{d}t - \int_0^1 F'(x + t(y - x))(\overline{x} - x)\, \mathrm{d}t \\
&= \int_0^1 \int_0^1 F''(x + t(y - x) + st(\overline{x} - y))(\overline{x} - x)(\overline{x} - y)t\, \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

The proof is complete. ∎

Lemma 3.5. *Suppose that* (*2.8*) *holds. Then the sequence* $\{x_n\}$ *generated by (1.4) with the initial point* x_0 *is well defined and the following estimates hold for any natural number* $n \geq 1$*:*

(i) $\|y_{n-1} - x_{n-1}\| \leq s_{n-1} - t_{n-1}$, $\|x_n - x_{n-1}\| \leq t_n - t_{n-1}$, $\|x_n - y_{n-1}\| \leq t_n - s_{n-1}$.

(ii) $\left\|[y_{n-1}, x_{n-1}; F]^{-1}F'(x_0)\right\| \leq -\dfrac{s_{n-1} - t_{n-1}}{h(s_{n-1}) - h(t_{n-1})}$.

(iii) $\left\|F'(x_0)^{-1}F(x_n)\right\| \leq h(t_n)\left(\dfrac{\|x_n - x_{n-1}\|}{t_n - t_{n-1}}\right)^2 \left(\dfrac{\|y_{n-1} - x_{n-1}\|}{s_{n-1} - t_{n-1}}\right)$.

Proof. For the case $n = 1$ in (i), it is clear that $\|y_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq \beta = s_0 - t_0$. It follows from (3.1) and Lemma 3.4 that

$$
\begin{aligned}
&\left\|F'(x_0)^{-1}([y_0, x_0; F] - F'(x_0))\right\| \\
&= \left\|F'(x_0)^{-1}\int_0^1 \int_0^1 F''(x_0 + st(y_0 - x_0))(y_0 - x_0)t\, ds\, dt\right\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{L}{6}\|y_0 - x_0\|^2 + \frac{1}{2}\eta\beta \\
&\leq \frac{L}{6}(s_0 - t_0)^2 + \frac{1}{2}\eta\beta \\
&= \frac{h(s_0) - h(t_0)}{s_0 - t_0} + 1.
\end{aligned}$$

Since $h'(t) < 0$ in $(0, r_0)$, we have $(h(s_0) - h(t_0))/(s_0 - t_0) < 0$. Thus, it follows from Banach lemma that $[y_0, x_0; F]^{-1}$ exists and satisfies

$$\left\|[y_0, x_0; F]^{-1}F'(x_0)\right\| \leq \frac{1}{1 - \left(\dfrac{h(s_0) - h(t_0)}{s_0 - t_0} + 1\right)} = -\frac{s_0 - t_0}{h(s_0) - h(t_0)}. \tag{3.3}$$

This together with the definitions of $\{s_n\}$ and $\{t_n\}$ given in (2.5) yields that

$$\begin{aligned}
\|x_1 - x_0\| &\leq \left\|[y_0, x_0; F]^{-1}F'(x_0)\right\| \left\|F'(x_0)^{-1}F(x_0)\right\| \\
&\leq -\frac{s_0 - t_0}{h(s_0) - h(t_0)} h(t_0) = t_1 - t_0.
\end{aligned}$$

By Lemma 3.4, we have

$$x_1 - y_0 = -F'(x_0)^{-1} \textstyle\int_0^1 \int_0^1 F''(x_0 + st(y_0 - x_0))(y_0 - x_0)(x_1 - x_0)t \, \mathrm{d}s\mathrm{d}t. \tag{3.4}$$

Combining this with the obtained bound of $\|y_0 - x_0\|$ and $\|x_1 - x_0\|$ gives

$$\begin{aligned}
\|x_1 - y_0\| &\leq \int_0^1 \int_0^1 \|F'(x_0)^{-1}F''(x_0 + st(y_0 - x_0))\| \|y_0 - x_0\| \|x_1 - x_0\| t \, \mathrm{d}s\mathrm{d}t \\
&\leq \left(\frac{L}{6}\|y_0 - x_0\| + \frac{\eta}{2}\right) \|y_0 - x_0\| \|x_1 - x_0\| \\
&\leq t_1 s_0 \left(\frac{L}{6}s_0 + \frac{\eta}{2}\right) \frac{\|y_0 - x_0\|}{s_0 - t_0} \frac{\|x_1 - x_0\|}{t_1 - t_0} \\
&= (t_1 - s_0)\frac{\|y_0 - x_0\|}{s_0 - t_0} \frac{\|x_1 - x_0\|}{t_1 - t_0}.
\end{aligned}$$

This implies that statement (i) holds for $n = 1$.

Statement (ii) for the case $n = 1$ is verified by (3.3). Below, we consider the case $n = 1$ for (iii). First we have the following expression of $F(x_1)$ due to Lemma 3.4:

$$F(x_1) = \int_0^1 \int_0^1 F''(x_0 + t(y_0 - x_0) + st(x_1 - y_0))(x_1 - x_0)(x_1 - y_0)t \, \mathrm{d}s\mathrm{d}t,$$

from which we can obtain the estimate

$$\begin{aligned}
&\|F'(x_0)^{-1}F(x_1)\| \\
&\le \left(\frac{L}{6}\|x_0-y_0\|+\frac{L}{6}\|x_1-x_0\|+\frac{\eta}{2}\right)\|x_1-x_0\|\|x_1-y_0\| \\
&\le \left(\frac{L}{6}s_0+\frac{L}{6}t_1+\frac{\eta}{2}\right)(t_1-t_0)(t_1-s_0)\left(\frac{\|x_1-x_0\|}{t_1-t_0}\right)^2\left(\frac{\|y_0-x_0\|}{s_0-t_0}\right) \\
&= h(t_1)\left(\frac{\|x_1-x_0\|}{t_1-t_0}\right)^2\left(\frac{\|y_0-x_0\|}{s_0-t_0}\right).
\end{aligned}$$

Therefore statement (iii) holds for $n=1$.

Assume that statements (i)-(iii) are true for $n=k(\ge 1)$. Blow, we use mathematical induction to prove that they also hold for $n=k+1$. First, by statement (i), we have

$$\|x_k-x_0\| \le \sum_{i=0}^{k-1}\|x_{i+1}-x_i\| \le \sum_{i=0}^{k-1}(t_{i+1}-t_i)=t_k<t^*<r_0. \tag{3.5}$$

Hence $F'(x_k)^{-1}$ exists by Lemma 2.1.

Noting that

$$\|F'(x_0)^{-1}F(x_k)\| \le h(t_k)$$

by the inductive hypothesis of (i) and (iii), it follows from Lemma 2.1 and (2.5) that

$$\|y_k-x_k\| \le \|F'(x_k)^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_k)\| \le -\frac{h(t_k)}{h'(t_k)}=s_k-t_k. \tag{3.6}$$

Hence by (3.1), Lemma 3.4 and the inductive hypothesis of (i), we have

$$\begin{aligned}
&\|F'(x_0)^{-1}([y_k,x_k;F]-F'(x_0))\| \\
&= \Big\|F'(x_0)^{-1}\int_0^1\int_0^1 F''(x_0+s[(x_k-x_0) \\
&\quad +t(y_k-x_k)])[(x_k-x_0)+t(y_k-x_k)]\,\mathrm{d}s\mathrm{d}t\Big\| \\
&\le \int_0^1\int_0^1 [L\,(s\|x_k-x_0\|+st\|y_k-x_k\|)+\eta](\|x_k-x_0\|+t\|y_k-x_k\|)\,\mathrm{d}s\mathrm{d}t \\
&\le \frac{L}{6}(s_k-t_k)^2+\frac{L}{2}t_k^2+\frac{L}{2}t_k(s_k-t_k)+\eta t_k+\frac{\eta}{2}(s_k-t_k)=\frac{h(s_k)-h(t_k)}{s_k-t_k}+1.
\end{aligned}$$

It follows from Lemma 2.3 and the monotonicity of h that $(h(s_k)-h(t_k))/(s_k-t_k)<0$. Thus, we have $\|F'(x_0)^{-1}([y_k,x_k;F]-F'(x_0))\|<1$ and by Banach lemma $[y_k,x_k;F]^{-1}F'(x_0)$ exists and satisfies

$$\|[y_k,x_k;F]^{-1}F'(x_0)\| \le \frac{1}{1-\left(\dfrac{h(s_k)-h(t_k)}{s_k-t_k}+1\right)}=-\frac{s_k-t_k}{h(s_k)-h(t_k)}. \tag{3.7}$$

Hence statement (ii) holds for $n = k+1$.

Combining (3.7) with the inductive hypothesis of (iii), one has that

$$\begin{aligned} &\|x_{k+1} - x_k\| \\ &\leq \|[y_k, x_k; F]^{-1} F'(x_0)\| \|F'(x_0)^{-1} F(x_k)\| \\ &\leq -\left(\frac{h(s_k) - h(t_k)}{s_k - t_k}\right)^{-1} h(t_k) \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \left(\frac{\|y_{k-1} - x_{k-1}\|}{s_{k-1} - t_{k-1}}\right) \\ &= (t_{k+1} - t_k) \left(\frac{\|x_k - x_{k-1}\|}{t_k - t_{k-1}}\right)^2 \left(\frac{\|y_{k-1} - x_{k-1}\|}{s_{k-1} - t_{k-1}}\right). \end{aligned} \tag{3.8}$$

This implies that $\|x_{k+1} - x_k\| \leq t_{k+1} - t_k$.

On the other hand, by (3.1), (3.6), (3.8) and Lemma 3.4, we conclude that

$$\begin{aligned} &\|x_{k+1} - y_k\| \\ &\leq \int_0^1 \int_0^1 \|F'(x_k)^{-1} F''(x_k + st(y_k - x_k))\| \|y_k - x_k\| \|x_{k+1} - x_k\| t \mathrm{d}s\mathrm{d}t \\ &\leq \|F'(x_k)^{-1} F'(x_0)\| \left(\frac{L}{2}\|x_k - x_0\| + \frac{L}{6}\|y_k - x_k\| + \frac{\eta}{2}\right) \|y_k - x_k\| \|x_{k+1} - x_k\| \\ &\leq -\frac{1}{h'(t_k)} \left(\frac{L}{3}t_k + \frac{L}{6}s_k + \frac{\eta}{2}\right) (s_k - t_k)(t_{k+1} - t_k) \left(\frac{\|y_k - x_k\| \|x_{k+1} - x_k\|}{(s_k - t_k)(t_{k+1} - t_k)}\right) \\ &= (t_{k+1} - s_k) \left(\frac{\|y_k - x_k\| \|x_{k+1} - x_k\|}{(s_k - t_k)(t_{k+1} - t_k)}\right), \end{aligned} \tag{3.9}$$

which leads to $\|x_{k+1} - y_k\| \leq t_{k+1} - s_k$. Hence, statement (i) holds for $n = k+1$.

Next, we will show that (iii) also holds for $n = k+1$. In fact, by using Lemma 3.4, together with (3.1), (3.6), (3.8) and (3.9), we obtain

$$\begin{aligned} &\|F'(x_0)^{-1} F(x_{k+1})\| \\ &\leq \int_0^1 \int_0^1 st^2 \, \mathrm{d}s\mathrm{d}t \cdot L \|x_{k+1} - x_k\| \|x_{k+1} - y_k\| \|y_k - x_k\| \\ &\quad + \int_0^1 \int_0^1 [L(t\|x_k - x_0\| + st^2 \|x_{k+1} - x_k\|) + \eta t] \|x_{k+1} - x_k\| \|x_{k+1} - y_k\| \, \mathrm{d}s\mathrm{d}t \\ &= \left(\frac{L}{6}\|y_k - x_k\| + \frac{L}{2}\|x_k - x_0\| + \frac{L}{6}\|x_{k+1} - x_k\| + \frac{\eta}{2}\right) \|x_{k+1} - x_k\| \|x_{k+1} - y_k\| \\ &\leq \left(\frac{L}{6}t_{k+1} + \frac{L}{6}t_k + \frac{L}{6}s_k + \frac{\eta}{2}\right) (t_{k+1} - t_k)(t_{k+1} - s_k) \left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2 \left(\frac{\|y_k - x_k\|}{s_k - t_k}\right) \\ &= h(t_{k+1}) \left(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\right)^2 \left(\frac{\|y_k - x_k\|}{s_k - t_k}\right). \end{aligned}$$

Therefore statement (iii) is confirmed for $n = k+1$. Hence (i)-(iii) hold for all $n \geq 0$.

Furthermore, by statement (i), one has, for any $n \geq 0$, $\|x_n - x_0\| \leq t_n < t^* < r_0$. Thus, by Lemma 2.1, we know that $F'(x_n)^{-1}$ exists for each $n \geq 1$, i.e. $\{x_n\}$ is well defined. The proof is complete. ■

Recall that the sequences $\{s_n\}$ and $\{t_n\}$ are defined by (2.5). We are now ready to prove a semilocal convergence theorem for Newton-Steffensen's method (1.4) under Lipschitz condition.

Theorem 3.1. *Suppose that (2.8) holds. Then the sequence $\{x_n\}$ generated by (1.4) with the initial point x_0 is well defined and converges to a solution $x^* \in \overline{\boldsymbol{B}(x_0, t^*)}$ of equation (1.1) with Q-cubic rate and x^* is unique in $\overline{\boldsymbol{B}(x_0, r_0)}$. Moreover, the following error bounds*

$$\|x^* - x_n\| \leq (t^* - t_n) \left(\frac{\|x^* - x_m\|}{t^* - t_m} \right)^{3^{n-m}} \quad \text{for all} \quad n \geq m \geq 0 \tag{3.10}$$

are valid, where t^* is defined in Lemma 2.2.

Proof. The uniqueness ball can be obtained by Proposition 2.2 (iii) and Theorem 3.2 in [18]. Moreover, it follows from Lemma 3.5 that $\{x_n\}$ is well defined. In addition, from Lemma 2.3 and Lemma 3.5 (i), we can see that $\{x_n\}$ is a Cauchy sequence, so it converges to a limit, say x^*. Below, we show that x^* is a solution of equation (1.1). It follows from Lemma 3.5 (iii) that

$$\|F'(x_0)^{-1}F(x_n)\| \leq h(t_n) \quad \text{for all } n \geq 0.$$

Letting $n \to \infty$ in the above relation gives that the limit x^* is a solution of equation (1.1). Moreover, Lemma 3.5 (i) gives

$$\|x^* - x_n\| \leq t^* - t_n. \tag{3.11}$$

Next, we verify that estimate (3.10) is true. It is clear that

$$x_{n+1} - x^* = \left([y_n, x_n; F]^{-1} F'(x_0)\right) \left(F'(x_0)^{-1} [y_n, x_n; F](x_{n+1} - x^*)\right). \tag{3.12}$$

Then we have

$$\begin{aligned}
\|x^* - y_n\| &= \left\| F'(x_n)^{-1} \left[\int_0^1 F'(x_n)(x^* - x_n)\,\mathrm{d}t + F(x_n) - F(x^*) \right] \right\| \\
&= \left\| F'(x_n)^{-1} \int_0^1 \int_0^1 F''(x_n + st(x^* - x_n))(x^* - x_n)^2 t \,\mathrm{d}s\mathrm{d}t \right\| \\
&\leq \|F'(x_n)^{-1} F'(x_0)\| \\
&\quad \left\| \int_0^1 \int_0^1 F'(x_0)^{-1} F''(x_n + st(x^* - x_n))(x^* - x_n)^2 t \,\mathrm{d}s\mathrm{d}t \right\|.
\end{aligned}$$

This together with (3.1), (3.2) and (3.11) gives the following estimate:

$$
\begin{aligned}
\|x^* - y_n\| &\leq -\frac{1}{h'(t_n)}\left(\frac{L}{6}t^* + \frac{L}{3}t_n + \frac{\eta}{2}\right)(t^* - t_n)^2\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^2 \\
&= (t^* - s_n)\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^2.
\end{aligned} \tag{3.13}
$$

In order to estimate $\|x_{n+1} - x^*\|$, we firstly notice that

$$
\begin{aligned}
& F'(x_0)^{-1}[y_n, x_n; F](x_{n+1} - x^*) \\
&= F'(x_0)^{-1}\left[F(x^*) - F(x_n) - \int_0^1 F'(x_n + t(y_n - x_n))(x^* - x_n)\,\mathrm{d}t\right] \\
&= F'(x_0)^{-1}\int_0^1\int_0^1 F''(x_n + t(y_n - x_n) \\
&\qquad + st(x^* - y_n))(x^* - y_n)(x^* - x_n)t\,\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Thus by (3.1), (3.2), (3.11) and (3.13), one has that

$$
\begin{aligned}
& \|F'(x_0)^{-1}[y_n, x_n; F](x_{n+1} - x^*)\| \\
&\leq \left(\frac{L}{6}t^* + \frac{L}{6}s_n + \frac{L}{6}t_n + \frac{\eta}{2}\right)(t^* - t_n)(t^* - s_n)\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3 \\
&= \frac{h(s_n) - h(t_n)}{s_n - t_n}(t_{n+1} - t^*)\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3.
\end{aligned} \tag{3.14}
$$

Combining (3.7) with (3.14), one gets from (3.12) that

$$
\|x_{n+1} - x^*\| \leq (t^* - t_{n+1})\left(\frac{\|x^* - x_n\|}{t^* - t_n}\right)^3.
$$

Therefore the error estimate (3.10) follows. Also, from the previous inequality, we know that the convergence rate of $\{x_n\}$ to x^* is Q-cubic. The proof is complete. ∎

4. Convergence Ball

Now we begin to study the local convergence properties for Newton-Steffensen's method (1.4). Recall that r_0 is defined by (2.7). Throughout this section, suppose $x^* \in D$ such that $F(x^*) = 0$, $\mathbf{B}(x^*, r_0) \subset D$ and the inverse $F'(x^*)^{-1}$ exists. Moreover, we also assume that $\|F'(x^*)^{-1}F''(x^*)\| \leq \eta$ and $F'(x^*)^{-1}F''(x)$ satisfies Lipschitz condition on $\mathbf{B}(x^*, r_0)$. For each $x \in \mathbf{B}(x^*, r_0)$, it follows from Lemma 2.1 that

$$
\|F'(x^*)^{-1}F''(x)\| \leq L\|x - x^*\| + \eta, \tag{4.1}
$$

$$
\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - \eta\|x - x^*\| - \frac{L}{2}\|x - x^*\|^2}. \tag{4.2}
$$

Let

$$r_1 := (-9\eta + \sqrt{81\eta^2 + 120L})/(10L). \tag{4.3}$$

Define the function G as follows:

$$G(t) = \frac{\frac{\eta}{2}t + \frac{L}{3}t^2}{1 - \eta t - \frac{L}{2}t^2}, \quad t \in (0, r_1). \tag{4.4}$$

It is clear that $r_1 \in (0, r_0)$ and $G(r_1) = 1$. Moreover, G increases in $(0, r_1)$.

Theorem 4.2. *For any $x_0 \in \boldsymbol{B}(x^*, r_1)$, the sequence $\{x_n\}$ generated by (1.4) converges to x^* and satisfies that*

$$\|x_n - x^*\| \le q^{3^n - 1}\|x_0 - x^*\|, \; n = 0, 1, \ldots, \tag{4.5}$$

where

$$q = G(t_0) < 1, \quad t_0 = \|x_0 - x^*\|. \tag{4.6}$$

Proof. For $n = 0, 1, \ldots$, we write $t_n = \|x_n - x^*\|$. It is sufficient to show that

$$t_{n+1} \le t_n \;\text{ and }\; \|x_{n+1} - x^*\| \le (G(t_n)/t_n)^2\|x_n - x^*\|^3, \quad n = 0, 1, \ldots. \tag{4.7}$$

In fact, by noticing the monotonicity of G/t, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (G(t_n)/t_n)^2 \|x_n - x^*\|^3 \le (G(t_0)/t_0)^2 \|x_n - x^*\|^3 \\ &= \left(\frac{q}{t_0}\right)^2 \|x_n - x^*\|^3, \quad n = 0, 1, \ldots.\end{aligned}$$

From this we can easily establish (4.5) by mathematical induction.

Now we prove (4.7) by mathematical induction. First we get the following expression of $x_{n+1} - x^*$:

$$\begin{aligned}&x_{n+1} - x^* \\ &= [y_n, x_n; F]^{-1}\left[\int_0^1 F'(x_n + t(y_n - x_n))(x_n - x^*)\mathrm{d}t - (F(x_n) - F(x^*))\right] \\ &= [y_n, x_n; F]^{-1}\int_0^1 \left[F'(x_n + t(y_n - x_n)) - F'(x^* + t(x_n - x^*))\right]\mathrm{d}t(x_n - x^*).\end{aligned} \tag{4.8}$$

Similarly, we also have

$$\begin{aligned}y_n - x^* &= x_n - F'(x_n)^{-1}F'(x_n) - x^* \\ &= F'(x_n)^{-1}\left[F(x^*) - F(x_n) + F'(x_n)(x_n - x^*)\right] \\ &= F'(x_n)^{-1}\int_0^1 \left[F'(x_n + t(x^* - x_n)) - F'(x_n)\right](x^* - x_n)\mathrm{d}t.\end{aligned} \tag{4.9}$$

On the other hand, we notice that

$$
\begin{aligned}
&F'(x^*)^{-1}([y_n, x_n; F] - F'(x^*)) \\
&= F'(x^*)^{-1} \int_0^1 \int_0^1 F''(x^* + s(1-t)(x_n - x^*) + st(y_n - x^*)) \\
&[(1-t)(x_n - x^*) + t(y_n - x^*)] \, \mathrm{d}s\mathrm{d}t.
\end{aligned} \tag{4.10}
$$

For the case $n = 0$, by (4.1) and (4.2), we get that

$$
\begin{aligned}
\|y_0 - x^*\| &\le \|F'(x_0)^{-1} F'(x^*)\| \left\| \int_0^1 \int_0^1 F'(x^*)^{-1} F''(x_0 + st(x^* - x_0)) t \, \mathrm{d}s\mathrm{d}t \right\| \|x^* - x_0\|^2 \\
&\le \frac{1}{1 - \eta\|x^* - x_0\| - \frac{L}{2}\|x^* - x_0\|^2} \\
&\quad \int_0^1 \int_0^1 [L(\|x_0 + st(x^* - x_0) - x^*\|) + \eta] \, \mathrm{d}s\mathrm{d}t \|x^* - x_0\|^2 \\
&= \frac{1}{1 - \eta t_0 - \frac{L}{2}t_0^2} \int_0^1 \int_0^1 (L(1 - st)t_0 + \eta) \, \mathrm{d}s\mathrm{d}t \|x^* - x_0\|^2 \\
&= \frac{\frac{\eta}{2} + \frac{L}{3}t_0}{1 - \eta t_0 - \frac{L}{2}t_0^2} \|x^* - x_0\|^2 \\
&= (G(t_0)/t_0) \|x^* - x_0\|^2.
\end{aligned}
$$

It follows from (4.6) that

$$
\|y_0 - x^*\| \le \|x^* - x_0\|. \tag{4.11}
$$

Combining Lemma 2.2 with (4.1) and (4.11), we obtain that

$$
\begin{aligned}
&\|F'(x^*)^{-1}([y_0, x_0; F] - F'(x^*))\| \\
&\le \int_0^1 \int_0^1 [L\,(s(1-t)\|x_0 - x^*\| + st\|y_0 - x^*\|) + \eta] \\
&\quad [(1-t)\|x_0 - x^*\| + t\|y_0 - x^*\|] \, \mathrm{d}s\mathrm{d}t \\
&\le \int_0^1 \int_0^1 [L\,(s(1-t)t_0 + stt_0) + \eta]\,[(1-t)t_0 + tt_0] \, \mathrm{d}s\mathrm{d}t \\
&= \eta t_0 + \frac{L}{2}t_0^2 < 1.
\end{aligned}
$$

It follows from Banach lemma that

$$
\|[y_0, x_0; F]^{-1} F'(x^*)\| \le \frac{1}{1 - \eta t_0 - \frac{L}{2}t_0^2}.
$$

This together with (4.8) and (4.11) yields that

$$
\begin{aligned}
&\|x_1 - x^*\| \\
\le\ & \|[y_0, x_0; F]^{-1} F'(x^*)\| \\
& \left\| F'(x^*)^{-1} \int_0^1 \int_0^1 F''\left(x^* + (1-t)(x_0 - x^*) + st(y_0 - x^*)\right) t\, \mathrm{d}s\mathrm{d}t \right\| \\
& \|y_0 - x^*\| \|x_0 - x^*\| \\
\le\ & \frac{1}{1 - \eta t_0 - \frac{L}{2} t_0^2} \int_0^1 \int_0^1 \left[L\left((1-t)t_0 + stt_0\right) + \eta\right] t\, \mathrm{d}s\mathrm{d}t G(t_0) \|x^* - x_0\|^3 \\
=\ & \frac{1}{1 - \eta t_0 - \frac{L}{2} t_0^2} \frac{1}{t_0} \left(\frac{\eta}{2} t_0 + \frac{L}{3} t_0^2\right) G(t_0) \|x^* - x_0\|^3 \\
=\ & \left(G(t_0)/t_0\right)^2 \|x^* - x_0\|^3 = q^2 t_0 \le t_0.
\end{aligned}
\tag{4.12}
$$

Hence the inequalities in (4.7) hold for $n = 0$.

Now assume that the inequalities in (4.7) hold for up to some $n \ge 1$. Thus $t_{n+1} \le t_n < r_1 < r_0$. By (4.1) and (4.2), we have

$$
\begin{aligned}
&\|y_{n+1} - x^*\| \\
\le\ & \|F'(x_{n+1})^{-1} F'(x^*)\| \\
& \left\| \int_0^1 \int_0^1 F'(x^*)^{-1} F''(x_{n+1} + st(x^* - x_{n+1})) t\, \mathrm{d}s\mathrm{d}t \right\| \|x^* - x_{n+1}\|^2 \\
\le\ & \frac{1}{1 - \eta \|x^* - x_{n+1}\| - \frac{L}{2} \|x^* - x_{n+1}\|^2} \\
& \int_0^1 \int_0^1 \left[L(\|x_{n+1} + st(x^* - x_{n+1}) - x^*\|) + \eta\right] \mathrm{d}s\mathrm{d}t \\
& \|x^* - x_{n+1}\|^2 \\
=\ & \frac{1}{1 - \eta t_{n+1} - \frac{L}{2} t_{n+1}^2} \int_0^1 \int_0^1 \left(L(1 - st)t_{n+1} + \eta\right) \mathrm{d}s\mathrm{d}t \|x^* - x_{n+1}\|^2 \\
=\ & \frac{\frac{\eta}{2} + \frac{L}{3} t_{n+1}}{1 - \eta t_{n+1} - \frac{L}{2} t_{n+1}^2} \|x^* - x_{n+1}\|^2 \\
=\ & (G(t_{n+1})/t_{n+1}) \|x^* - x_{n+1}\|^2.
\end{aligned}
\tag{4.13}
$$

It follows from the monotonicity of G and (4.6) that

$$
\|y_{n+1} - x^*\| \le G(t_{n+1}) t_{n+1} \le G(t_0) t_{n+1} \le t_{n+1}.
$$

Thus (4.10) can be further reduced to

$$\begin{aligned} &\|F'(x^*)^{-1}([y_{n+1}, x_{n+1}; F] - F'(x^*))\| \\ &\leq \int_0^1 \int_0^1 [L\left(s(1-t)t_{n+1} + stt_{n+1}\right) + \eta]\left[(1-t)t_{n+1} + tt_{n+1}\right] \mathrm{d}s\mathrm{d}t \\ &= \eta t_{n+1} + \frac{L}{2}t_{n+1}^2 < 1. \end{aligned} \tag{4.14}$$

Using Banach lemma again, one has that

$$\|[y_{n+1}, x_{n+1}; F]^{-1}F'(x^*)\| \leq \frac{1}{1 - \eta t_{n+1} - \frac{L}{2}t_{n+1}^2}.$$

This together with (4.1), (4.2), (4.8) and (4.13) yields that

$$\begin{aligned} &\|x_{n+2} - x^*\| \\ &\leq \|[y_{n+1}, x_{n+1}; F]^{-1}F'(x^*)\| \left\| F'(x^*)^{-1} \int_0^1 \int_0^1 F''(x^* + (1-t)(x_{n+1} - x^*) \right. \\ &\quad + st(y_{n+1} - x^*)t \,\mathrm{d}s\mathrm{d}t\| \,\|y_{n+1} - x^*\| \|x_{n+1} - x^*\| \\ &\leq \frac{1}{1 - \eta t_{n+1} - \frac{L}{2}t_{n+1}^2} \\ &\quad \int_0^1 \int_0^1 [L\left((1-t)t_{n+1} + stt_{n+1}\right) + \eta]\, t \,\mathrm{d}s\mathrm{d}t G(t_{n+1})\|x^* - x_{n+1}\|^3 \\ &= \frac{1}{1 - \eta t_{n+1} - \frac{L}{2}t_{n+1}^2} \frac{1}{t_{n+1}} \left(\frac{\eta}{2}t_{n+1} + \frac{L}{3}t_{n+1}^2\right) G(t_{n+1})\|x^* - x_{n+1}\|^3 \\ &= \left(G(t_{n+1})/t_{n+1}\right)^2 \|x^* - x_{n+1}\|^3. \end{aligned}$$

This together with the monotonicity of G yields that

$$\|x_{n+2} - x^*\| \leq G(t_0)^2 t_{n+1} \leq t_{n+1}.$$

Thus the inequalities in (4.7) hold for $n+1$ and hence they hold for each n. The proof is complete. ■

5. Numerical Example

In this section, we give numerical examples to illustrate the application of convergence results of Newton-Steffensen's method in high dimensions. To this end, we apply our semilocal convergence result to an example which also appears in [1, 12]. Consider the following nonlinear boundary value problem of second order:

$$\begin{cases} x'' + x^3 + \mu x^2 = 0, \ \mu \in \mathbb{R}, \\ x(0) = x(1) = 0. \end{cases} \tag{5.1}$$

To solve this problem by finite differences, we divide interval $[0,1]$ into n subintervals and let $h = 1/n$. We denote the points of subdivision by $t_i = ih$ and $x(t_i) = x_i$, $i = 0, 1, \ldots, n$. Notice that x_0 and x_n are given by the boundary conditions, i.e., $x_0 = 0 = x_n$. We first approximate the second derivative $x''(t)$ in the differential equation by

$$\begin{aligned} x''(t) &\approx \frac{x(t+h) - 2x(t) + x(t-h)}{h^2}, \\ x''(t_i) &\approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}, \ i = 1, 2, \ldots n-1. \end{aligned}$$

By substituting this expression into the differential equation, we have the following system of nonlinear equations:

$$\text{(5.2)} \qquad \begin{cases} 2x_1 - h^2 x_1^3 - h^2 \mu x_1^2 - x_2 = 0, \\ -x_{i-1} + 2x_i - h^2 x_i^3 - h^2 \mu x_i^2 - x_{i+1} = 0, \ i = 2, 3, \ldots n-2, \\ -x_{n-2} + 2x_{n-1} - h^2 x_{n-1}^3 - h^2 \mu x_{n-1}^2 = 0. \end{cases}$$

Therefore, an operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ can be defined by $F(x) = Hx - h^2 g(x) - h^2 \mu f(x)$, where

$$x = (x_1, x_2, \ldots, x_{n-1})^T, \ g(x) = (x_1^3, x_2^3, \ldots, x_{n-1}^3)^T, \ f(x) = (x_1^2, x_2^2, \ldots, x_{n-1}^2)^T$$

and

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Thus,

$$F'(x) = H - 3h^2 \begin{pmatrix} x_1^2 & 0 & \cdots & 0 \\ 0 & x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1}^2 \end{pmatrix} - 2h^2 \mu \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{n-1} \end{pmatrix}.$$

Then, we apply Newton-Steffensen's method (1.4) to find a solution x^* of the equation

$$F(x) = 0.$$

Let $x \in \mathbb{R}^{n-1}$, and choose the norm $\|x\| = \max_{1 \le i \le n-1} |x_i|$. Then the induced norm for an $(n-1) \times (n-1)$ matrix A is

$$\|A\| = \max_{1\le i\le n-1} \sum_{j=1}^{n-1} |a_{ij}|.$$

It is known (see [11]) that F has a divided difference at the points $v, w \in \mathbb{R}^{n-1}$, which is defined by the matrix whose entries are

$$[v,w;F]_{ij} = \frac{1}{v_j - w_j}(F_i(v_1,\ldots,v_j,w_{j+1},\ldots,w_{n-1}) - F_i(v_1,\ldots,v_{j-1},w_j,\ldots,w_{n-1})).$$

Therefore

$$[v,w;F] = H - h^2 \begin{pmatrix} \frac{v_1^3-w_1^3+\mu(v_1^2-w_1^2)}{v_1-w_1} & 0 & \cdots & 0 \\ 0 & \frac{v_2^3-w_2^3+\mu(v_2^2-w_2^2)}{v_2-w_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{v_{n-1}^3-w_{n-1}^3+\mu(v_{n-1}^2-w_{n-1}^2)}{v_{n-1}-w_{n-1}} \end{pmatrix}.$$

In this case, we have $[v,w;F] = \int_0^1 F'(v+t(w-v))\,\mathrm{d}t$. Note that

$$F''(x)u_1u_2 = -2h^2(u_1^1u_2^1(3x_1+\mu),\ldots,u_1^{n-1}u_2^{n-1}(3x_{n-1}+\mu))^T,$$
$$(F''(x)-F''(y))u_1u_2 = -6h^2(u_1^1u_2^1(x_1-y_1),\ldots,u_1^{n-1}u_2^{n-1}(x_{n-1}-y_{n-1}))^T,$$

where $u_i = (u_i^1,\ldots,u_i^{n-1}) \in \mathbb{R}^{n-1}$ for each $i = 1,2$. Hence

$$\|F''(x)\| = 2h^2 \max_{1\le i\le n-1}\{|3x_i+\mu|\} \text{ and } \|F''(x)-F''(y)\| = 6h^2\|x-y\|. \tag{5.3}$$

Thus, one has that

$$\|F'(x_0)^{-1}(F''(x)-F''(y))\| \le \|F'(x_0)^{-1}\|\|F''(x)-F''(y)\| \le 6h^2\|F'(x_0)^{-1}\|\|x-y\|.$$

This means that $F'(x_0)^{-1}F''(x)$ satisfies Lipschitz condition (2.1) with the Lipschitz constant

$$L = 6h^2\|F'(x_0)^{-1}\|. \tag{5.4}$$

Moreover, we set

$$\beta := \|F'(x_0)^{-1}\|\|F(x_0)\| \text{ and } \eta := \|F'(x_0)^{-1}\|\|F''(x_0)\|. \tag{5.5}$$

We study the problem in two conditions: $\mu = 0$ and $\mu \ne 0$. All our tests were done in Matlab. The stopping tolerance for the iterations is $\|F(x_k)\| \le eps$, where $eps \approx 2.2204e-16$ is the machine precision in Matlab. For $n = 10$, the convergence performances of Newton-Steffensen's method with $\mu = 0$ and $\mu \neq 0$ are illustrated in Table 1 (shown at the end of subsection 5.2) from which we can see that Newton-Steffensen's method converges to the root very effectively.

5.1. When $\mu = 0$

Problem (5.1) becomes

$$\begin{cases} x'' + x^3 = 0, \\ x(0) = x(1) = 0. \end{cases} \tag{5.6}$$

Now we apply Newton-Steffensen's method to approximate the solution of $F(x) = 0$. Since a solution of $F(x) = 0$ would vanish at the end points of $[0, 1]$ and be positive in the interior of $[0, 1]$, a reasonable choice of initial approximation seems to be $k \sin \omega\pi t$, where k and ω are some constants. Below we consider two cases: convergence criterion (2.8) holds or not.

Case I. Let $n = 10,\ k = 9,\ \omega = 1$. This approximation gives us the following initial point:

$$x_0 = \begin{pmatrix} 2.781152949374527 \\ 5.290067270632259 \\ 7.281152949374527 \\ 8.559508646656381 \\ 9.000000000000000 \\ 8.559508646656383 \\ 7.281152949374527 \\ 5.290067270632259 \\ 2.781152949374528 \end{pmatrix}.$$

With the notations in (5.4) and (5.5), we get

$$L = 0.465586357135854,\ \beta = 3.208079306744555,\ \eta = 4.190277214222688$$

and

$$\beta \geq \triangle(L; \eta) = 0.118289602834299.$$

Hence convergence criterion (2.8) doesn't hold in this case. We obtain x^* after eight iterations:

$$x^* = \begin{pmatrix} 6.554680420487635 \\ 10.293218733270551 \\ 3.126055577275099 \\ -4.346592715592183 \\ -10.998044973982024 \\ -4.346592715592183 \\ 3.126055577275099 \\ 10.293218733270551 \\ 6.554680420487635 \end{pmatrix}.$$

If x^* is interpolated by polynomials of order eight, then the approximation $\bar{x}^*$ to the solution of (5.6) can be shown in Figure 1.

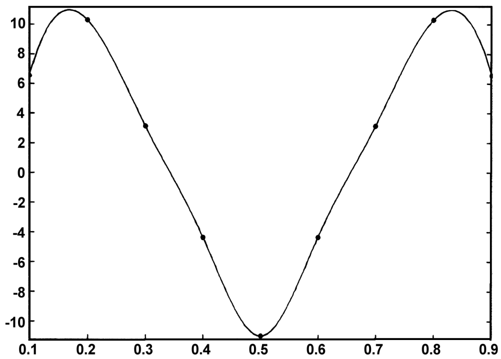

Fig. 1. x^* and the related approximation $\bar{x}^*$ for Case I.

Case II. Let $n = 10,\ k = 10,\ \omega = 8.$ This approximation gives us the following initial point:

$$x_0 = \begin{pmatrix} 5.877852522924733 \\ -9.510565162951536 \\ 9.510565162951535 \\ -5.877852522924728 \\ -0.000000000000005 \\ 5.877852522924735 \\ -9.510565162951538 \\ 9.510565162951533 \\ -5.877852522924725 \end{pmatrix}.$$

We obtain x^* after five iterations:

$$x^* = \begin{pmatrix} 17.656977750521808 \\ -19.735003321788422 \\ 19.735003321788422 \\ -17.656977750521808 \\ 0.000000000000000 \\ 17.656977750521808 \\ -19.735003321788422 \\ 19.735003321788422 \\ -17.656977750521808 \end{pmatrix}.$$

Moreover, we have

$$L = 0.200444924940861,\ \beta = 28.035067577717861,\ \eta = 1.906344520232991$$

and

$$\beta > \triangle(L;\eta) = 0.257649074900776.$$

Hence convergence criterion (2.8) doesn't hold in this case. If x^* is interpolated by polynomials of order eight, then the approximation $\bar{x}^*$ to the solution of (5.6) can be shown in Figure 2.

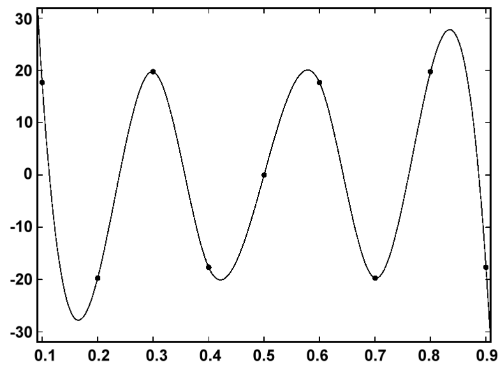

Fig. 2. x^* and the related approximation $\bar{x}^*$ for Case II

5.2. When $\mu \neq 0$

We consider, for example, $\mu = 1$. Then, as in the previous example, we can choose $k \sin \omega\pi t$ as an initial approximation, where k and ω are some constants. We consider the following cases in the same way as in the last subsection:

Case III. Let $n = 10,\ k = 3,\ \omega = 1$. Then the initial point is:

$$x_0 = \begin{pmatrix} 0.927050983124842 \\ 1.763355756877419 \\ 2.427050983124842 \\ 2.853169548885461 \\ 3.000000000000000 \\ 2.853169548885461 \\ 2.427050983124842 \\ 1.763355756877420 \\ 0.927050983124843 \end{pmatrix}.$$

With the notations in (5.4) and (5.5), we get

$$L = 0.529992660987916,\ \beta = 0.165443680845126,\ \eta = 1.766642203293053$$

and

$$\beta \leq \triangle(L;\eta) = 0.268752756830446.$$

Hence Theorem (3.1) is applicable. We obtain x^* after three iterations:

$$x^* = \begin{pmatrix} 0.832478544048846 \\ 1.652257635647339 \\ 2.399631279496108 \\ 2.951246325683993 \\ 3.158713551155907 \\ 2.951246325683993 \\ 2.399631279496108 \\ 1.652257635647339 \\ 0.832478544048846 \end{pmatrix}.$$

If x^* is interpolated by polynomials of order eight, then the approximation $\bar{x}^*$ to the solution of (5.1) can be shown in Figure 3.

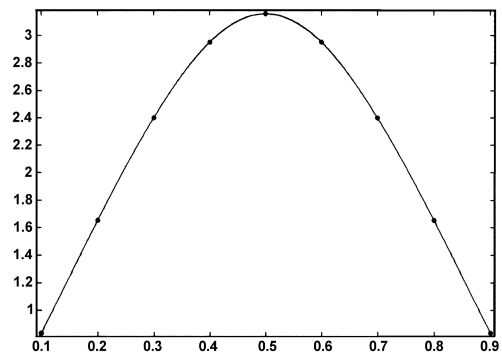

Fig. 3. x^* and the related approximation $\bar{x}^*$ for Case III.

Case IV. Let $n = 10,\ k = 0.5,\ \omega = 15$. Then the initial point is:

$$x_0 = \begin{pmatrix} -0.500000000000000 \\ 0.000000000000000 \\ 0.500000000000000 \\ -0.000000000000000 \\ -0.500000000000000 \\ 0.000000000000002 \\ 0.500000000000000 \\ -0.000000000000001 \\ -0.500000000000000 \end{pmatrix}.$$

We obtain $x^* = 0$ after three iterations. Moreover, we have

$$L = 0.779137639293556,\ \beta = 0.515755101109577,\ \eta = 0.649281366077963$$

and

$$\beta \leq \triangle(L; \eta) = 0.546034010164211.$$

That is, (2.8) holds. Hence Theorem 3.1 is applicable.

Table 1: Values of $\|x_k - x^*\|$

k	Case I	Case II	Case III	Case IV
0	$1.9998e+1$	$1.1779e+1$	$1.5871e-1$	$5.0000e-1$
1	$1.2554e+1$	$7.3840e+0$	$2.3358e-4$	$2.8561e-4$
2	$9.3105e+0$	$8.0944e-1$	$2.8928e-12$	$1.9628e-13$
3	$7.6714e+0$	$3.0813e-3$	0.0000	0.0000
4	$7.2214e+0$	$1.8769e-1$	0.0000	0.0000
5	$4.2121e-1$	0.0000	0.0000	0.0000
6	$1.5713e-3$	0.0000	0.0000	0.0000
7	$8.0344e-11$	0.0000	0.0000	0.0000
8	0.0000	0.0000	0.0000	0.0000

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Shaohua Yu and Xiubin Xu*
Department of Mathematics
Zhejiang Normal University
Jinhua 321004
P. R. China
E-mail: yushaohua00@126.com
xxu@zjnu.cn

Jianqiu Li
Hangzhou Institute of Commerce
Zhejiang Gongshang University
Hangzhou 310012
P. R. China
E-mail: lijianqiu566@163.com

Yonghui Ling
Department of Mathematics
Zhejiang Normal University
Jinhua 321004
P. R. China
E-mail: lingyinghui@163.com