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ON THE BOUNDEDNESS OF MULTILINEAR OPERATORS ON WEIGHTED HERZ-MORREY SPACES

Xiangxing Tao* and Huihui Zhang

Abstract. Boundedness of multilinear operators on weighted Herz-Morrey spaces is established in this paper. The weak estimates on endpoints are also derived. As a special case, the conclusions can apply to multilinear Calderón-Zygmund operators.

1. INTRODUCTION

In recent years, the study of multilinear integrals has been received more attention, which is motivated not only as the generalization of the linear theory but also the natural appearance of multilinear singular integral theory. In 2002, Grafakos and Torres [2] studied the boundedness of multilinear Calderón-Zygmund operators on the product of Lebesgue spaces and the endpoint weak estimates; and they extended it to a weighted version in [3]. Recently, the authors have studied the boundedness of the multilinear fractional integrals on Herz-Morrey spaces in [6] [7] and [10]. In this paper, we focus on the boundedness of the multilinear singular integral operators on weighted Herz-Morrey spaces and their weak estimates on endpoints. As a special case, our conclusions can apply to multilinear Calderón-Zygmund operators and thus extend the related theorems appeared in [2] and [3], and also include the related results in [9] and [11] in which the boundedness on weighted Herz spaces were considered.

In fact, we consider the following general multilinear operator

$$\mathcal{T}\left(f_1,\ldots,f_m\right)(x)=\int_{(\mathbb{R}^n)^m}\mathcal{K}(x,y_1,\ldots,y_m)f_1(y_1)\ldots f_m(y_m)dy_1\ldots dy_m \tag{1.1}$$

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with the kernel $\mathcal{K}$ satisfying the size condition

$$|\mathcal{K}(x, y_1, \ldots, y_m)| \le C \left(\sum_{i=1}^{m} |x - y_i|\right)^{-mn}. \tag{1.2}$$

This class of multilinear operators include the m-linear Calderón-Zygmund operators, which are associated with an m-linear Calderón-Zygmund kerneal $\mathcal{K}$ defined away from the diagonal $y_0 = y_1 = \ldots = y_m$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|\mathcal{K}(y_0, y_1, \ldots, y_m)| \le C \left(\sum_{i,j=0}^{m} |y_i - y_j|\right)^{-mn}$$

and

$$|\nabla\mathcal{K}(y_0, y_1, \ldots, y_m)| \le C \left(\sum_{i,j=0}^{m} |y_i - y_j|\right)^{-(mn+1)}.$$

Definition 1.1. A non-negative function u is said to be a Muckenhoupt A_p weight, denote by $u \in A_p$, if for any ball B on $\mathbb{R}^n$, there exists a constant $C > 0$ independent of the ball B such that

$$\frac{1}{|B|}\int_B u(x)dx \left(\frac{1}{|B|}\int_B u(x)^{-\frac{1}{p-1}}dx\right)^{p-1} \le C, \qquad \text{when } \ 1 < p < \infty$$

$$\frac{1}{|B|}\int_B u(x)dx \le C \inf_{x\in B} u(x), \qquad \text{when } \ p = 1.$$

It's well known that $A_p \subset A_\infty$ for any $1 \le p < \infty$, thus for any $u \in A_p$ there exists $0 < \delta_u \le p$ such that

$$C_1\left(\frac{|B_1|}{|B_2|}\right)^p \le \frac{u(B_1)}{u(B_2)} \le C_2\left(\frac{|B_1|}{|B_2|}\right)^{\delta_u} \quad \text{for any } B_1 \subset B_2. \tag{1.3}$$

From the works of Grafakos and Torres in [2] and [3], we have

Theorem 1.2. *Let $1 < q_1, \ldots, q_m, q < \infty$ and $\frac{1}{q} = \frac{1}{q_1} + \ldots + \frac{1}{q_m}$. Then the m-linear Calderón-Zygmund operator $\mathcal{T}(f_1, \ldots, f_m)$ is bounded from $L^{q_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Moreover, if $u_i \in A_{q_i}(\mathbb{R}^n), i = 1, 2, \ldots, m$ and $u^{\frac{1}{q}} = u_1{}^{\frac{1}{q_1}} u_2{}^{\frac{1}{q_2}} \ldots u_m{}^{\frac{1}{q_m}}$, then

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{q,u} \le C \prod_{i=1}^{m} \|f_i\|_{q_i,u_i},$$

where we denote by $\|f\|_{q,u} = \|f\|_{L^q_u} = \left(\int_{\mathbb{R}^n} |f(x)|^q u(x)dx\right)^{\frac{1}{q}}$.

In order to state our theorems, we first introduce the weighted Herz-Morrey spaces, which are the generalization of the classical Morrey space and Herz space, date back to Beurling [1] and Herz [4]. One may see [5] for the weighted Herz spaces. Throughout the whole paper, let $B_k = \{x \in \mathbb{R}^n : |x| < 2^k\}$, $E_k = B_k \setminus B_{k-1}$ and $\chi_k = \chi_{E_k}$.

Definition 1.3. Suppose $\alpha \in R, 0 \le q < \infty, 0 \le p < \infty$, $0 \le \lambda < \infty$, and υ, u are two nonnegative weights on $\mathbb{R}^n$, then the homogeneous weighted Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ is defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u) = \left\{ f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, u) : \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} < \infty \right\}$$

with the norm

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} = \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda}{n}} \left\{ \sum_{k=-\infty}^{k_0} \upsilon(B_k)^{\frac{\alpha p}{n}} \|f\chi_k\|^p_{L^q_u} \right\}^{\frac{1}{p}}.$$

The homogeneous weighted weak Herz-Morrey space $WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ is defined by

$$WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u) = \left\{ f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, u) : \|f\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} < \infty \right\}$$

with the norm

$$\|f\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} = \sup_{\substack{k_0 \in Z \\ t>0}} \upsilon(B_{k_0})^{\frac{-\lambda}{n}} t \left\{ \sum_{k=-\infty}^{k_0} \upsilon(B_k)^{\frac{\alpha p}{n}} \left[u\left(\{x \in E_k : |f(x)| > t\}\right)\right]^{\frac{p}{q}} \right\}^{\frac{1}{p}}$$

Remark 1.4. If $\upsilon = u = 1$, then $M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ is the homogeneous Herz-Morrey space $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$, $WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ is the homogeneous weak Herz-Morrey space $WM\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$. If $\lambda = 0$, then $M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ reduces to the weighted homogeneous Herz space $\dot{K}^{\alpha,p}_q(\upsilon, u)$, $WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon, u)$ becomes the homogeneous weighted weak Herz space $W\dot{K}^{\alpha,p}_q(\upsilon, u)$.

Next we always suppose that $\mathcal{T}$ is a multilinear operator defined as (1.1) with the condition (1.2); And always assume that

$$u^{\frac{1}{q}} = u_1^{\frac{1}{q_1}} u_2^{\frac{1}{q_2}} \dots u_m^{\frac{1}{q_m}}$$

and

$$\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i}, \quad \frac{1}{q} = \sum_{i=1}^m \frac{1}{q_i}, \quad \alpha = \sum_{i=1}^m \alpha_i, \quad \lambda = \sum_{i=1}^m \lambda_i.$$

We will prove the following three theorems.

Theorem 1.5. *Let* $0 < p_i < \infty$, $1 < q_i < \infty$, $\upsilon \in A_{q_\upsilon}$ *with* $1 \le q_\upsilon < \infty$, $u_i \in A_{q_{u_i}}$ *with* $1 \le q_{u_i} < q_i$, *and let* $\alpha_i q_\upsilon < n(1 - \frac{q_{u_i}}{q_i})$, $\alpha_i \delta_\upsilon > \frac{-n\delta_{u_i}}{q_i} + \lambda_i q_\upsilon$ *for* $i = 1, 2, \ldots, m$. *If* $\mathcal{T}$ *is bounded from* $L^{q_1}_{u_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}_{u_m}(\mathbb{R}^n)$ *to* $L^q_u(\mathbb{R}^n)$, *then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}(\upsilon,u_i)}$$

with the constant C *independent of* $(f_1, \ldots, f_m)$.

Theorem 1.6. *Let* $0 < p_i < \infty$, $1 \le q_i < \infty$, $\upsilon \in A_{q_\upsilon}$ *with* $1 \le q_\upsilon < \infty$, $u_i \in A_{q_{u_i}}$ *with* $1 \le q_{u_i} < q_i$, *and let* $\alpha_i q_\upsilon < n(1 - \frac{q_{u_i}}{q_i})$, $\alpha_i \delta_\upsilon > \frac{-n\delta_{u_i}}{q_i} + \lambda_i q_\upsilon$ *for* $i = 1, 2, \ldots, m$. *If* $\mathcal{T}$ *is bounded from* $L^{q_1}_{u_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}_{u_m}(\mathbb{R}^n)$ *to* $WL^q_u(\mathbb{R}^n)$, *then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}(\upsilon,u_i)}$$

with the constant C *independent of* $(f_1, \ldots, f_m)$.

Theorem 1.7. *Let* $0 < p_i < \infty$, $1 < q_i < \infty$, $\upsilon \in A_{q_\upsilon}$ *with* $1 \le q_\upsilon < \infty$, $u_i \in A_{q_{u_i}}$ *with* $1 \le q_{u_i} < q_i$, *and let* $\alpha_i q_\upsilon \le n(1 - \frac{q_{u_i}}{q_i})$, $\alpha_i \delta_\upsilon > \frac{-n\delta_{u_i}}{q_i} + \lambda_i \delta_\upsilon$ *for* $i = 1, 2, \ldots, m$. *If* $\mathcal{T}$ *is bounded from* $L^{q_1}_{u_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}_{u_m}(\mathbb{R}^n)$ *to* $WL^q_u(\mathbb{R}^n)$, *then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}(\upsilon,u_i)}$$

with the constant C *independent of* $(f_1, \ldots, f_m)$.

Letting $\upsilon = u_1 = u_2 = \cdots = u_m = 1$, we obtain the following unweighted boundedness for $\mathcal{T}$.

Corollary 1.8. *Let* $0 < p_i < \infty$, $1 < q_i < \infty$, *and* $\frac{-n}{q_i} + \lambda_i < \alpha_i < n(1 - \frac{1}{q_i})$ *for* $i = 1, 2, \ldots, m$. *If* $\mathcal{T}$ *is bounded from* $L^{q_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$, *then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}}$$

with the constant C *independent of* $(f_1, \ldots, f_m)$.

Corollary 1.9. *Let* $0 < p_i < \infty$, $1 \le q_i < \infty$, *and* $\frac{-n}{q_i} + \lambda_i < \alpha_i < n(1 - \frac{1}{q_i})$ *for* $i = 1, 2, \ldots, m$. *If* $\mathcal{T}$ *is bounded from* $L^{q_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}(\mathbb{R}^n)$ *to* $WL^q(\mathbb{R}^n)$, *then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}} \le C \prod_{i=1}^m \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}}$$

with the constant C *independent of* $(f_1, \ldots, f_m)$.

Corollary 1.10. *Let $0 < p_i < \infty$, $1 < q_i < \infty$, and $\frac{-n}{q_i} + \lambda_i < \alpha_i \le n(1 - \frac{1}{q_i})$ for $i = 1, 2, \ldots, m$. If $\mathcal{T}$ is bounded from $L^{q_1}(\mathbb{R}^n) \times \ldots \times L^{q_m}(\mathbb{R}^n)$ to $WL^q(\mathbb{R}^n)$, then*

$$\|\mathcal{T}(f_1, \ldots, f_m)\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}} \le C \prod_{i=1}^{m} \|f_i\|_{M\dot{K}^{\alpha_i,\lambda_i}_{p_i,q_i}}$$

with the constant C independent of $(f_1, \ldots, f_m)$.

2. The Proof of Theorem 1.5

Without loss of generality, we only prove the theorems for the case $m = 2$. We introduce some notations for convenience. For $k, i \in Z$, $\tau = 1, 2$, set

$$E_\tau(k,i) = \begin{cases} 2^{(k-i)[-n+\alpha_\tau q_\upsilon + n q_{u_\tau}/q_\tau]}, & \text{if } i \le k-2 \\ 1, & \text{if } k-1 \le i \le k+1 \\ 2^{(k-i)[\alpha_\tau \delta_\upsilon + n\delta_{u_\tau}/q_\tau]}, & \text{if } i \ge k+2 \end{cases}.$$

Lemma 2.1. *Let $\upsilon \in A_{q_\upsilon}$ with $1 \le q_\upsilon < \infty$, and $u_\tau \in A_{q_{u_\tau}}$ with $1 \le q_{u_\tau} < q_\tau$, $\tau = 1, 2$. If $\mathcal{T}$ is bounded from $L^{q_1}_{u_1}(\mathbb{R}^n) \times L^{q_2}_{u_2}(\mathbb{R}^n)$ to $L^q_u(\mathbb{R}^n)$, then*

$$\begin{aligned} &\upsilon(B_k)^{\alpha/n} \|\mathcal{T}(f\chi_i, g\chi_j)\chi_k\|_{q,u} \\ &\le C E_1(k,i) \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \times E_2(k,j) \upsilon(B_j)^{\alpha_2/n} \|g\chi_j\|_{q_2,u_2} \end{aligned}$$

for each k , i , j, where the constant C is independent of k , i , j.

Proof. In the case $k-1 \le i \le k+1$, $k-1 \le j \le k+1$, we note that $\upsilon(B_k) \sim \upsilon(B_i) \sim \upsilon(B_j)$ because of $\upsilon \in A_\infty$. So by the boundedness of $\mathcal{T}$ we obtain

$$\begin{aligned} &\upsilon(B_k)^{\alpha/n} \|\mathcal{T}(f\chi_i, g\chi_j)\chi_k\|_{q,u} \\ &\le C\, [\upsilon(B_i)]^{\alpha_1/n} \, \|f\chi_i\|_{q_1,u_1} \, [\upsilon(B_j)]^{\alpha_2/n} \, \|g\chi_j\|_{q_2,u_2}. \end{aligned}$$

In the other cases, we see that $|x - y_1| + |x - y_2| \sim 2^{\max(k,i,j)}$ for $x \in E_k$, $y_1 \in E_i$, $y_2 \in E_j$. Thus applying the Hölder inequality and the condition $u_\tau \in A_{q_{u_\tau}}(\mathbb{R}^n) \subseteq A_{q_\tau}(\mathbb{R}^n)$, we have

$$\begin{aligned} &|\mathcal{T}(f\chi_i, g\chi_j)\chi_k(x)| \le C 2^{-2\max(k,i,j)n} \|f\chi_i\|_1 \|g\chi_j\|_1 \\ &\le C 2^{-2\max(k,i,j)n} \|f\chi_i\|_{q_1,u_1} \left[u_1^{-\frac{1}{q_1-1}}(B_i) \right]^{1/q_1'} \|g\chi_j\|_{q_2,u_2} \left[u_2^{-\frac{1}{q_2-1}}(B_j) \right]^{1/q_2'} \\ &\le \frac{C 2^{-2\max(k,i,j)n} 2^{(i+j)n}}{[u_1(B_i)]^{1/q_1} \, [u_2(B_j)]^{1/q_2}} \|f\chi_i\|_{q_1,u_1} \|g\chi_j\|_{q_2,u_2}, \end{aligned}$$

Observing $[u(B_k)]^{1/q} \le [u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}$, so we get

$$\begin{aligned}
&\upsilon(B_k)^{\alpha/n}\|\mathcal{T}(f\chi_i, g\chi_j)\chi_k\|_{q,u} \\
&\le C2^{-\max(k,i)n}2^{in}\left[\frac{\upsilon(B_k)}{\upsilon(B_i)}\right]^{\alpha_1/n}\left[\frac{u_1(B_k)}{u_1(B_i)}\right]^{1/q_1}[\upsilon(B_i)]^{\alpha_1/n}\,\|f\chi_i\|_{q_1,u_1} \\
&\quad\times 2^{-\max(k,j)n}2^{jn}\left[\frac{\upsilon(B_k)}{\upsilon(B_j)}\right]^{\alpha_2/n}\left[\frac{u_2(B_k)}{u_2(B_j)}\right]^{1/q_2}[\upsilon(B_j)]^{\alpha_2/n}\,\|g\chi_j\|_{q_2,u_2}.
\end{aligned}$$

Recall the property (1.3) we can see that

$$2^{-\max(k,i)n}2^{in}\left[\frac{\upsilon(B_k)}{\upsilon(B_i)}\right]^{\alpha_1/n}\left[\frac{u_1(B_k)}{u_1(B_i)}\right]^{1/q_1} \le CE_1(k,i),$$

and

$$2^{-\max(k,j)n}2^{jn}\left[\frac{\upsilon(B_k)}{\upsilon(B_j)}\right]^{\alpha_2/n}\left[\frac{u_2(B_k)}{u_2(B_j)}\right]^{1/q_2} \le CE_2(k,j).$$

Thus we obtain the lemma. ∎

Now we give **the proof of Theorem 1.5.** If $q \ge 1$, by the Minkowski inequality and Lemma 2.1, we have

$$\begin{aligned}
&\upsilon(B_k)^{\alpha/n}\|\mathcal{T}(f,g)\chi_k\|_{q,u} = \upsilon(B_k)^{\alpha/n}\left\|\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\mathcal{T}(f\chi_i, g\chi_j)\chi_k\right\|_{q,u} \\
&\le \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\upsilon(B_k)^{\alpha/n}\|\mathcal{T}(f\chi_i, g\chi_j)\chi_k\|_{q,u} \\
&\le C\sum_{i=-\infty}^{\infty}E_1(k,i)\upsilon(B_i)^{\alpha_1/n}\|f\chi_i\|_{q_1,u_1}\sum_{j=-\infty}^{\infty}E_2(k,j)\upsilon(B_j)^{\alpha_2/n}\|g\chi_j\|_{q_2,u_2} \\
&\le C\sum_{i=-\infty}^{\infty}E_1(k,i)^{1-\varepsilon}\upsilon(B_i)^{\alpha_1/n}\|f\chi_i\|_{q_1,u_1}\sum_{j=-\infty}^{\infty}E_2(k,j)^{1-\varepsilon}\upsilon(B_j)^{\alpha_2/n}\|g\chi_j\|_{q_2,u_2}
\end{aligned}$$

for any $0 < \varepsilon < 1$, since $E_1(k,i) + E_2(k,j) \to 0$ whenever $i, j \to \pm\infty$.

If $0 < q < 1$, by the inequality $(\sum |a_i|)^q \le \sum |a_i|^q$ and Lemma 2.1 we have

$$\begin{aligned}
&\upsilon(B_k)^{\alpha/n}\|\mathcal{T}(f,g)\chi_k\|_{q,u} \\
&= \upsilon(B_k)^{\alpha/n}\left\{\int_{E_k}\left|\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\mathcal{T}(f\chi_i, g\chi_j)(x)\right|^q u(x)^q dx\right\}^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\leq \upsilon(B_k)^{\alpha/n}\left\{\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\int_{E_k}|\mathcal{T}(f\chi_i,g\chi_j)(x)|^q u(x)^q dx\right\}^{1/q}\\
&\leq \left\{\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}\Big[\upsilon(B_k)^{\alpha/n}\|\mathcal{T}(f\chi_i,g\chi_j)\chi_k\|_{q,u}\Big]^q\right\}^{1/q}\\
&\leq \left\{\sum_{i=-\infty}^{\infty}E_1(k,i)^q\upsilon(B_i)^{\alpha_1 q/n}\|f\chi_i\|_{q_1,u_1}^q\sum_{j=-\infty}^{\infty}E_2(k,j)^q\upsilon(B_j)^{\alpha_2 q/n}\|g\chi_j\|_{q_2,u_2}^q\right\}^{1/q}\\
&\leq C\sum_{i=-\infty}^{\infty}E_1(k,i)^{1-\varepsilon}\upsilon(B_i)^{\alpha_1 q/n}\|f\chi_i\|_{q_1,u_1}^q\left\{\sum_{i=-\infty}^{\infty}[E_1(k,i)]^{\frac{\varepsilon}{(1-q)}}\right\}^{\frac{(1-q)}{q}}\\
&\qquad\times\sum_{j=-\infty}^{\infty}E_2(k,j)^{1-\varepsilon}\upsilon(B_j)^{\alpha_2 q/n}\|g\chi_j\|_{q_2,u_2}^q\left\{\sum_{j=-\infty}^{\infty}[E_2(k,j)]^{\frac{\varepsilon}{(1-q)}}\right\}^{\frac{(1-q)}{q}}\\
&\leq C\sum_{i=-\infty}^{\infty}E_1(k,i)^{1-\varepsilon}\upsilon(B_i)^{\alpha_1 q/n}\|f\chi_i\|_{q_1,u_1}^q\sum_{j=-\infty}^{\infty}E_2(k,j)^{1-\varepsilon}\upsilon(B_j)^{\alpha_2 q/n}\|g\chi_j\|_{q_2,u_2}^q
\end{aligned}
$$

for any $0<\varepsilon<1$, where in the last inequality we have used the fact that

$$
\sum_{i=-\infty}^{\infty}[E_1(k,i)]^{\gamma}+\sum_{j=-\infty}^{\infty}[E_2(k,j)]^{\gamma}<\infty\quad\text{for any }\gamma>0 \tag{2.1}
$$

because of the condition $\alpha_\tau q_\upsilon<n(1-\frac{q_{u_\tau}}{q_\tau})$ and $\alpha_\tau\delta_\upsilon+\frac{n\delta_{u_\tau}}{q_\tau}>\lambda_\tau q_\upsilon>0$ for $\tau=1,2$.

Therefore, for any $q>0$ and any $0<\varepsilon<1$ we have that

$$
\begin{aligned}
&\|\mathcal{T}(f,g)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)}=\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\left\{\sum_{k=-\infty}^{k_0}\upsilon(B_k)^{\frac{\alpha p}{n}}\|\mathcal{T}(f,g)\chi_k\|_{q,u}^p\right\}^{\frac{1}{p}}\\
&\leq C\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\left\{\sum_{k=-\infty}^{k_0}\left\{\sum_{i=-\infty}^{\infty}E_1(k,i)^{1-\varepsilon}\upsilon(B_i)^{\alpha_1/n}\|f\chi_i\|_{q_1,u_1}\right\}^p\right.\\
&\qquad\left.\times\left\{\sum_{j=-\infty}^{\infty}E_2(k,j)^{1-\varepsilon}\upsilon(B_j)^{\alpha_2/n}\|g\chi_j\|_{q_2,u_2}\right\}^p\right\}^{\frac{1}{p}}\\
&\leq C\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\left\{\sum_{k=-\infty}^{k_0}\left\{\sum_{i=-\infty}^{\infty}E_1(k,i)^{1-\varepsilon}\upsilon(B_i)^{\alpha_1/n}\|f\chi_i\|_{q_1,u_1}\right\}^{p_1}\right\}^{\frac{1}{p_1}}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{ \sum_{k=-\infty}^{k_0} \left\{ \sum_{j=-\infty}^{\infty} E_2(k,j)^{1-\varepsilon} \upsilon(B_j)^{\alpha_2/n} \|g\chi_j\|_{q_2,u_2} \right\}^{p_2} \right\}^{\frac{1}{p_2}} \\
&\leq C \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left\{ \sum_{i=-\infty}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right\}^{p_1} \right\}^{\frac{1}{p_1}} \\
&\quad \times \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_2}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left\{ \sum_{j=-\infty}^{\infty} E_2(k,j)^{1-\varepsilon} \upsilon(B_j)^{\alpha_2/n} \|g\chi_j\|_{q_2,u_2} \right\}^{p_2} \right\}^{\frac{1}{p_2}} \\
&:= J_1 \times J_2.
\end{aligned}
$$

It's enough to show that $J_1 \leq C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}$ and $J_2 \leq C\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}$. By the symmetry, we only give the estimate for J_1. We first write that

$$
\begin{aligned}
J_1 &\leq \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=-\infty}^{k-2} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right]^{p_1} \right\}^{1/p_1} \\
&\quad + \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k-1}^{k+1} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right]^{p_1} \right\}^{1/p_1} \\
&\quad + \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k+2}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right]^{p_1} \right\}^{1/p_1} \\
&\quad := J_{11} + J_{12} + J_{13}.
\end{aligned}
$$

If $0 < p_1 < 1$, then by the inequality $(\sum |a_i|)^{p_1} \leq \sum |a_i|^{p_1}$ and the inequality (2.1), one can see that

$$
\begin{aligned}
J_{11} &\leq \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=-\infty}^{k-2} E_1(k,i)^{p_1(1-\varepsilon)} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|^{p_1}_{q_1,u_1} \right] \right\}^{1/p_1} \\
&\leq \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0-2} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|^{p_1}_{q_1,u_1} \right] \left[\sum_{k=i+2}^{\infty} E_1(k,i)^{p_1(1-\varepsilon)} \right] \right\}^{1/p_1} \\
&\leq C \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0-2} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|^{p_1}_{q_1,u_1} \right] \right\}^{1/p_1} \\
&\leq C \sup_{k_0 \in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|^{p_1}_{q_1,u_1} \right] \right\}^{1/p_1} \\
&= C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.
\end{aligned}
$$

If $p_1 > 1$, the Hӧlder inequality and the inequality (2.1) yield

$$
\begin{aligned}
J_{11} \le & \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=-\infty}^{k-2} E_1(k,i)^{p_1(1-\varepsilon)} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right] \right. \\
& \left. \times \left[\sum_{i=-\infty}^{k-2} E_1(k,i)^{p_1'(1-\varepsilon)} \right]^{\frac{p_1}{p_1'}} \right\}^{1/p_1} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \sum_{i=-\infty}^{k-2} E_1(k,i)^{p_1(1-\varepsilon)} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right\}^{1/p_1} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0-2} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right] \left[\sum_{k=i+2}^{\infty} E_1(k,i)^{p_1(1-\varepsilon)} \right] \right\}^{1/p_1} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0-2} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right\}^{1/p_1} \\
& \le C\|f\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\upsilon,u_1)}.
\end{aligned}
$$

Thus we have obtained for any $p_1 > 0$ that $J_{11} \le C\|f\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\upsilon,u_1)}$.

For J_{12}, we have

$$
\begin{aligned}
J_{12} \le & \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \sum_{i=k-1}^{k+1} E_1(k,i)^{p_1(1-\varepsilon)} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right\}^{1/p_1} \\
& \le \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0+1} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \left[\sum_{k=i-1}^{i+1} E_1(k,i)^{p_1(1-\varepsilon)} \right] \right\}^{1/p_1} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0+1} \upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right\}^{1/p_1} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0+1})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0+1} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right] \right\}^{1/p_1} \left[\frac{\upsilon(B_{k_0})}{\upsilon(B_{k_0+1})} \right]^{\frac{-\lambda_1}{n}} \\
& \le C \sup_{k_0\in Z} \upsilon(B_{k_0+1})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0+1} \left[\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|_{q_1,u_1}^{p_1} \right] \right\}^{1/p_1} \\
& = C\|f\|_{M\dot{K}_{p_1,q_1}^{\alpha_1,\lambda_1}(\upsilon,u_1)}.
\end{aligned}
$$

For J_{13}, we decompose it further into two parts as follows

$$
J_{13} = \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k+2}^{k_0} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right. \right.
$$

$$
\begin{aligned}
&\quad + \sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \Bigg]^{p_1} \Bigg\}^{1/p_1} \\
&\le \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k+2}^{k_0} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right]^{p_1} \right\}^{1/p_1} \\
&\quad + \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \right]^{p_1} \right\}^{1/p_1} \\
&:= J_{131} + J_{132}.
\end{aligned}
$$

Follow the same way used in the estimation for J_{11}, we get $J_{131} \le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}$. It's left to estimate the term J_{132}. First noting that

$$
\begin{aligned}
J_{132} &\le \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \\
&\quad \times \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \left(\upsilon(B_i)^{\alpha_1 p_1/n} \|f\chi_i\|^{p_1}_{q_1,u_1} \right)^{\frac{1}{p_1}} \right]^{p_1} \right\}^{1/p_1} \\
&\le \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\frac{\lambda_1}{n}} \|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}}(\upsilon,u_1) \right]^{p_1} \right\}^{1/p_1} \\
&\le \|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \upsilon(B_i)^{\frac{\lambda_1}{n}} \right]^{p_1} \right\}^{1/p_1} \\
&\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \sup_{k_0\in Z} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} \left(\frac{\upsilon(B_i)}{\upsilon(B_{k_0})} \right)^{\frac{\lambda_1}{n}} \right]^{p_1} \right\}^{1/p_1} \\
&\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \sup_{k_0\in Z} \left\{ \sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} 2^{(i-k_0)\lambda_1 q_\upsilon} \right]^{p_1} \right\}^{1/p_1} .
\end{aligned}
$$

We remark that the condition $\alpha_i\delta_\upsilon > \frac{-n\delta_{u_i}}{q_i} + \lambda_i q_\upsilon$ implies thet $(\alpha_i\delta_\upsilon + \frac{n\delta_{u_i}}{q_i})(1-\varepsilon) - \lambda_i q_\upsilon > 0$ for some small enough positive number ε. Hence, from the definition of $E_1(k,i)$ we see that

$$
\sum_{k=-\infty}^{k_0} \left[\sum_{i=k_0+1}^{\infty} E_1(k,i)^{1-\varepsilon} 2^{(i-k_0)\lambda_1 q_\upsilon} \right]^{p_1}
$$

$$= \sum_{k=-\infty}^{k_0}\left[2^{k(1-\varepsilon)\left(\alpha_1\delta_\upsilon+n\delta_{u_1}/q_1\right)-k_0\lambda_1 q_\upsilon}\sum_{i=k_0+1}^{\infty}2^{-i(1-\varepsilon)\left(\alpha_1\delta_\upsilon+n\delta_{u_1}/q_1\right)+i\lambda_1 q_\upsilon}\right]^{p_1}$$
$$\le C\sum_{k=-\infty}^{k_0}\left[2^{(k-k_0)(1-\varepsilon)\left(\alpha_1\delta_\upsilon+n\delta_{u_1}/q_1\right)}\right]^{p_1}\le C$$

with the positive constant C independent of k_0. Thus we obtain

$$J_{132}\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.$$

Now combining all the estimates above we reach at

$$J_1\le J_{11}+J_{12}+J_{131}+J_{132}\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.$$

By symmetry, we have $J_2\le C\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}$. Then we get

$$\|\mathcal{T}(f,g)\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)}\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.$$

The proof of Theorem 1.5 is complete.

3. The Proof of Theorem 1.6 and 1.7

We first give **the proof of Theorem 1.6**. Let $\Lambda_1=\{(i,j):k-1\le i,j\le k+1\}$ and $\Lambda_2=\mathbb{Z}^2\setminus\Lambda_1$, then

$$\begin{aligned}|\mathcal{T}(f,g)(x)|&\le|\mathcal{T}(\sum_{i=k-1}^{k+1}f\chi_i,\sum_{j=k-1}^{k+1}g\chi_j)(x)|+\sum_{i,j\in\Lambda_2}|\mathcal{T}(f\chi_i,g\chi_j)(x)|\\&:=L_1(x)+L_2(x).\end{aligned}$$

We can write the following decomposition

$$\begin{aligned}&\|\mathcal{T}(f,g)\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\upsilon,u)}\\&\le\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\sup_{t>0}t\left\{\sum_{k=-\infty}^{k_0}\upsilon(B_k)^{\frac{\alpha p}{n}}\left[u\left(\{x\in E_k:|\mathcal{T}(f,g)(x)|>t\}\right)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\&\le\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\sup_{t>0}t\left\{\sum_{k=-\infty}^{k_0}\upsilon(B_k)^{\frac{\alpha p}{n}}\left[u\left(\{x\in E_k:|L_1(x)|>t/2\}\right)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\&\quad+\sup_{k_0\in Z}\upsilon(B_{k_0})^{\frac{-\lambda}{n}}\sup_{t>0}t\left\{\sum_{k=-\infty}^{k_0}\upsilon(B_k)^{\frac{\alpha p}{n}}\left[u\left(\{x\in E_k:|L_2(x)|>t/2\}\right)\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}}\\&:=H_1+H_2.\end{aligned}$$

By the weak boundedness of $\mathcal{T}$ and the Hölder inequality, and using the same method as for J_{12}, we get

$$\begin{aligned}
H_1 &\le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda}{n}} \left\{ \sum_{k=-\infty}^{k_0} \upsilon(B_k)^{\frac{\alpha p}{n}} \left(\sum_{i=k-1}^{k+1} \|f\chi_i\|_{q_1,u_1} \cdot \sum_{j=k-1}^{k+1} \|g\chi_j\|_{q_2,u_2} \right)^p \right\}^{\frac{1}{p}} \\
&\le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \sum_{i=k-1}^{k+1} \upsilon(B_i)^{\frac{\alpha_1 p_1}{n}} \|f\chi_i\|_{q_1,u_1}^{p_1} \left[\frac{\upsilon(B_k)}{\upsilon(B_i)} \right]^{\frac{\alpha_1 p_1}{n}} \right\}^{\frac{1}{p_1}} \\
&\quad \times \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_2}{n}} \left\{ \sum_{k=-\infty}^{k_0} \sum_{j=k-1}^{k+1} \upsilon(B_j)^{\frac{\alpha_2 p_2}{n}} \|g\chi_j\|_{q_2,u_2}^{p_2} \left[\frac{\upsilon(B_k)}{\upsilon(B_j)} \right]^{\frac{\alpha_2 p_2}{n}} \right\}^{\frac{1}{p_2}} \\
&\le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{i=-\infty}^{k_0+1} \upsilon(B_i)^{\frac{\alpha_1 p_1}{n}} \|f\chi_i\|_{q_1,u_1}^{p_1} \sum_{k=i-1}^{i+1} \left[\frac{\upsilon(B_k)}{\upsilon(B_i)} \right]^{\frac{\alpha_1 p_1}{n}} \right\}^{\frac{1}{p_1}} \\
&\quad \times \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_2}{n}} \left\{ \sum_{j=-\infty}^{k_0+1} \upsilon(B_j)^{\frac{\alpha_2 p_2}{n}} \|g\chi_j\|_{q_2,u_2}^{p_2} \sum_{k=j-1}^{j+1} \left[\frac{\upsilon(B_k)}{\upsilon(B_j)} \right]^{\frac{\alpha_2 p_2}{n}} \right\}^{\frac{1}{p_2}} \\
&\le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.
\end{aligned}$$

As for H_2, It is easy to verify that the estimates in Lemma 2.1 still hold for $(i,j)\in\Lambda_2$ even if q_1 or q_2 equals to 1. Thus we have

$$\begin{aligned}
&t\upsilon(B_k)^{\alpha/n} u(\{x\in E_k : |L_1(x)|>t/2\})^{\frac{1}{q}} \le C\upsilon(B_k)^{\alpha/n} \left\| \sum_{(i,j)\in\Lambda_2} \mathcal{T}(f\chi_i, g\chi_j)\chi_k \right\|_{q,u} \\
&\le C \sum_{(i,j)\in\Lambda_2} E_1(k,i)\upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1} \times E_2(k,j)\upsilon(B_j)^{\alpha_2/n} \|g\chi_j\|_{q_2,u_2}
\end{aligned}$$

and so

$$\begin{aligned}
H_2 &\le C \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{i\le k-2, i\ge k+2} E_1(k,i)\upsilon(B_i)^{\frac{\alpha_1}{n}} \|f\chi_i\|_{q_1,u_1} \right)^{p_1} \right\}^{\frac{1}{p_1}} \\
&\quad \times \sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_2}{n}} \left\{ \sum_{k=-\infty}^{k_0} \left(\sum_{j\le k-2, j\ge k+2} E_2(k,j)\upsilon(B_j)^{\frac{\alpha_2}{n}} \|g\chi_i\|_{q_2,u_2} \right)^{p_2} \right\}^{\frac{1}{p_2}} \\
&:= I_1 \cdot I_2.
\end{aligned}$$

Further, one has

$$\begin{aligned}I_1 &\le C\sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{i=-\infty}^{k-2} E_1(k,i)\upsilon(B_i)^{\frac{\alpha_1}{n}}\|f\chi_i\|_{q_1,u_1}\right)^{p_1}\right\}^{\frac{1}{p_1}}\\ &\quad + C\sup_{k_0\in Z} \upsilon(B_{k_0})^{\frac{-\lambda_1}{n}}\left\{\sum_{k=-\infty}^{k_0}\left(\sum_{i=k+2}^{\infty} E_1(k,i)\upsilon(B_i)^{\frac{\alpha_1}{n}}\|f\chi_i\|_{q_1,u_1}\right)^{p_1}\right\}^{\frac{1}{p_1}}\\ &:= I_{11}+I_{12}.\end{aligned}$$

Now we use the same method as for J_{11} to estimate I_{11}, and use the same method as for J_{13} to estimate I_{12}. Then we can deduce that

$$I_1 \le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.$$

By symmetry, we have

$$I_2 \le C\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.$$

Hence

$$H_2 \le C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.$$

Together with the estimate of H_1, we finish the proof of Theorem 1.6.

Finally we turn to **the proof of Theorem 1.7**. Using the same notions defined in the proof of Theorem 1.6. We have the same estimates for H_1 as above, and we need only give the estimates for H_2 under the current hypotheses. Without loss of generality, let $\alpha_1 q_\upsilon = n(1-q_{u_1}/q_1)$ and $\alpha_2 q_\upsilon < n(1-q_{u_2}/q_2)$, and $\alpha_\tau\delta_\upsilon > -n\delta_{u_\tau}/q_\tau + \lambda_\tau\delta_\upsilon$ for $\tau = 1,2$.

When $(i,j)\in\Lambda_2$, by the estimates in the proof of Lemma 2.1, we obtain

$$|L_2(x)\chi_k(x)| \le C\sum_{(i,j)\in\Lambda_2} 2^{-2\max(k,i,j)n}2^{(i+j)n}\frac{\|f\chi_i\|_{q_1,u_1}\|g\chi_j\|_{q_2,u_2}}{[u_1(B_i)]^{1/q_1}\,[u_2(B_j)]^{1/q_2}}$$

$$\begin{aligned}&\le C\sum_{(i,j)\in\Lambda_2}\frac{2^{-2\max(k,i,j)n}2^{(i+j)n}}{\left\{[u_1(B_k)]^{1/q_1}\,[u_2(B_k)]^{1/q_2}\,[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}}\\ &\qquad\times\left\{\left[\frac{\upsilon(B_k)}{\upsilon(B_i)}\right]^{(\alpha_1-\lambda_1)/n}\left[\frac{u_1(B_k)}{u_1(B_i)}\right]^{1/q_1}\upsilon(B_i)^{(\alpha_1-\lambda_1)/n}\|f\chi_i\|_{q_1,u_1}\right\}\\ &\qquad\quad\times\left\{\left[\frac{\upsilon(B_k)}{\upsilon(B_j)}\right]^{(\alpha_2-\lambda_2)/n}\left[\frac{u_2(B_k)}{u_2(B_j)}\right]^{1/q_2}\upsilon(B_j)^{(\alpha_2-\lambda_2)/n}\|g\chi_j\|_{q_2,u_2}\right\}\\ &\le \frac{C}{\left\{[u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}\,[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}}\end{aligned}$$

$$\times \left\{ \sum_{i=-\infty}^{k-2} E_1(k,i) 2^{-(k-i)\lambda_1 q_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \right.$$
$$\left. + \sum_{i=k+2}^{\infty} E_1(k,i) 2^{-(k-i)\lambda_1 \delta_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \right\}$$
$$\times \left\{ \sum_{j=-\infty}^{k-2} E_2(k,j) 2^{-(k-j)\lambda_2 q_\upsilon} \upsilon(B_j)^{(\alpha_2-\lambda_2)/n} \|g\chi_j\|_{q_2,u_2} \right.$$
$$\left. + \sum_{j=k+2}^{\infty} E_2(k,j) 2^{-(k-j)\lambda_2 \delta_\upsilon} \upsilon(B_j)^{(\alpha_2-\lambda_2)/n} \|g\chi_j\|_{q_2,u_2} \right\}.$$

Since $\alpha_1 q_\upsilon = n(1 - q_{u_1}/q_1)$, one has $E_1(k,i) = 1$ for $i \le k-2$, and so

$$\begin{aligned}
&\sum_{i=-\infty}^{k-2} E_1(k,i) 2^{-(k-i)\lambda_1 q_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \\
&= \begin{cases} \displaystyle\sum_{i=-\infty}^{k-2} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1}, & \text{if } \lambda_1 = 0 \\ \displaystyle\sum_{i=-\infty}^{k-2} 2^{-(k-i)\lambda_1 q_\upsilon} \upsilon(B_i)^{-\lambda_1/n} \upsilon(B_i)^{\alpha_1/n} \|f\chi_i\|_{q_1,u_1}, & \text{if } \lambda_1 > 0 \end{cases} \\
&\le \begin{cases} \|f\|_{M\dot K^{\alpha_1,0}_{p_1,q_1}(\upsilon,u_1)}, & \text{if } \lambda_1 = 0 \\ \|f\|_{M\dot K^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \displaystyle\sum_{i=-\infty}^{k-2} 2^{-(k-i)\lambda_1 q_\upsilon}, & \text{if } \lambda_1 > 0 \end{cases} \\
&\le C\|f\|_{M\dot K^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.
\end{aligned}$$

On the other hand, notice the condition $\alpha_1\delta_\upsilon > \frac{-n\delta_{u_1}}{q_1} + \lambda_1\delta_\upsilon$, we have

$$\begin{aligned}
&\sum_{i=k+2}^{\infty} E_1(k,i) 2^{-(k-i)\lambda_1 \delta_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \\
&\le \|f\|_{M\dot K^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)} \sum_{i=k+2}^{\infty} E_1(k,i) 2^{-(k-i)\lambda_1\delta_\upsilon} \le C\|f\|_{M\dot K^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)},
\end{aligned}$$

so we get

$$\begin{aligned}
&\sum_{i=-\infty}^{k-2} E_1(k,i) 2^{-(k-i)\lambda_1 q_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \\
&+ \sum_{i=k+2}^{\infty} E_1(k,i) 2^{-(k-i)\lambda_1 \delta_\upsilon} \upsilon(B_i)^{(\alpha_1-\lambda_1)/n} \|f\chi_i\|_{q_1,u_1} \le C\|f\|_{M\dot K^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}.
\end{aligned}$$

And in the same way we get

$$\sum_{j=-\infty}^{k-2} E_2(k,j) 2^{-(k-j)\lambda_2 q_\upsilon} \upsilon(B_j)^{(\alpha_2-\lambda_2)/n} \|g\chi_j\|_{q_2,u_2} + \sum_{j=k+2}^{\infty} E_2(k,j) 2^{-(k-j)\lambda_2 \delta_\upsilon} \upsilon(B_j)^{(\alpha_2-\lambda_2)/n} \|g\chi_j\|_{q_2,u_2} \le C\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.$$

Hence we have

$$\begin{aligned}
|L_2(x)\chi_k(x)| &\le \frac{C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}}{\left\{[u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}} \\
&\le \frac{C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}}{\left\{[u_1(B_0)]^{1/q_1}[u_2(B_0)]^{1/q_2}[\upsilon(B_0)]^{(\alpha-\lambda)/n}\right\}} \\
&\quad \times \left\{\left[\frac{u_1(B_0)}{u_1(B_k)}\right]^{1/q_1}\left[\frac{\upsilon(B_0)}{\upsilon(B_k)}\right]^{\alpha_1/n}\left[\frac{u_2(B_0)}{u_2(B_k)}\right]^{1/q_2}\left[\frac{\upsilon(B_0)}{\upsilon(B_k)}\right]^{\alpha_2/n}\right\} \\
&\le \frac{C2^{-k\left[(\alpha-\lambda)\delta_\upsilon+\frac{n\delta_{u_1}}{q_1}+\frac{n\delta_{u_2}}{q_2}\right]}}{\left\{[u_1(B_0)]^{1/q_1}[u_2(B_0)]^{1/q_2}[\upsilon(B_0)]^{(\alpha-\lambda)/n}\right\}}\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)} \\
&\to 0
\end{aligned}$$

whenever $k \to +\infty$. Now for any given $t > 0$, we set

$$k_t = \max\left\{k \in \mathbb{Z} : t/2 < \frac{C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}}{\left\{[u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}}\right\}$$

Obviously, if $k \ge k_t + 1$, then the set $\{x \in E_k : |L_2(x)| > t/2\}$ is empty. Thus

$$\begin{aligned}
H_2 &= \sup_{k_0 \le k_t} \upsilon(B_{k_0})^{\frac{-\lambda}{n}} \sup_{t>0} t \left\{\sum_{k=-\infty}^{k_0} \upsilon(B_k)^{\frac{\alpha p}{n}} \left[u(\{x \in E_k : |L_2(x)| > t/2\})\right]^{\frac{p}{q}}\right\}^{\frac{1}{p}} \\
&\le C \sup_{k_0 \le k_t, t>0} \upsilon(B_{k_0})^{\frac{-\lambda}{n}} \left\{\sum_{k=-\infty}^{k_0} \upsilon(B_k)^{\frac{\alpha p}{n}} [u(B_k)]^{\frac{p}{q}}\right\}^{\frac{1}{p}} \\
&\qquad \times \frac{\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}}{\left\{[u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}}
\end{aligned}$$

$$\leq C \sup_{k_0\leq k_t, t>0} \frac{\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}}{\left\{[u_1(B_k)]^{1/q_1}[u_2(B_k)]^{1/q_2}[\upsilon(B_k)]^{(\alpha-\lambda)/n}\right\}} \times \left\{\sum_{k=-\infty}^{k_0} \upsilon(B_{k_0})^{\frac{-\lambda p}{n}}\upsilon(B_k)^{\frac{\alpha p}{n}}[u(B_k)]^{\frac{p}{q}}\right\}^{\frac{1}{p}}$$

$$\leq C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)} \sup_{k_0\leq k_t,t>0}\left\{\sum_{k=-\infty}^{k_0}\left[\frac{\upsilon(B_{k_0})}{\upsilon(B_k)}\right]^{\frac{-\lambda p}{n}}\right\}^{\frac{1}{p}}$$

$$\leq C\|f\|_{M\dot{K}^{\alpha_1,\lambda_1}_{p_1,q_1}(\upsilon,u_1)}\|g\|_{M\dot{K}^{\alpha_2,\lambda_2}_{p_2,q_2}(\upsilon,u_2)}.$$

Combining the estimates for H_1 and H_2, we finish the proof of Theorem 1.7.

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