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EXTENDED WELL-POSEDNESS OF VECTOR OPTIMIZATION PROBLEMS: THE CONVEX CASE

Giovanni P. Crespi,* Melania Papalia and Matteo Rocca

Abstract. In this paper we investigate a notion of extended well-posedness in vector optimization. Appropriate asymptotically minimizing sequences, when both the objective function and the feasible region are subject to perturbation are introduced. We show that convex problems, i.e. problems in which both the objective function and the perturbations are $C-$convex, are extended well-posed. Further, we characterize the proposed well-posedness notion both in terms of linear and nonlinear scalarization.

1. INTRODUCTION

Well-posedness of a scalar minimization problem is a classical concept deeply studied in different fields of scalar optimization such as calculus of variations and optimal control (see for example the monographs [6, 14, 17] for a review on this topic). Applications in the field of variational analysis are also studied, for details and references see [23, 24]. Two different approaches to scalar well-posedness are well known: the first one, proposed by Hadamard ([7]), at the beginning of the last century, studies well-posedness as a form of continuous dependence of the optimal solution from the data of the problem, i.e. the feasible region or the objective function.

In the early sixties Tykhonov ([28]) introduced another well-posedness notion based on the construction of appropriate minimizing sequences converging to the unique solution. The relationships between these two approaches have been widely studied ([2, 10]).

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*Corresponding author.

Nevertherless most applications require a form of convergent behaviour of minimizing sequences obtained from perturbed problems and this led to the statement of extended well-posedness as a combination of Hadamard and Tykhonov approaches. This notion was introduced by Zolezzi ([30]) in the context of scalar optimization.

In the framework of vector optimization several approaches gave rise to various well-posedness notions. New definitions of Tykhonov's type have been proposed and the search for classes of functions that enjoy such well-posedness properties has been carried on (see for example [16, 1, 2, 21, 20, 5]), together with the development of scalarization techniques. In this last case, the aim is to investigate the equivalence between the well-posedness of vector problems and the well-posedness of scalarized ones (see for example [19, 22, 25] as recent contributions).

The generalization of extended well-posedness to the vector case is less developed. The first attempt to consider parametric models for vector valued functions has been made by Huang ([10, 11, 12]), who extended the notion introduced by Zolezzi.

In this paper we study in a finite dimensional setting, a vector notion of extended well-posedness which considers the perturbation of both the objective function and the feasible region. We show that C-convex functions enjoy such extended well-posedness property. As a consequence C-convex functions enjoy also the extended well-posedness property in the sense of Huang. Finally we study the links between the introduced well-posedness notion and well-posedness of scalarized problems, both in the linear and in the nonlinear case.

2. Preliminaries

Consider a function $f : \mathbb{R}^m \to \mathbb{R}^l$ and let $C \subseteq \mathbb{R}^l$ be a closed, convex, pointed cone with nonempty interior endowing $\mathbb{R}^l$ with a partial order relation in the following way:

$$y \leq_C z \iff z - y \in C,$$

$$y <_C z \iff z - y \in \operatorname{int} C.$$

We consider the vector optimization problem

$$VP(f,X) \qquad \min_C f(x), \quad x \in X$$

where X is a closed subset of $\mathbb{R}^m$.
A point $x \in X$ is an *efficient solution* of problem $VP(f,X)$ when

$$(f(X) - f(x)) \cap (-C) = \{0\}.$$

We denote by Eff (f,X) the set of all efficient solutions of the problem $VP(f,X)$ and by Min (f,X) the set of all *minimal points*, i.e. the image of Eff (f,X) through

the objective function f.
A point $x \in X$ is a *weakly efficient solution* for problem $VP(f,X)$ when

$$(f(X) - f(x)) \cap (-\mathrm{int}\, C) = \emptyset.$$

We denote by WEff (f,X) the set of weakly efficient solutions of the original problem and by WMin (f,X) the set of all *weakly minimal points*.
Consider now a sequence of functions $f_n : \mathbb{R}^m \to \mathbb{R}^l$ and a sequence of closed subsets $X_n \subseteq \mathbb{R}^m$. We denote by

$$VP(f_n, X_n) \qquad\qquad \min_C f_n(x), \quad x \in X_n$$

the perturbation of $VP(f,X)$ and we wish to investigate the behaviour of the sets WEff (f_n, X_n) and WMin (f_n, X_n) when f_n and X_n "approach" to f and X respectively.

In the following, we set Lev $(f,y,X) = \{x \in X : f(x) \in y - C\}$ and denote by B the unit ball both in $\mathbb{R}^m$ and $\mathbb{R}^l$; from the context will be clear to which space we refer.

A tool that plays a key role in the sequel is the *Kuratowski-Painlevé set-convergence* ([3, 17]). Let A_n be a sequence of subsets of $\mathbb{R}^m$. Set

$$\mathrm{Ls}\, A_n := \left\{ x \in \mathbb{R}^m \;:\; x = \lim_{k \to +\infty} x^k,\ x^k \in A_{n_k},\ n_k \text{ a selection of the integers} \right\}$$

and

$$\mathrm{Li}\, A_n := \left\{ x \in \mathbb{R}^m \;:\; x = \lim_{k \to +\infty} x^k,\ x^k \in A_k,\ \text{eventually} \right\} .$$

The set Ls A_n is called the upper limit of the sequence of sets A_n, while the set Li A_n is called the lower limit of A_n. We say that the sequence A_n converges in the sense of Kuratowski to the set $A \subseteq \mathbb{R}^m$, when

$$\mathrm{Ls}\, A_n \subseteq A \subseteq \mathrm{Li}\, A_n\,,$$

and we denote this convergence by $A_n \xrightarrow{K} A$.

We close this section recalling some stability results involving the level sets Lev (f,y,X) and the efficient solutions of problem $VP(f,X)$. Both these results will play a crucial role in the next section.

Definition 2.1. ([15]). A function $f : \mathbb{R}^m \to \mathbb{R}^l$ is said to be C-convex on $\mathbb{R}^m$ if

$$f(\lambda x + (1-\lambda) z) - \lambda f(x) - (1-\lambda) f(z) \in -C,$$

for every $x, z \in \mathbb{R}^m$ and $\lambda \in [0,1]$.

Lemma 2.1. ([18]). *Let* $f_n : \mathbb{R}^m \to \mathbb{R}^l$ *and* $f : \mathbb{R}^m \to \mathbb{R}^l$ *be* $C-$*convex functions. Suppose*

(i) $f_n \to f$ *in the continuous convergence,*

(ii) $X_n \xrightarrow{K} X$,

(iii) Lev (f, y, X) *is nonempty and bounded for some* $y \in \mathbb{R}^l$.

Then $\forall \varepsilon > 0$ *it holds:*

$$\text{Lev } (f_n, y, X_n) \subseteq \text{Lev } (f, y, X) + \varepsilon B,$$

eventually.

Theorem 2.1. ([18]). *Let* $f_n : \mathbb{R}^m \to \mathbb{R}^l$ *and* $f : \mathbb{R}^m \to \mathbb{R}^l$ *be* $C-$*convex functions, with* $f_n \to f$ *in the continuous convergence and* $X_n \xrightarrow{K} X$. *Assume* Lev (f, y, X) *is nonempty and bounded for some* $y \in \mathbb{R}^l$.

(i) *If* $y \in \text{Min}\,(f, X)$, *there exists a sequence* $y^n \in \text{Min}\,(f_n, X_n)$ *such that* $y^n \to y$, *i.e.* $\text{Li}\,\text{Min}\,(f_n, X_n) \subseteq \text{Min}\,(f, X)$;

(ii) *if* $y \in \text{Min}\,(f, X)$, *there exist* $\bar{x} \in f^{-1}(y)$ *and* $x^n \in \text{Eff}\,(f_n, X_n)$ *such that* $x^n \to \bar{x}$.

3. Extended Well-posedness of Convex Vector Minimization Problems

Throughout this section and the following one, we assume WEff $(f, X) \neq \emptyset$ and we denote by $d(y, A) = \inf\{\|y - a\|, a \in A\}$ the distance of a point $y \in \mathbb{R}^l$ from a set $A \subseteq \mathbb{R}^l$.

Definition 3.1. Let $e \in \text{int}\,C$, let $f : \mathbb{R}^m \to \mathbb{R}^l$, let $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be a sequence of functions, and let X_n be a sequence of subsets of $\mathbb{R}^m$. Problem $VP(f, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n) when for every sequence $x^n \in X_n$ such that

$$(f_n(X_n) - f_n(x^n)) \cap (-\text{int}\,C - \varepsilon_n e) = \emptyset, \tag{1}$$

for some sequence $\varepsilon_n \to 0^+$, there exists a subsequence x^{n_k} of x^n such that $d(x^{n_k}, \text{WEff}\,(f, X)) \to 0$, as $k \to +\infty$.

It can be shown that the previous definition does not depend on the choice of the vector $e \in \text{int}\,C$. The proof of this statement follows along the lines of that of Proposition 3.3 in [5].

When WEff (f, X) is compact, the requirement $d(x^{n_k}, \text{WEff}\,(f, X)) \to 0$, amounts to the existence of a point $\bar{x} \in \text{WEff}\,(f, X)$ such that x^{n_k} converges to $\bar{x}$. Sequences x^n satisfying condition (1) are called asymptotically minimizing sequences.

Remark 3.1. Definition 3.1 considers sequences of points which can fail to be feasible for problem $VP(f, X)$ and hence recalls the notion of Levitin-Polyak well-posedness [13]. Indeed, if x^n is a minimizing sequence in the Levitin-Polyak sense, and hence $d(x^n, X) \to 0$, then it is also minimizing in the sense specified above, if we choose $X_n = X + \alpha_n B$ for a convenient sequence $\alpha_n \to 0^+$.

The next example shows a problem $VP(f, X)$ which is not well-posed according to Definition 3.1.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = \sqrt{|x|}$, let $X = \mathbb{R}$ and $C = \mathbb{R}_+$. We consider the sequence of functions

$$f_n(x) = \begin{cases} \sqrt{n}\,, & x \in (-\infty, -n] \\ \sqrt{|x|}\,, & x \in (-n, n] \\ \sqrt{n} + \sqrt{n}(n - x)\,, & x \in (n, n+1] \\ \sqrt{n} - \sqrt{n}(n + 2 - x)\,, & x \in (n+1, n+2] \\ \sqrt{n}\,, & x \in (n+2, +\infty) \end{cases}$$

and the sequence of sets $X_n = [-n-2, n+2]$. Problem $VP(f, X)$ is not well-posed with respect to the perturbations f_n and X_n. Indeed $x^n = n + 1$ is a minimizing sequence, but it does not converge to the unique solution $x^* = 0$. It is clear that $\text{WEff}\,(f, X) = \{0\} \neq \emptyset$.

For the reader's convenience, we recall the notion of connected set that will play a key role in the proof of the next theorem.

Definition 3.2. A set $A \subseteq \mathbb{R}^m$ is said to be connected when there are no open subsets U and V of $\mathbb{R}^m$ such that:

$$A \subseteq U \cup V, \quad A \cap U \neq \emptyset, \quad A \cap V \neq \emptyset \quad \text{and} \quad A \cap U \cap V = \emptyset\,.$$

Theorem 3.1. *Let $f : \mathbb{R}^m \to \mathbb{R}^l$ and $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be $C-$convex functions, with $f_n \to f$ in the continuous convergence, let X_n, X be closed and convex sets such that $X_n \xrightarrow{K} X$, and assume* $\text{WEff}\,(f, X)$ *is nonempty and bounded. Then problem $VP(f, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n).*

Proof. Let

$$\text{WEff}_{\varepsilon_n e}\,(f_n, X_n) = \{x \in X_n : (f_n\,(X_n) - f_n(x)) \cap (-\text{int}\, C - \varepsilon_n e) = \emptyset\}\,.$$

Assume that $VP(f, X)$ is not well-posed (with respect to the perturbations defined by the sequences f_n and X_n). Then we can find sequences $\varepsilon_n \to 0^+$, $x^n \in$

WEff $_{\varepsilon_n e}(f_n, X_n)$, such that, for some $\delta > 0$ it holds $x^n \notin$ WEff $(f, X) + \delta B$.
We claim that for every n there exists a point $z^n \in \partial$[WEff $(f, X) + \delta B$], the boundary of WEff $(f, X) + \delta B$, such that $z^n \in$ WEff $_{\varepsilon_n e}(f_n, X_n)$. Indeed, if such a z^n does not exist, we would have for every n

$$\text{WEff}_{\varepsilon_n e}(f_n, X_n) \subseteq \text{int}\,[\text{WEff}\,(f, X) + \delta B] \cup [\text{WEff}\,(f, X) + \delta B]^c. \tag{2}$$

Clearly WEff $_{\varepsilon_n e}(f_n, X_n) \cap$ [WEff $(f, X) + \delta B]^c \neq \emptyset$. We now prove that

$$\text{WEff}_{\varepsilon_n e}(f_n, X_n) \cap \text{int}\,[\text{WEff}\,(f, X) + \delta B] \neq \emptyset, \tag{3}$$

eventually. Since

$$\text{WEff}\,(f_n, X_n) \subseteq \text{WEff}_{\varepsilon_n e}(f_n, X_n),$$

it is enough to prove

$$\text{WEff}\,(f_n, X_n) \cap \text{int}\,[\text{WEff}\,(f, X) + \delta B] \neq \emptyset, \tag{4}$$

eventually.

It is known [27] that functions which are $C-$convex on $\mathbb{R}^m$ are also continuous on $\mathbb{R}^m$. If $\bar{y} = f(\bar{x})$, with $\bar{x} \in$ WEff (f, X), the level set Lev $(f, \bar{y}, X)$ is clearly nonempty and further we have

$$\text{Lev}\,(f, \bar{y}, X) \subseteq \text{WEff}\,(f, X).$$

Indeed, assume there exists a point $x' \in$ Lev $(f, \bar{y}, X) \setminus$WEff (f, X). Hence $f(x') \in f(\bar{x}) - C$ and we can find a point $x'' \in X$ such that $f(x'') \in f(x') - \text{int}\,C$. This entails $f(x'') \in f(x') - \text{int}\,C \subseteq f(\bar{x}) - \text{int}\,C$, which contradicts to $\bar{x} \in$ WEff (f, X).
The inclusion Lev $(f, \bar{y}, X) \subseteq$ WEff (f, X) proves Lev $(f, \bar{y}, X)$ is bounded. Since f is continuous, we conclude that Lev $(f, \bar{y}, X)$ is compact.

It follows, see Corollary 3.11 in [15] that Min $(f,$ Lev $(f, y, X))$ is nonempty and since

$$\text{Min}\,(f, \text{Lev}\,(f, \bar{y}, X)) \subseteq \text{Min}\,(f, X),$$

also Min (f, X) is nonempty. Further, the compactness of Lev $(f, \bar{y}, X)$ and the C-convexity of f implies Lev (f, y, X) is compact $\forall y \in \mathbb{R}^l$ (when nonempty) [18]. Let $y \in$ Min (f, X). From Theorem 2.1 ii), we get the existence of a point $x \in f^{-1}(y) \subseteq$ Eff (f, X) and a sequence $v^n \in$ Eff (f_n, X_n), with $v^n \to x$.
Recalling Eff $(f, X) \subseteq$ WEff (f, X) and Eff $(f_n, X_n) \subseteq$ WEff (f_n, X_n), it follows easily that (4) holds and hence (3) holds.

From Lemma 2.1, we get

$$\text{Lev}\,(f_n, y, X_n) \subseteq \text{Lev}\,(f, y, X) + \varepsilon B, \tag{5}$$

eventually. Hence there exists $\bar{n} \in \mathbb{N}$ such that Lev (f_n, y, X_n) is bounded for $n > \bar{n}$. Since f_n are $C-$convex, this implies that for $n > \bar{n}$ all the level sets of f_n are bounded [18]. This means that the sets WEff $_{\varepsilon_n e}(f_n, X_n)$ are connected, nonempty and closed for $n > \bar{n}$ (see Theorem 4.1 in [5]) and hence (2) cannot hold. It follows the existence of a sequence $z^n \in \partial\,[\text{WEff }(f, X) + \delta B] \cap \text{WEff}_{\varepsilon_n e}\,(f_n, X_n)$.

Since WEff (f, X) is compact, we can assume z^n converges to a point $\bar{z}$ and since $X_n \xrightarrow{K} X$, it follows $\bar{z} \in X$. Since $z^n \in \text{WEff}_{\varepsilon_n e}\,(f_n, X_n)$ it follows $\bar{z} \in \text{WEff}\,(f, X)$. Indeed, if $\bar{z} \notin \text{WEff}\,(f, X)$, there exists $x \in X$ such that $f(x) - f(\bar{z}) \in -\text{int}\,C$ and hence we can find a positive number $\bar{\delta}$, such that

$$f(x) - f(\bar{z}) \in -\text{int}\,C - \bar{\delta}e. \tag{6}$$

Since $x \in X$, there exists a sequence $w^n \to x,\ w^n \in X_n$ and from (6), we obtain $f_n(w^n) - f_n(z^n) \in -\text{int}\,C - \bar{\delta}e$, eventually, which contradicts to $z^n \in \text{WEff}_{\varepsilon_n e}$ (f_n, X_n). To complete the proof it is enough to observe that from $z^n \in \partial[\text{WEff}$ $(f, X) + \delta B]$ we get the contradiction $\bar{z} \notin \text{WEff}\,(f, X)$. ■

The boundedness assumption on WEff (f, X) cannot be avoided, as the following example shows.

Example 3.2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, z) = (z^2, e^x),\ C = \mathbb{R}^2_+$ and $X = \mathbb{R}^2,\ f_n = f$ and $X_n = X,\ \forall n$. Function f is $C-$convex, and WMin $(f, X) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\}$ while WEff $(f, X) = \{(x, z) \in \mathbb{R}^2 : z = 0\}$. The asymptotically minimizing sequence $(x^n, z^n) = (-n, -n)$ doesn't admit any subsequence (x^{n_k}, z^{n_k}) such that $d(f(x^{n_k}, z^{n_k}), \text{WEff}\,(f, X)) \to 0$.

Huang ([10, 11]) introduced three different types of extended well-posedness for a vector optimization problem, considering a fixed feasible region and a parametric model for the objective function. Let (P, ρ) be a metric space and let $p^* \in P$ be a fixed point. Let L be a closed ball with center p^* and positive radius. Let $F : X \times L \to Y$ be such that $F(x, p^*) = f(x),\ \forall x \in X$.
The following definitions are quoted from [10] and [11].

Definition 3.3. A sequence $x^n \in X$ is a strongly asymptotically minimizing sequence if for any sequence $p^n \to p^*$, it holds

$$(F(X, p^n) - F(x^n, p^n) + \varepsilon_n e) \cap (-C \setminus \{0\}) = \emptyset, \tag{7}$$

for some $\varepsilon_n > 0,\ \varepsilon_n \to 0^+$.

Consider $e \in \text{int}\,C$ and let $\xi(y) = \min\{t \in \mathbb{R} : y \in te - C\},\ \forall y \in Y$.

Definition 3.4. Problem $VP(f, X)$ is called well-posed in the strongly extended sense (with respect to the perturbation defined by F) if

(i) $\mathrm{WEff}\,(f, X) \neq \emptyset$,

(ii) for any $p \in L,\ \xi(F(x,p))$ is bounded from below on X,

(iii) for every strongly asymptotically minimizing sequence $x^n \in X$, there exist a subsequence x^{n_k} and some point $x^* \in \mathrm{WEff}\,(f, X)$ such that $x^{n_k} \to x^*$.

Next result shows that if $F\,(\cdot, p)$ is C-convex for $p \in L$ and F is continuous, then $VP\,(f, X)$ is well posed in the sense of Definition 3.4.

Corollary 3.1. *Assume $F(x,p)$ are $C-$convex for all $p \in L,$ $\mathrm{WEff}\,(f,X) \neq \emptyset$ and for any $p \in L,\ \xi(F(x,p))$ is bounded below on X*

Assume further that $\forall p^n \to p^*,\ \forall x^n \to x^*,\ F(x^n, p^n) \to F(x^*, p^*)$.

Then $VP(f, X)$ is well-posed in the sense of Definition 3.4, (with respect to the perturbation defined by F).

Proof. Let $f_n(x) = F(x, p^n),\ \forall x \in X$. Then the proof follows easily from Theorem 3.1. ■

4. Well-posedness of Scalarized Problems

We divide the results of this section in two subsections. The first one is devoted to nonlinear scalarization in which no convexity assumption is needed, while the second one concerns linear scalarization under $C-$convexity hypothesis.
We recall the notion of oriented distance (for more details see [8, 9, 29, 23]).

Definition 4.1. For a set $A \subseteq \mathbb{R}^l$, the oriented distance function $\Delta_A : \mathbb{R}^l \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$\Delta_A(y) = d(y, A) - d(y, A^c).$$

The main properties of function Δ_A are gathered in the following proposition.

Proposition 4.1. ([29]).

(i) *If $A \neq \emptyset$ and $A \neq \mathbb{R}^l$ then Δ_A is real valued;*

(ii) $\Delta_A(y) < 0$ *for every* $y \in \mathrm{int}\, A$, $\Delta_A(y) = 0$ *for every* $y \in \partial A$ *and* $\Delta_A(y) > 0$ *for every* $y \in \mathrm{int}\, A^c$*;*

(iii) *if A is closed, then it holds $A = \{y :\ \Delta_A(y) \leq 0\}$*

(iv) *if A is a closed convex cone, then Δ_A is nonincreasing with respect to the ordering relation induced on $\mathbb{R}^l$, i.e. the following is true: if $y_1, y_2 \in \mathbb{R}^l$ then*

$$y_1 - y_2 \in A \quad \Longrightarrow \qquad \Delta_A(y_1) \leq \Delta_A(y_2)$$

If A has nonempty interior, then

$$y_1 - y_2 \in \mathrm{int}\, A \quad \Longrightarrow \qquad \Delta_A(y_1) < \Delta_A(y_2)$$

4.1. Nonlinear scalarization

We relate to $VP(f, X)$ the scalar optimization problem

$$SP(h, X) \qquad \min h(x), \quad x \in X$$

where $h(x) = -\inf_{z \in X} \Delta_{-C}(f(z) - f(x))$ and X is a closed subset of $\mathbb{R}^m$. Function h is always nonnegative and can be written ([4, 25]) as

$$h(x) = \sup_{z \in X} \min_{\lambda \in C^+ \cap \partial B} \langle \lambda, f(x) - f(z) \rangle$$

where $C^+ = \{v \in \mathbb{R}^l : \langle v, c \rangle \geq 0, \forall c \in C\}$.
We denote by $\mathrm{S}\,(h, X)$ the set of solutions of problem $SP(h, X)$. Further, for f_n and X_n defined as in Section 2, we consider the scalar perturbed problem

$$SP(h_n, X_n) \qquad \min h_n(x), \quad x \in X_n$$

where $h_n(x) = -\inf_{z \in X_n} \Delta_{-C}(f_n(z) - f_n(x))$.
We investigate the behaviour of solutions of problem $SP(h, X)$ when it is subject to the perturbation $SP(h_n, X_n)$ induced by f_n and X_n.

Definition 4.2. A sequence $x^n \in X_n$ is called asymptotically minimizing for problem $SP(h, X)$ when

$$h_n(x^n) \to \inf_{x \in X} h(x).$$

Definition 4.3. Problem $SP(h, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n) when for every asymptotically minimizing sequence x^n there exists a subsequence x^{n_k} such that $d(x^{n_k}, \mathrm{S}\,(h, X)) \to 0$ as $k \to \infty$.

Next result characterizes solutions of $VP(f, X)$ in terms of solutions of the scalar problem $SP(h, X)$.

Theorem 4.2. ([25]). *Let $\bar{x} \in X$. Then $\bar{x} \in \mathrm{WEff}\,(f, X)$ if and only if $h(\bar{x}) = 0$ (and hence $\bar{x} \in \mathrm{S}\,(h, X)$).*

Remark 4.2. When $\mathrm{WEff}\,(f, X) \neq \emptyset$, Theorem 4.2 states $\inf_{x \in X} h(x) = 0$ and hence Definition 4.2 recalls the notion of extended well-posedness introduced in [30].

Theorem 4.3. *Problem $VP(f, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n) if and only if $SP(h, X)$ is well-posed (with respect to the perturbations defined by the sequences h_n and X_n).*

Proof. Since $\mathrm{WEff}\,(f, X) \neq \emptyset$, it follows $\inf_{x \in X} h(x) = 0$. We show that the asymptotically minimizing sequences of the two problems coincide.

Let x^n be an asymptotically minimizing sequence for problem $VP(f, X)$, i.e.

$$f_n(x) - f_n(x^n) + \varepsilon_n e \notin -\operatorname{int} C,\ \forall x \in X_n.$$

This is equivalent to $\Delta_{-C}(f_n(x) - f_n(x^n) + \varepsilon_n e) \geq 0,\ \forall x \in X_n$ and hence by subadditivity of $\Delta_{-C}(\cdot)$,

$$\Delta_{-C}(f_n(x) - f_n(x^n)) \geq -\Delta_{-C}(\varepsilon_n e) := -\gamma_n,\ \forall x \in X_n,$$

where $\gamma_n \geq 0$ and $\gamma_n \to 0$. It follows

$$0 \leq h_n(x^n) = -\inf_{z \in X_n} \Delta_{-C}(f_n(z) - f_n(x^n)) \leq \gamma_n,$$

hence $h_n(x^n) \to 0$, that is x^n is asymptotically minimizing for problem $SP(h, X)$.

Assume now x^n is an asymptotically minimizing sequence for problem $SP(h, X)$, i.e.

$$h_n(x^n) \to 0,$$

which implies $h_n(x^n) \leq \beta_n$, for some sequence $\beta_n \geq 0, \beta_n \to 0$. It holds,

$$-\inf_{z \in X_n} \Delta_{-C}(f_n(z) - f_n(x^n)) \leq \beta_n,$$

that is $\Delta_{-C}(f_n(z) - f_n(x^n)) \geq -\beta_n,\ \forall z \in X_n$.
Since $\Delta_{-C}(y) = \max_{\lambda \in C^+ \cap \partial B} \langle \lambda, y \rangle$, choosing a vector $e \in \operatorname{int} C$ with $\langle \lambda, e \rangle \geq 1,\ \forall \lambda \in C^+ \cap \partial B$, we obtain, $\forall z \in X_n$:

$$\begin{aligned}
\Delta_{-C}(f_n(z) - f_n(x^n) + \beta_n e) &= \max_{\lambda \in C^+ \cap \partial B} \langle \lambda, f_n(z) - f_n(x^n) + \beta_n e \rangle \\
&\geq \max_{\lambda \in C^+ \cap \partial B} \langle \lambda, f_n(z) - f_n(x^n) \rangle + \min_{\lambda \in C^+ \cap \partial B} \langle \lambda, \beta_n e \rangle \\
&\geq \max_{\lambda \in C^+ \cap \partial B} \langle \lambda, f_n(z) - f_n(x^n) \rangle + \beta_n \\
&\geq -\beta_n + \beta_n = 0.
\end{aligned}$$

Hence $\Delta_{-C}(f_n(z) - f_n(x^n) + \beta_n e) \geq 0,\ \forall z \in X_n$ and this is equivalent to say $f_n(z) - f_n(x^n) + \beta_n e \notin -\operatorname{int} C,\ \forall z \in X_n$.
Thus x^n is an asymptotically minimizing sequence for the vector problem $VP(f, X)$.
Recalling $\operatorname{WEff}(f, X) = \{x \in X : h(x) = 0\} = \operatorname{S}(h, X)$, the proof of the result is easily completed. ∎

4.2. Linear scalarization

Consider the scalar optimization problem

$$SP(g, X) \qquad \min g(\lambda, x), \quad x \in X$$

where $g(\lambda, x) = \langle \lambda, f(x) \rangle,\ \lambda$ is a fixed vector belonging to the set $C^+ \cap \partial B$ and X is the feasible region of function f.
We denote by $\mathrm{S}\,(g, X)$ the set of solutions of $SP(g, X)$. We consider the perturbed problem

$$SP(g_n, X_n) \qquad\qquad \min g_n(\lambda, x), \quad x \in X_n$$

where $g_n(\lambda, x) = \langle \lambda, f_n(x) \rangle$.

Definition 4.4. A sequence $x^n \in X_n$ is called asymptotically minimizing for problem $SP(g, X)$ when

$$g_n(\lambda, x^n) \to \inf_{x \in X} g(\lambda, x).$$

Definition 4.5. Problem $SP(g, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n) when

(i) $\mathrm{S}\,(g, X) \neq \emptyset$;

(ii) for every asymptotically minimizing sequence x^n there exists a subsequence x^{n_k} such that $d(x^{n_k}, \mathrm{S}\,(g, X)) \to 0$ as $k \to \infty$.

Theorem 4.4. ([15, 26]). *Let $f : \mathbb{R}^m \to \mathbb{R}^l$ be a C-convex function and let $\bar{x} \in X$. Then $\bar{x} \in \mathrm{WEff}\,(f, X)$ if and only if*

$$\bar{x} \in \bigcup_{\lambda \in C^+ \cap \partial B} \mathrm{S}\,(g, X).$$

Theorem 4.5. *Let $f : \mathbb{R}^m \to \mathbb{R}^l$ and $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be $C-$convex functions with $f_n \to f$ in the continuous convergence. Let X_n, X be closed and convex sets such that $X_n \xrightarrow{K} X$. If, for every $\lambda \in C^+ \cap \partial B$, problem $SP(g, X)$ is well-posed (with respect to the perturbations defined by the sequences g_n and X_n), then problem $VP(f, X)$ is well-posed (with respect to the perturbations defined by the sequences f_n and X_n).*

Proof. We know that an asymptotically minimizing sequence for problem $VP(f, X)$, is always asymptotically minimizing for problem $SP(h, X)$ defined in the previous subsection.
Let x^n be an asymptotically minimizing sequence for problem $VP(f, X)$. Then $h_n(x^n) \to 0$ and by the compactness of $C^+ \cap \partial B$, there exists a sequence $\lambda^n \to \lambda^* \in C^+ \cap \partial B$ such that

$$\min_{\lambda \in C^+ \cap \partial B} \langle \lambda, f_n(x^n) - f_n(x) \rangle = \langle \lambda^n, f_n(x^n) - f_n(x) \rangle,\ \forall n,$$

and hence

$$\sup_{x \in X_n} \langle \lambda^n, f_n(x^n) - f_n(x) \rangle \to 0,$$

i.e.

$$\langle \lambda^n, f_n(x^n) \rangle - \inf_{x \in X_n} \langle \lambda^n, f_n(x) \rangle \to 0.$$

Since f is C-convex, we observe that $g(\lambda, x)$ is a convex function for every $\lambda \in C^+ \cap \partial B$ (see [15]) and since $\lambda^n \to \lambda^*$, it follows $\langle \lambda^n, f_n \rangle \to \langle \lambda^*, f \rangle$ in the continuous convergence. Hence (see e.g. [17]),

$$g_n(\lambda^n, x^n) = \langle \lambda^n, f_n(x^n) \rangle \to \inf_{x \in X} \langle \lambda^*, f(x) \rangle = \inf_{x \in X} g(\lambda^*, x).$$

We claim that $g_n(\lambda^*, x^n) \to \inf_{x \in X} g(\lambda^*, x)$.
Since $\lambda^n \to \lambda^*,\ \forall \varepsilon > 0,\ \exists \bar{n}$ such that $\forall n > \bar{n}$

$$|\langle \lambda^*, f_n(x^n) \rangle - \langle \lambda^n, f_n(x^n) \rangle| < \frac{\varepsilon}{2},$$

i.e. $|\langle \lambda^* - \lambda^n, f_n(x^n) \rangle| < \frac{\varepsilon}{2}$. Hence, $\forall n > \bar{n}$

$$\begin{aligned} 0 &\leq \langle \lambda^*, f_n(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) \rangle \\ &= g_n(\lambda^*, x^n) - \inf_{x \in X} \langle \lambda^* \rangle, f(x) \\ &= \langle \lambda^n, f_n(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) + \langle \lambda^* \rangle - \lambda^n, f_n(x^n) \rangle \\ &\leq \langle \lambda^n, f_n(x^n) \rangle - \inf_{x \in X} \langle \lambda^*, f(x) \rangle + \frac{\varepsilon}{2}. \end{aligned}$$

Since $\langle \lambda^n, f_n(x^n) \rangle \to \inf_{x \in X} \langle \lambda^*, f(x) \rangle$ and ε is arbitrary, we prove the claim. Hence recalling the assumption of scalar well-posedness on $SP(g, X)$ the proof is completed. ∎

The following example shows that the converse of Theorem 4.5 is not true in general.

Example 4.3. Let $f : \mathbb{R} \to \mathbb{R}^2,\ f(x) = (x^2, e^x),\ C = \mathbb{R}^2_+$ and $X = \mathbb{R},\ f_n = f$ and $X_n = X, \forall n$.

Function f is $C-$convex and WEff $(f, X) = \{x \in \mathbb{R} : x \leq 0\}$. Problem $VP(f, X)$ is well-posed (with respect to the perturbations defined by f_n and X_n), but the scalarized function $g(\lambda, x) = e^x$ obtained by $\lambda = (0, 1)$ is not well-posed, since S $(g, X) = \emptyset$.

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Giovanni P. Crespi
Université de la Vallé d'Aoste
Faculty of Economics
Strada dei Cappuccini 2A
11100 Aosta,
Italia
E-mail: g.crespi@univda.it

Melania Papalia
Department of Economics
Università dell'Insubria
via Ravasi 2, 21100 Varese
Italia
E-mail: melaniapa@libero.it

Matteo Rocca
Department of Economics
Università dell'Insubria
via Ravasi 2, 21100 Varese
Italia
E-mail: mrocca@eco.uninsubria.it