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ON DOMINATING SETS FOR NEVANLINNA CLASS (I)

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Abstract. In this article we prove that a subset E of the open unit disc U is a dominating set for the subclass N^+ of the Nevanlinna class N if and only if E is nontangentially dense. When E is a compact subset of U, we also give a complete characterization of E to be a dominating set for the subclass N^-. Here N^- denotes the proper subclass of N that consists of all of the reciprocal of the singular inner functions.

1. Introduction

Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$ be the open unit disc on the complex plane, and let $\mathbb{T}$ be the boundary of U. Denote by $\mathcal{O}(U)$ the space of holomorphic functions on U. Define the Hardy spaces H^p, $0 < p < \infty$, and Nevanlinna class N as follows:

$$H^p = \{f \in \mathcal{O}(U) \mid \| f \|_p^p = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty\},$$

and

$$N = \{f \in \mathcal{O}(U) \mid \| f \|_0 = \sup_{0 \le r < 1} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \right\} < \infty\}.$$

As usual, H^∞ denotes the space of bounded holomorphic functions with supremum norm. For instance, see Rudin[8]. Then, we make the following definitions.

Definition 1.1. Let X be a subspace of $\mathcal{O}(U)$ such that a number $\| f \|_X$ is associated with each function f in X. A subset E of U is called a dominating set for X with respect to $\| \cdot \|_X$ if every two functions $f, g \in X$ with $|f(z)| \le |g(z)|$ for $z \in E$ implies $\| f \|_X \le \| g \|_X$.

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Definition 1.2. A subset E of U is said to be nontangentially dense (n.t.d.) if almost every point $e^{i\theta}$ of $\mathbb{T}$ is a nontangential limit of E.

The following result is proved by Brown, Shields, and Zeller[2] for $p=\infty$, and by Danikas and Hayman[5] for general p. See also Hayman[7].

Theorem 1.3. *A subset E of U is a dominating set for H^p, $0<p\le\infty$, if and only if E is nontangentially dense.*

In this article we shall investigate the dominating phenomenon of Nevanlinna class. Let us recall briefly some facts about Nevanlinna class. For details the reader is referred to Duren[6] and Rudin[8].

Definition 1.4. Let $S(z)$ be a holomorphic function on U. $S(z)$ is called a singular inner function if

$$S(z)=\lambda\exp\left\{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\},\quad z\in U, \tag{1.1}$$

where $|\lambda|=1$ and μ is a finite positive Borel measure on $\mathbb{T}$ which is singular with respect to Lebesgue measure.

Definition 1.5. A function $F(z)$ is called an outer function for the Nevanlinna class N if $F(z)$ is of the form

$$F(z)=\lambda\exp\left\{\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log\phi(t)dt\right\},\quad z\in U, \tag{1.2}$$

where $|\lambda|=1$, $\phi(t)\ge 0$ and $\log\phi(t)\in L^1(\mathbb{T})$.

Then we have the factorization theorem for the Nevanlinna class N as follows. For instance, see Duren [6].

Theorem 1.6. *A function $f\ne 0$ belongs to the Nevanlinna class N if and only if f can be expressed in the form*

$$f(z)=B(z)[S_1(z)/S_2(z)]F(z), \tag{1.3}$$

where $B(z)$ is a Blaschke product, $S_1(z)$ and $S_2(z)$ are singular inner functions, and $F(z)$ is an outer function for the Nevanlinna class.

Definition 1.7. The subclass N^+ of the Nevanlinna class N is defined by

$$N^+=\{f\in N \mid\ f(z)=B(z)S(z)F(z)\}, \tag{1.4}$$

where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function, and $F(z)$ is an outer function for the Nevanlinna class.

That is, $f\in N^+$ if $f\in N$ and $S_2(z)\equiv 1$ in Theorem 1.6. In a sense, N^+ is the natural limit of H^p as $p\to 0$. It is clear that $H^p\subsetneq N^+\subsetneq N$. Also, it is well-known that every function $f \in N$ has nontangential limit almost everywhere on $\mathbb{T}$. Then, we have

Theorem 1.8. *A function $f \in N$ belongs to the class N^+ if and only if*

$$\lim_{r\to 1^-}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta = \int_0^{2\pi} \log^+|f(e^{i\theta})|d\theta. \tag{1.5}$$

For related results concerning the dominating phenomena for various function spaces, the reader is referred to Chen[3] and Chen and Lu[4].

2. The Class N^+

In this section, following the line developed in Danikas and Hayman[5], we shall first characterize the dominating sets for the class N^+.

Theorem 2.1. *A subset E of U is a dominating set for the class N^+ if and only if E is nontangentially dense.*

The following result due to Bonsall[1] will be needed for proving Theorem 2.1.

Theorem 2.2. *Let E be a subset of U. Then (i) and (ii) are equivalent:*

(i) *E is nontangentially dense.*

(ii) *$\sup_{z\in E} h(z) = \sup_{z\in U} h(z)$, for every real-valued bounded harmonic function $h(z)$ on U.*

Proof of Theorem 2.1. First, suppose E is nontangentially dense. Let f and g be two functions in N^+ satisfying $|f(z)| \le |g(z)|$ for all $z \in E$. Since both f and g have nontangential limits almost everywhere on $\mathbb{T}$, it follows from the hypothesis of E that $|f(e^{i\theta})| \le |g(e^{i\theta})|$ for almost everywhere $e^{i\theta} \in \mathbb{T}$. Therefore, by Theorem 1.8, we obtain

$$\| f \|_0 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(e^{i\theta})|d\theta\right\} \le \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+|g(e^{i\theta})|d\theta\right\} = \| g \|_0 .$$

This proves that E is a dominating set for the class N^+.

On the other hand, suppose now that E is not nontangentially dense. Thus, according to Theorem 2.2, there exists a real-valued bounded harmonic function $u(z)$ on U such that

$$\sup_{z\in E} u(z) = \alpha < \beta = \sup_{z\in U} u(z).$$

After a linear transformation, if necessary, we may assume that $\alpha = 0$, $\beta = 1$. Hence,

$$u(z) \leq 0 \text{ for } z \in E, \quad \sup_{z\in U} u(z) = 1, \quad \inf_{z\in U} u(z) = -\gamma,$$

for some $\gamma > 0$.

Now, let $v(z)$ be a harmonic conjugate of $u(z)$ on U. Thus, $h(z) = u(z) + iv(z)$ is a holomorphic function on U. It follows that $H(z) = e^{h(z)}$ is a bounded holomorphic function on U. Therefore, if $g \in N^+$, not identically zero, then $g_m(z) = g(z)e^{mh(z)}$ is also in the class N^+ for every real number $m > 0$. Since $u(z) \leq 0$ for $z \in E$, we have

$$|g_m(z)| = |g(z)|e^{mu(z)} \leq |g(z)|, \quad \text{for } z \in E,$$

and, by Theorem 1.8,

$$\| g_m \|_0 = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+|g_m(e^{i\theta})|d\theta\right\} = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log^+(|g(e^{i\theta})|e^{mu(e^{i\theta})})d\theta\right\}.$$

Clearly, $|g(e^{i\theta})| > 0$ almost everywhere on $\mathbb{T}$ and $u(e^{i\theta}) > 0$ on a set of positive measure on $\mathbb{T}$, since $\sup_{z\in U} u(z) = 1$. Hence, there exist a set $\Sigma \subset \mathbb{T}$ of positive measure δ and positive constants η and ϵ such that

$$u(e^{i\theta}) \geq \eta > 0, \quad |g(e^{i\theta})| > \epsilon > 0, \quad \text{for } e^{i\theta} \in \Sigma.$$

It follows that

$$\begin{aligned}\lim_{m\to\infty} \| g_m \|_0 &\geq \lim_{m\to\infty} \exp\left\{\frac{1}{2\pi}\int_\Sigma \log^+(|g(e^{i\theta})|e^{mu(e^{i\theta})})d\theta\right\} \\ &\geq \lim_{m\to\infty} \exp\left\{\frac{1}{2\pi}\int_\Sigma \log^+(\epsilon e^{m\eta})d\theta\right\} = \infty.\end{aligned}$$

This shows that $\| g_m \|_0 > \| g \|_0$ for sufficiently large m. It contradicts the fact that E is a dominating set for the class N^+. The proof of the theorem is thus completed.

The next result shows that a nontangentially dense subset, in general, is not sufficient to dominate the Nevanlinna class N.

Theorem 2.3. *There exist a nontangentially dense subset E of U and two functions f, g in the Nevanlinna class N such that $|f(z)| \leq |g(z)|$ for $z \in E$, but $\| f \|_0 > \| g \|_0$.*

Proof. Let $g(z) \equiv 2$ and $f(z) = \exp\{\frac{1+z}{1-z}\}$. Observe that $f(z)$ is the reciprocal of the singular inner function $s(z)$ generated by the unit point measure at 1. Hence, $f \in N \setminus N^+$. A direct calculation shows $\| g \|_0 = 2$ and $\| f \|_0 = e$. Thus, $\| f \|_0 > \| g \|_0$.

Observe also that $|f(e^{i\theta})| = 1$ for $\theta \neq 2k\pi$, $k \in \mathbb{Z}$. Now, a direct calculation shows that the set

$$C = \{z \in U \mid |f(z)| = |g(z)|\}$$

forms a circle centered at $(\frac{\ln 2}{1+\ln 2}, 0)$ with radius $\frac{1}{1+\ln 2}$. Clearly, it shows that C is a smaller circle in U tangent to $\mathbb{T}$ at 1. Thus, the crescent region E in U bounded by C and $\mathbb{T}$ is nontangentially dense such that $|f(z)| < |g(z)|$ for $z \in E$. This proves the theorem.

One should observe that the crucial fact in the proof of Theorem 2.3 is that the singular inner function $s(z)$ appears in the denominator of the dominated side. If the singular inner function $s(z)$ appears in the denominator of the dominating side, we can still prove the following theorem, a slight generalization of Theorem 2.1.

Theorem 2.4. *Let E be a subset of U. The following two statements are equivalent:*

(1) *E is a nontangentially dense subset of U.*

(2) *If $f \in N^+$ and $g \in N$ satisfy $|f(z)| \leq |g(z)|$ for $z \in E$, then $\| f \|_0 \leq \| g \|_0$.*

Proof. Following from Theorem 2.1, we have $(2) \Rightarrow (1)$. To show $(1) \Rightarrow (2)$, by Theorem 2.1 again, we may assume that $g \in N \setminus N^+$. Write $g = h/s$ where $h \in N^+$ and s is a singular inner function. It follows that $|f(e^{i\theta})| \leq |g(e^{i\theta})| = |h(e^{i\theta})|$ almost everywhere on $\mathbb{T}$. Hence, by Theorem 1.8, we obtain $\| f \|_0 \leq \| h \|_0 \leq \| g \|_0$. This proves the theorem.

3. The Class N^-

In this section we shall discuss the dominating phenomenon for the class N^- which is defined to be the subclass of N that consists of the reciprocal of the singular inner functions. N^- is also a proper and important subclass of the Nevanlinna class N. First, we have the following lemma.

Lemma 3.1. *If S is a singular inner function generated by a finite positive singular Borel measure μ, then*

(1) *$S \circ \varphi$ is also a singular inner function for every $\varphi \in Aut(U)$.*

(2)
$$\frac{1}{2\pi}\int_0^{2\pi} log^+ \left|\frac{1}{S(re^{i\theta})}\right| d\theta = \mu(\mathbb{T}), \quad \text{for all } 0 \leq r < 1.$$

In particular, $\| \frac{1}{S} \|_0 = e^{\mu(\mathbb{T})} = |\frac{1}{S(0)}|$.

Proof. To prove (1), it is trivial if φ is a rotation. If $\varphi(z) = \frac{A-z}{1-\overline{A}z}$ for some $A \in U$, then $S \circ \varphi$ is generated by the singular Borel measure μ_φ defined by

$$\mu_\varphi(E) = \int_{\varphi(E)} \frac{1-|A|^2}{|1-\overline{A}e^{i\theta}|^2} d\mu(\theta),$$

where E is a measurable subset of $\mathbb{T}$. (2) follows from a direct calculation.

Next, we recall the definition of holomorphic hull of a compact subset K of U.

Definition 3.2. Let K be a compact subset of U. The holomorphic hull $\hat{K}_U$ of K is defined to be the set

$$\hat{K}_U = \{z \in U \mid \ |f(z)| \leq \sup_K |f|, \text{for all } f \in \mathcal{O}(U)\}.$$

For one complex variable, $\hat{K}_U$ is still a compact subset of U that contains all of the bounded components of $\mathbb{C} \setminus K$. For several complex variables, the compactness of $\hat{K}_\Omega$ is equivalent to the concept of domain of holomorphy of Ω.

Now, we prove the following theorem which completely characterize the necessary and sufficient conditions for a compact subset E of U to be the dominating set for the class N^-.

Theorem 3.3. *Let E be a compact subset of U. Then the following two statements are equivalent:*

(1) $0 \in \hat{E}_U$.

(2) *E is a dominating set for the class N^-. That is, if $S_j(z)$, $j = 1, 2$, are two singular inner functions, generated by finite positive singular Borel measure μ_j respectively, such that $1/|S_1(z)| \leq 1/|S_2(z)|$ for $z \in E$, then $\| 1/S_1 \|_0 \leq \| 1/S_2 \|_0$, i.e., $\mu_1(\mathbb{T}) = \log|\frac{1}{S_1(0)}| \leq \log|\frac{1}{S_2(0)}| = \mu_2(\mathbb{T})$.*

To prove Theorem 3.3 we shall need the lemma.

Lemma 3.4. *Let $0 < \eta \ll 1$ be a sufficiently small positive number, and let $\mu = \frac{\eta}{\pi}\delta(1)$, where $\delta(1)$ is the unit point mass measure at $e^{i0} = 1$. Then, for $|z| \leq r < 1$,*

$$\left| P[\mu](z) - \frac{1}{2\pi}\int_{-\eta}^{\eta} \frac{1-|z|^2}{|e^{it}-z|^2} dt \right| \leq \frac{\eta^2}{\pi(1-r)^4}.$$

Here $P[\mu](z)$ denotes the Poisson integral generated by μ.

Proof. A straightforward calculation shows, for $|z| \leq r < 1$,

$$\begin{aligned}
\left| P[\mu](z) - \tfrac{1}{2\pi}\textstyle\int_{-\eta}^{\eta} \frac{1-|z|^2}{|e^{it}-z|^2} dt \right| &= \left| \frac{\eta}{\pi}\frac{1-|z|^2}{|1-z|^2} - \frac{1}{2\pi}\int_{-\eta}^{\eta} \frac{1-|z|^2}{|e^{it}-z|^2} dt \right| \\
&= \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|1-z|^2} \left| \int_{-\eta}^{\eta} \left(1 - \left| \frac{1-z}{e^{it}-z} \right|^2 \right) dt \right| \\
&\leq \frac{1}{2\pi(1-r)^4} \int_{-\eta}^{\eta} |z(1-e^{-it}) + \overline{z}(1-e^{it})| dt \\
&\leq \frac{2}{\pi(1-r)^4} \int_0^{\eta} t dt = \frac{\eta^2}{\pi(1-r)^4}.
\end{aligned}$$

Here we have used $|1 - e^{it}| \le t$ for $t \ge 0$. This proves the lemma.

Proof of Theorem 3.3. $(1) \Rightarrow (2)$. This follows immediately from the maximum modulus principle and Lemma 3.1.

Now, suppose (2) holds. If (1) is not true, then the origin belongs to the unbounded component of $\mathbb{C} \setminus E$. We shall assume that, if $z \in E$, $|z| \le r < 1$ for some r. Hence, by Mergelyan's approximation theorem, there exists a polynomial $h(z)$ such that

$$|h(0) - 1| < 0.1, \quad \text{and} \quad |h(z) + 1| < 0.1 \quad \text{for } z \in E.$$

Let $u(z) = \mathrm{Re}h(z)$, then we have

$$|u(0) - 1| < 0.1, \quad \text{and} \quad |u(z) + 1| < 0.1 \quad \text{for } z \in E. \tag{3.1}$$

Clearly, $u(z)$ is a real polynomial and is equal to the Poisson integral of the boundary value $u(e^{it})$. We shall assume that $|u(e^{it})| \le M$ for some positive constant M and all t.

Therefore, given $0 < \epsilon < 0.1$, there exists $n_0 \in \mathbb{N}$ such that

$$|u(e^{i\alpha}) - u(e^{i\beta})| < \epsilon \quad \text{if} \quad |\alpha - \beta| \le \frac{2\pi}{n_0}.$$

Now, let $n \ge n_0$ be any positive integer. Divide the unit circle $\mathbb{T}$ into n equal subarcs, say, by the points $e^{i\theta_j}$, $\theta_j = \frac{2\pi j}{n}$, $j = 1, \cdots, n$. Also, let $\theta_0 = 0$. Next, let e^{it_j}, $t_j = \frac{1}{2}(\theta_{j-1} + \theta_j) = \frac{2\pi}{n}(j - \frac{1}{2})$, $j = 1, \cdots, n$, be the midpoint of the arc from $e^{i\theta_{j-1}}$ to $e^{i\theta_j}$. Define

$$g(e^{it}) = \sum_{j=1}^{n} u(e^{it_j})\chi_{[\theta_{j-1},\theta_j)},$$

where $\chi_{[\theta_{j-1},\theta_j)}$ denotes the characteristic function of the half open arc from $e^{i\theta_{j-1}}$ to $e^{i\theta_j}$.

A direct estimate shows, for $|z| \le r < 1$,

$$|u(z) - P[g](z)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} (u(e^{it}) - g(e^{it}))dt \right| \le \epsilon. \tag{3.2}$$

Next, define a singular measure μ by

$$\mu(t) = \sum_{j=1}^{n} \frac{u(e^{it_j})}{n} \delta(e^{it_j}), \tag{3.3}$$

where $\delta(e^{it_j})$ is the unit point mass measure at e^{it_j}. Hence, by Lemma 3.4 for large n, we obtain, for $|z| \le r < 1$,

$$
\begin{aligned}
|P[\mu](z)-P[g](z)| &= \left|\sum_{j=1}^{n} u(e^{it_j})\left(\frac{1}{n}\frac{1-|z|^2}{|e^{it_j}-z|^2}-\frac{1}{2\pi}\int_{\frac{2\pi(j-1)}{n}}^{\frac{2\pi j}{n}}\frac{1-|z|^2}{|e^{it}-z|^2}dt\right)\right| \\
&\leq \sum_{j=1}^{n}|u(e^{it_j})|\frac{(\frac{\pi}{n})^2}{\pi(1-r)^4} \qquad (3.4)\\
&\leq \frac{M\pi}{n(1-r)^4}.
\end{aligned}
$$

Thus, if n is chosen large enough, we can achieve that, for $|z|\leq r<1$,

$$
|P[\mu](z)-P[g](z)|<\epsilon. \tag{3.5}
$$

Hence, by combining estimates (3.2), (3.4) and (3.5), we see that, for $|z|\leq r<1$,

$$
|u(z)-P[\mu](z)|<2\epsilon. \tag{3.6}
$$

Namely, there exists a real harmonic function $P[\mu](z)$ generated by a singular measure μ such that

$$
P[\mu](0)>\frac{1}{2}, \quad \text{and} \quad P[\mu](z)<-\frac{1}{2} \quad \text{for } z\in E. \tag{3.7}
$$

In particular, it shows that $\mu(\mathbb{T})=\mu^+(\mathbb{T})-\mu^-(\mathbb{T})=P[\mu](0)>\frac{1}{2}$. We also note from (3.3) that $|\mu|(\mathbb{T})\leq\| u \|_\infty\leq M$.

Finally, we let $\mu_1=\mu^+$ and $\mu_2=\mu^-$, and let the singular inner functions $S_1(z)$, $S_2(z)$ be generated by μ_1 and μ_2 respectively. It follows that, for $z\in E$,

$$
\left|\frac{1}{S_1(z)}\right| = \exp\left\{\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}d\mu_1(t)\right\} < \exp\left\{\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}d\mu_2(t)\right\} = \left|\frac{1}{S_2(z)}\right|.
$$

However,

$$
\| \frac{1}{S_1(z)} \|_0 = e^{\mu_1(\mathbb{T})} > e^{\mu_2(\mathbb{T})} = \| \frac{1}{S_2(z)} \|_0 .
$$

This is a contradiction. Therefore, (2) implies (1), and the proof of the theorem is now completed.

Theorem 3.5. *Let $P[f](z)>0$ be the Poisson integral generated by a nonnegative L^1-function f on $\mathbb{T}$. Then, given $\epsilon>0$ and $0<r<1$, there is a finite positive Borel singular measure μ on $\mathbb{T}$ such that $2\pi\mu(\mathbb{T})\leq\| f \|_1+\epsilon$ and*

$$
|P[f](z)-P[\mu](z)|<\epsilon
$$

for $|z|\leq r<1$.

Proof. Note first that f can be approximated by continuous function h in L^1-norm on $\mathbb{T}$. If f is nonnegative, we may also assume that h is nonnegative. Then the assertion follows immediately from the proof of Theorem 3.3.

Now, we consider the case when the set E is not compact in U, i.e., E contains a point sequence tending to the boundary. We shall give an alternate proof of Theorem 2.3. In order to do so, we need the following lemma.

Lemma 3.6. *Given $\epsilon > 0$, for any $0 < \alpha < \beta < 2\pi$, there exist positive numbers $\eta > 0$, $\delta > 0$, and finitely many points e^{it_j}, $j = 1, \cdots, k = k(\epsilon)$, such that*

(i) $\alpha \le t_1 < t_2 < \cdots < t_k \le \beta$,

(ii) $k\eta^2 \le \epsilon$,

(iii) *for* $z \in E = \{z = re^{i\theta} \in U \mid \ 1-\delta \le r < 1,\ \alpha \le \theta \le \beta\}$,

$$\frac{1-|z|^2}{|1-z|^2} \le \eta^2 \sum_{j=1}^{k} \frac{1-|z|^2}{|e^{it_j}-z|^2}.$$

Proof. We first consider the case when $k = 1$. It suffices to see where the equality of (iii) holds, i.e.,

$$\eta|1-z| = |e^{it}-z|, \quad z \in U. \tag{3.8}$$

We shall assume that η is sufficiently small, i.e., $0 < \eta \ll \min\{\alpha, 2\pi-\beta\}$. A little simplification shows (3.8) is equivalent to

$$\left|z - \frac{e^{it}-\eta^2}{1-\eta^2}\right| = \frac{\eta|1-e^{it}|}{1-\eta^2}, \quad z \in U, \tag{3.9}$$

which means equation (3.8) defines a circle C in the complex plane that intersects transversally the unit circle $\mathbb{T}$ at two different points $e^{i\theta_1}$ and $e^{i\theta_2}$ with $\theta_1 < t < \theta_2$. Following from (3.8), we see that θ_1, θ_2 satisfy the equation

$$\eta^2(1-\cos\theta) = 1-\cos(\theta - t). \tag{3.10}$$

Since η is very small, θ_1, θ_2 must be very close to t. Thus, using Taylor's expansion of cosine, we solve (3.10) for θ and get

$$(\theta - t)^2 = F(\theta, t)\eta^2,$$

where $F(\theta, t) = \frac{1-\cos\theta}{\frac{1}{2}+\circ(|\theta - t|)}$. Obviously, $F(\theta, t) \ge m^2 > 0$ for some positive constant m depending only on α and β. It follows that

$$\theta_1 = t - \sqrt{F(\theta_1, t)}\eta,$$
$$\theta_2 = t + \sqrt{F(\theta_2, t)}\eta.$$

Hence, we obtain

$$\theta_2 - \theta_1 = (\sqrt{F(\theta_2, t)} + \sqrt{F(\theta_1, t)})\eta \geq 2m\eta.$$

Now, given $\epsilon > 0$, we simply choose k points $t_1, \cdots, t_k$ satisfying (i) for sufficiently large k such that $k\eta^2 = \epsilon$. For such k, η will be sufficiently small, and we can achieve

$$\sum_{j=1}^{k}(\theta_{j2} - \theta_{j1}) \geq \sum_{j=1}^{k} 2m\eta = 2mk\eta = \frac{2m\epsilon}{\eta} > \beta - \alpha.$$

Therefore, one can easily arrange these k points so that the corresponding open arcs $I_j = \{e^{i\theta} \mid \ \theta_{j1} < \theta < \theta_{j2}\}$ satisfy

(1) I_j intersects I_{j-1} and I_{j+1}, where $2 \leq j \leq k-1$,

(2) $0 < \theta_{11} < \alpha < \beta < \theta_{k2} < 2\pi$.

Then, it is clear that one can choose a small $\delta > 0$ such that (iii) holds. This proves the lemma.

Example 3.7. (*an alternate proof of Theorem 2.3*). Now, using Lemmas 3.1 and 3.6, we can provide an alternate proof of Theorem 2.3 by constructing two singular inner functions with the required properties as follows. Let $\theta_0 = \frac{\pi}{4}$ and $\theta_j = \theta_0/2^j$, $j \in \mathbb{N}$. According to Lemma 3.6, for each $j \in \mathbb{N}$, we can choose $\eta_j > 0$, $\delta_j > 0$ and k_j points $t_{j1}, \cdots, t_{jk_j}$ such that

(i) $2^{-j}\theta_0 \leq t_{j1} < \cdots < t_{jk_j} \leq 2^{-j+1}\theta_0$,

(ii) $k_j\eta_j^2 \leq 10^{-j}$,

(iii) for $z \in E_j = \{z = re^{i\theta} \in U \mid \ 1 - \delta_j \leq r < 1, \ 2^{-j}\theta_0 \leq \theta \leq 2^{-j+1}\theta_0\}$,

$$\frac{1-|z|^2}{|1-z|^2} \leq \eta_j^2 \sum_{m=1}^{k_j} \frac{1-|z|^2}{|e^{it_{jm}} - z|^2}.$$

Similarly, we can also choose $\eta_0 > 0$, $\delta_0 > 0$ and k_0 points $t_{01}, \cdots, t_{0k_0}$ such that

(i) $\theta_0 \leq t_{01} < \cdots < t_{0k_0} \leq 2\pi - \theta_0$,

(ii) $k_0\eta_0^2 \leq 9^{-1}$,

(iii) for $z \in E_0 = \{z = re^{i\theta} \in U \mid \ 1 - \delta_0 \leq r < 1, \ \theta_0 \leq \theta \leq 2\pi - \theta_0\}$,

$$\frac{1-|z|^2}{|1-z|^2} \leq \eta_0^2 \sum_{m=1}^{k_0} \frac{1-|z|^2}{|e^{it_{0m}} - z|^2}.$$

Clearly, if we denote the conjugate set of E_j by $\overline{E}_j$, then

$$E = E_0 \cup \Big(\bigcup_{j=1}^{\infty} E_j\Big) \cup \Big(\bigcup_{j=1}^{\infty} \overline{E}_j\Big) \tag{3.11}$$

is a closed nontangentially dense subset of U.

Now, we define the singular measures

$$\begin{aligned} \mu_1 &= \delta(1), \\ \mu_2 &= \eta_0^2 \sum_{m=1}^{k_0} \delta(e^{it_{0m}}) + \sum_{j=1}^{\infty}\Bigg(\eta_j^2 \sum_{m=1}^{k_j} (\delta(e^{it_{jm}}) + \delta(e^{-it_{jm}}))\Bigg), \end{aligned} \tag{3.12}$$

where $\delta(e^{i\theta})$ is the Dirac unit point mass measure concentrated at $e^{i\theta}$. Then,

$$S_j(z) = \exp\Bigg\{-\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} d\mu_j(t)\Bigg\}, \quad j = 1, 2, \tag{3.13}$$

are singular inner functions. Hence, $1/S_j(z)$, $j = 1, 2$, belong to the Nevanlinna class.

Since, on E,

$$\frac{1-|z|^2}{|1-z|^2} \le \eta_0^2 \sum_{m=1}^{k_0} \frac{1-|z|^2}{|e^{it_{0m}}-z|^2} + \sum_{j=1}^{\infty}\Bigg(\eta_j^2 \sum_{m=1}^{k_j} \Big(\frac{1-|z|^2}{|e^{it_{jm}}-z|^2} + \frac{1-|z|^2}{|e^{-it_{jm}}-z|^2}\Big)\Bigg),$$

we have

$$\Big|\frac{1}{S_1(z)}\Big| \le \Big|\frac{1}{S_2(z)}\Big|, \quad \text{for } z \in E.$$

On the other hand, by Lemma 3.1, we obtain

$$\Big\|\frac{1}{S_2}\Big\|_0 = \exp\Bigg\{k_0\eta_0^2 + 2\sum_{j=1}^{\infty} k_j\eta_j^2\Bigg\} < \exp\Bigg\{\frac{1}{9} + 2\sum_{j=1}^{\infty}\frac{1}{10^j}\Bigg\} = e^{1/3} < e = \Big\|\frac{1}{S_1}\Big\|_0.$$

Thus, a nontangentially dense subset of U, in general, is not sufficient to dominate the Nevanlinna class N.

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