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MEROMORPHIC SOLUTIONS OF CERTAIN FUNCTIONAL EQUATIONS

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Abstract. By utilizing Nevanlinna's value distribution theory, we study the existence or solvability of meromorphic solutions of functional equations of the type $P(f)f'P(g)g' = 1$, where $P(z)$ is a polynomial with two distinct zeros at least. We show that such type of equations have no meromorphic solutions f and g when $P(z)$ has at least three distinct zeros. Moreover, for some polynomials $P(z)$ with two distinct zeros only, such type of equations possess transcendental meromorphic solutions which can be expressed by Weierstrass $\wp$ function.

1. Introduction

In this paper, meromorphic functions are always defined in the complex plane $\mathbb{C}$. Let $f(z)$ be a nonconstant meromorphic function. We shall use the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, $N(r,f)$ and $\overline{N}(r,f)$ (see, e.g., [4]). We use $S(r,f)$ to denote the quantity $o\big(T(r,f)\big)$, $(r\to\infty,\ r\notin E)$, where the letter E is a set of $r\in(0,\infty)$ with finite linear measure. A meromorphic function $a(z)(\not\equiv\infty)$ is called a small function with respect to $f(z)$ provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and c a finite complex number. If $f(z)-c$ and $g(z)-c$ have the same zeros counting multiplicity, then we say that $f(z)$ and $g(z)$ share the value c CM. Let a, b be two constants. We use $\overline{N}(r, f=a, g=b)$ to denote the reduced counting function of the common zeros of $f-a$ and $g-b$. If

$$\overline{N}\left(r,\frac{1}{f-a}\right) - \overline{N}(r, f=a, g=a) = S(r,f),$$

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and

$$\overline{N}\left(r, \frac{1}{g-a}\right) - \overline{N}(r, f=a, g=a) = S(r,g),$$

then we say that f and g share the value a CM*. It is obvious that f and g share a CM implies that f and g share a CM*.

Nevanlinna's value distribution theory has been used to study the Fermat type of equations of meromorphic functions since 1960s (see e.g. [2, 8]). And we refer the reader to [3, 5] and [10] for some recent developments of value sharing and more general type equation $P(f) = Q(g)$ of meromorphic functions, where P, Q are two polynomials in $\mathbb{C}[z]$.

In 1977, C.-C. Yang and X.-H. Hua [9] proved the following theorem.

Theorem A. *Suppose that f, g are two nonconstant meromorphic functions and $n \geq 6$ is an integer. If $f^n f' g^n g' = 1$, then $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -1$.*

In this paper, by using Nevanlinna's value distribution theory, we study the existence or solvability of the meromorphic solutions of functional equations of the type

$$P(f)f'P(g)g' = 1, \tag{1}$$

where P is a polynomial with two distinct zeros at least. We shall prove the following result.

Theorem 1. *Suppose that $P(z)$ is a polynomial with three distinct zeros. Then equation* (1) *has no meromorphic solutions f and g.*

We point out that equation (1) may or may not have meromorphic solutions, if the polynomial P just has two distinct zeros. In fact, we can prove the following results.

Theorem 2. *Let d be a nonzero constant, k be a positive integer. Then the functional equation*

$$f^{2k+1}(f-1)^k f' g^{2k+1}(g-1)^k g' = d \tag{2}$$

has no nonconstant meromorphic solution.

Theorem 3. *Suppose that c is a nonzero constant, k is a positive integer. Then the pair (f, g) of nonconstant meromorphic solution of the following equation*

$$(f-1)^k(f+1)^k f'(g-1)^k(g+1)^k g' = c \tag{3}$$

must be

$$f = 1 + \frac{(\wp - c/2)^2}{c\wp}, \tag{4}$$

$$g = -\frac{\wp^2 + c^2/4}{(\wp - c/2)(\wp + c/2)}, \tag{5}$$

where $\wp$ *is the* $\wp$*-function satisfies*

$$(\wp')^2 = \wp(\wp - c/2)(\wp + c/2). \tag{6}$$

As an application of Theorem 1, we prove the following result.

Corollary 1. *Suppose that* f *and* g *are two nonconstant meromorphic functions. Let* m, n *be two relatively prime positive integers satisfying* $n + m \geq 18$*, and* a, b *two distinct constants. Let* $H(z) = (z - a)^n(z - b)^m$*. If the derivatives of the two functions* $H(f)$ *and* $H(g)$ *share* 1 *CM, then*

$$f = \frac{bh^{m+n} + (a - b)h^m - a}{h^{m+n} - 1}, \quad g = \frac{ah^{m+n} + (b - a)h^m - b}{h^{m+n} - 1},$$

where h *is a nonconstant meromorphic function.*

2. Lemmas

The following lemmas will be used in the proofs of our theorems. Lemma 1 can be derived easily by the lemma of logarithmic derivative, $i.e.$, $m(r, f'/f) = S(r, f)$, see e.g. [4]. Lemma 2 is well-known.

Lemma 1. *Let* $f(z)$ *be a nonconstant meromorphic function, and let* $P_k(f)$ *be a polynomial in* f *of degree* k*, and* a_i*,* $i = 1, 2 \ldots, n$ *be distinct complex numbers in* $\mathbb{C}$*, and* j *be a positive integer. Let*

$$g = \frac{P_k(f)f^{(j)}}{(f - a_1)\cdots(f - a_n)}.$$

If $k < n$*, then* $m(r, g) = S(r, f)$*.*

Lemma 2. ([11]). *Let* $f(z)$ *be a nonconstant meromorphic function. If*

$$R(f) = \frac{P_1(f)}{Q_1(f)} = \frac{a_p f^p + a_{p-1}f^{p-1} + \cdots + a_0}{b_q f^q + b_{q-1}f^{q-1} + \cdots + b_0},$$

where $P_1(f)$ *and* $Q_1(f)$ *are two relatively prime polynomials of degree* p *and* q*, respectively, and all the coefficients* $a_i(z)$ *and* $b_j(z)$ *are small functions of* $f(z)$ *with* $a_p(z) \not\equiv 0$*,* $b_q(z) \not\equiv 0$*, then we have*

$$T(r, R(f)) = \max\{p, q\}T(r, f) + S(r, f). \tag{7}$$

Lemma 3. ([6] or [7]). *Suppose that f and g are two nonconstant meromorphic functions sharing the value 1 CM. If $f \neq g$ and $fg \neq 1$, then the following inequality holds:*

$$T(r,f) \leq N_2(r,f) + N_2(r,g) + N_2\left(r,\frac{1}{f}\right) + N_2\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g), \tag{8}$$

where $N_2(r,f) = \overline{N}(r,f) + \overline{N}_{(2}(r,f)$.

Lemma 4. ([1] or [2]). *Any functions $F(z)$ and $G(z)$, which are meromorphic in the plane and satisfy $F^3 + G^3 = 1$, have the form*

$$F = \frac{1}{2\sqrt{3}}\frac{\sqrt{3}+\wp'(h)}{\wp(h)},\ G = \frac{\eta}{2\sqrt{3}}\frac{\sqrt{3}-\wp'(h)}{\wp(h)},$$

where η is a cube-root of unity, h is a nonconstant entire function, and $\wp$ is the $\wp$-function satisfying

$$(\wp')^2 = 4\wp^3 - 1. \tag{9}$$

Lemma 5. *Let c be a nonzero constant. Then the meromorphic solution of the equation*

$$(f')^4 = c^2(f-1)^3(f+1)^3, \tag{10}$$

must be the following function

$$f = 1 + \frac{(\wp - c/2)^2}{c\wp}, \tag{11}$$

where $\wp$ is the $\wp$-function satisfying

$$(\wp')^2 = \wp(\wp - c/2)(\wp + c/2). \tag{12}$$

Proof. Suppose that f is a meromorphic solution of equation (10). It is easily seen that the function $cf(f^2-1) - (f')^2$ cannot be identically zero. Let

$$g = \frac{(c^2/2)(f^2-1)}{cf(f^2-1) - (f')^2}. \tag{13}$$

Then we have

$$(f')^2 = \frac{(c/2)(f^2-1)(2fg-c)}{g}. \tag{14}$$

It follows from (10) and (14) that $f^2 - 1 = (f - c/(2g))^2$. Therefore,

$$f = 1 + \frac{(g - c/2)^2}{cg}, \tag{15}$$

and thus

$$f' = \frac{g'(g - c/2)(g + c/2)}{cg^2}. \tag{16}$$

By (15), we have

$$2fg - c = (2/c)(g - c/2)(g + c/2). \tag{17}$$

By substituting (15), (16) and (17) into (14), we get $(g')^2 = g(g - c/2)(g + c/2)$. This also completes the proof of Lemma 5. ∎

3. Proof of Theorem 1

Suppose that (f, g) is a pair of nonconstant meromorphic solution of equation (1), where $P(z)$ has three distinct zeros r_1, r_2, r_3. Let $P(z) = (z - r_1)^{k_1}(z - r_2)^{k_2}(z - r_3)^{k_3}Q(z)$, where $Q(z)$ is a polynomial of degree p. If z is a r_1 point of f of multiplicity m, then it is a pole of g of multiplicity n, and $mk_1 + m - 1 = n(k_1 + k_2 + k_3 + p) + n + 1$. Therefore, $m \geq (k_1 + k_2 + k_3 + 3 + p)/(k_1 + 1)$. Similarly, the multiplicities of all r_i points of f are greater than or equal to $n_i := (k_1 + k_2 + k_3 + 3 + p)/(k_i + 1)$, $i = 1, 2, 3$. If $p > 0$, then $1/n_1 + 1/n_2 + 1/n_3 < 1$. By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned}
T(r,f) &\leq \overline{N}\left(r, \frac{1}{f - r_1}\right) + \overline{N}\left(r, \frac{1}{f - r_2}\right) + \overline{N}\left(r, \frac{1}{f - r_3}\right) + S(r,f) \\
&\leq \frac{1}{n_1} N\left(r, \frac{1}{f - r_1}\right) + \frac{1}{n_2} N\left(r, \frac{1}{f - r_2}\right) + \frac{1}{n_3} N\left(r, \frac{1}{f - r_3}\right) + S(r,f) \\
&\leq \left(\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}\right) T(r,f) + S(r,f),
\end{aligned}$$

Therefore, $1/n_1 + 1/n_2 + 1/n_3 \geq 1$, a contradiction. Hence $p = 0$, i.e., $Q(z)$ is a nonzero constant. The last inequality above also shows that f has no Nevanlinna exceptional value. Let $m_i := (k_1 + k_2 + k_3 + 3)/(k_i + 1)$, $i = 1, 2, 3$. Then the multiplicities of all r_i points of f are greater than or equal to m_i. If m_1 is not an integer, then the multiplicities of all r_1 points of f are greater than or equal to $[m_1] + 1$. Since $1/([m_1] + 1) + 1/m_2 + 1/m_3 < 1$, we can still derive a contradiction by Nevanlinna's second fundamental theorem. Hence m_1 is a positive integer. Similarly, m_2 and m_3 are positive integers, too.

Without loss of generality, we assume $m_1 \le m_2 \le m_3$. Since

$$k_1 + k_2 + k_3 + 3 = m_i(k_i + 1),\ i = 1, 2, 3, \tag{18}$$

we have $3(k_1 + k_2 + k_3 + 3) \ge m_1(k_1 + k_2 + k_3 + 3)$, that is $m_1 \le 3$. Note that

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1. \tag{19}$$

We have $m_1 > 1$. Therefore, $m_1 = 2$ or $m_1 = 3$.

If $m_1 = 3$, then by $m_1 \le m_2 \le m_3$ and (19) we get $m_1 = m_2 = m_3 = 3$. Therefore, there exists a positive integer k such that $k_i = k$, $i = 1, 2, 3$.

If $m_1 = 2$, then we have $k_1 = k_2 + k_3 + 1$, by (18) and $m_1 \le m_2 \le m_3$, we get $2(k_1+k_2+k_3+3) \ge m_2(k_2+k_3+2)$. Therefore, $4(k_2+k_3+2) \ge m_2(k_2+k_3+2)$, and thus $m_2 \le 4$. From (19), it is easily seen that $m_2 \ne 2$. Hence we have $m_2 = 3$ or $m_2 = 4$. If $m_2 = 3$, then we have $m_3 = 6$. Thus there exists a positive integer k such that $k_1 = 3k + 2$, $k_2 = 2k + 1$, $k_3 = k$. If $m_2 = 4$, then we have $m_3 = 4$, Thus there exists a positive integer k such that $k_1 = 2k+1$, $k_2 = k$, $k_3 = k$. Hence equation (1) becomes one of the following three equations:

$$(f-r_1)^k(f-r_2)^k(f-r_3)^k f'(g-r_1)^k(g-r_2)^k(g-r_3)^k g' = d; \tag{20}$$

$$(f-r_1)^{2k+1}(f-r_2)^k(f-r_3)^k f'(g-r_1)^{2k+1}(g-r_2)^k(g-r_3)^k g' = d; \tag{21}$$

$$(f-r_1)^{3k+2}(f-r_2)^{2k+1}(f-r_3)^k f'(g-r_1)^{3k+2}(g-r_2)^{2k+1}(g-r_3)^k g' = d, \tag{22}$$

Since all r_j points $(j = 1, 2, 3)$ of f are poles of g. By Nevanlinna's second fundamental theorem, we have $T(r, f) \le T(r, g) + S(r, f)$. Symmetrically, we have $T(r, g) \le T(r, f) + S(r, g)$. Hence $T(r, f) = T(r, g) + S(r)$, where $S(r) := S(r, f) = S(r, g)$. By the above discussion, we see that the multiplicities of all r_i points of f are greater than or equal to m_i. By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r, f) &\le \overline{N}\left(r, \frac{1}{f-r_1}\right) + \overline{N}\left(r, \frac{1}{f-r_2}\right) + \overline{N}\left(r, \frac{1}{f-r_3}\right) + S(r) \\ &\le \frac{1}{m_1} N\left(r, \frac{1}{f-r_1}\right) + \frac{1}{m_2} N\left(r, \frac{1}{f-r_2}\right) + \frac{1}{m_3} N\left(r, \frac{1}{f-r_3}\right) + S(r) \\ &\le T(r, f) + S(r), \end{aligned}$$

which implies that

$$N\left(r, \frac{1}{f-r_i}\right) = m_i \overline{N}\left(r, \frac{1}{f-r_i}\right) + S(r) \ne S(r),\ i = 1, 2, 3.$$

Therefore, "almost all" r_i points of f have multiplicity m_i, and thus "almost all" poles of g are simple. Symmetrically, "almost all" r_i points of g have multiplicity m_i, and "almost all" poles of f are simple. The above equation also shows that r_i and ∞ cannot be the exceptional values of f and g.

In the following, we study the three functional equations (20), (21) and (22), respectively.

3.1. Solution of equation (20)

By a transformation, we need to study equation

$$f^k(f-r_1)^k(f-r_2)^k f' g^k(g-r_1)^k(g-r_2)^k g' = d \tag{23}$$

only, where d, r_1, r_2 are nonzero constant, and $r_1 \neq r_2$. Suppose that f and g are nonconstant meromorphic functions of the equation. Then the multiplicities of "almost all" zeros, r_1 points and r_2 points of f and g are 3, and "almost all" the poles of f and g are simple. Let

$$\varphi_1 = \frac{(f')^3}{f^2(f-r_1)^2(f-r_2)^2}, \quad \varphi_2 = \frac{(g')^3}{g^2(g-r_1)^2(g-r_2)^2}. \tag{24}$$

Then we have $\varphi_i \not\equiv 0$ and $N(r, \varphi_i) = S(r)$, $i = 1, 2$. We can rewrite the first equation in (24) as

$$\varphi_1 = \frac{f'}{f(f-r_1)} \cdot \frac{f'}{f(f-r_2)} \cdot \frac{f'}{(f-r_1)(f-r_2)}.$$

By Lemma 1, we get $m(r, \varphi_1) = S(r)$. Therefore, $T(r, \varphi_1) = S(r)$. Similarly, we have $T(r, \varphi_2) = S(r)$. By rewriting the first equation in (24) as

$$f = \frac{1}{\varphi_1}\frac{f'}{f}\left(\frac{f'}{(f-r_1)(f-r_2)}\right)^2,$$

and by Lemma 1, we get $m(r, f) = S(r)$. Similarly, we have $m(r, g) = S(r)$. Let

$$\varphi = f(f-r_1)(f-r_2)g(g-r_1)(g-r_2). \tag{25}$$

It is obvious that $\varphi \not\equiv 0$. From (23), we see that poles of f must be zeros of $g^2(g-r_1)^2(g-r_2)^2 g'$. The second equation in (24) tell us that zeros of g' must be zeros of φ_2 provided that they are not the zeros of $g^2(g-r_1)^2(g-r_2)^2$. Hence "almost all" poles of f are zeros of $g^2(g-r_1)^2(g-r_2)^2$. Symmetrically, "almost all" poles of g are zeros of $f^2(f-r_1)^2(f-r_2)^2$. This investigation shows that $N(r, \varphi) = S(r)$. Hence $T(r, \varphi) = S(r)$. By equation (23), we get

$$\varphi^k f' g' = d. \tag{26}$$

Suppose that z_1 is a zero of g of multiplicity 3. Then z_1 must be a simple pole of f. We have the following Laurent expansions in a neighborhood of z_1,

$$f(z) = \frac{A_1}{z - z_1} + O(1), \quad g(z) = A_2(z - z_1)^3 + O((z - z_1)^4),$$

where A_1 and A_2 are nonzero constants. By taking derivatives in the above expansions, we get

$$f'(z) = \frac{-A_1}{(z - z_1)^2} + O(1), \quad g'(z) = 3A_2(z - z_1)^2 + O((z - z_1)^3).$$

From (26), we get

$$\varphi^k(z_1)(-3A_1A_2) = d. \tag{27}$$

From (25), we get

$$\varphi(z_1) = A_1^3 A_2 r_1 r_2. \tag{28}$$

On the other hand, from the first equation in (24), we get

$$\varphi(z_1)\varphi_1(z_1) = -A_2 r_1 r_2. \tag{29}$$

Therefore,

$$A_1^3 = -\frac{1}{\varphi_1(z_1)}, \quad A_2 = -\frac{\varphi(z_1)\varphi_1(z_1)}{r_1 r_2}.$$

From (27) and the above two equations, we obtain

$$\varphi^{3k}(z_1)\left(-\frac{27}{\varphi_1(z_1)}\frac{\varphi^3(z_1)\varphi_1^3(z_1)}{r_1^3 r_2^3}\right) = d^3. \tag{30}$$

It follows that

$$27\varphi^{3k+3}(z_1)\varphi_1^2(z_1) = -(dr_1r_2)^3, \tag{31}$$

which means that z_1 is a zero of the function $27\varphi^{3k+3}\varphi_1^2 + (dr_1r_2)^3$. Note that this function is small with respect to f and g. If $27\varphi^{3k+3}\varphi_1^2 \not\equiv -(dr_1r_2)^3$, then we get $N(r, 1/g) = S(r)$. This means that 0 is an Nevanlinna exceptional value of g, which is impossible. Hence

$$27\varphi^{3k+3}\varphi_1^2 = -(dr_1r_2)^3. \tag{32}$$

Similarly we have

$$27\varphi^{3k+3}\varphi_2^2 = -(dr_1r_2)^3. \tag{33}$$

Therefore, $\varphi_1 = \varphi_2$ or $\varphi_1 = -\varphi_2$.

By (23), (24), (25) and (26), we deduce that

$$\varphi^{3k+2}\varphi_1\varphi_2 = d^3. \tag{34}$$

From (32) and (34), we obtain

$$\varphi = -\frac{\varphi_2}{\varphi_1}\left(\frac{r_1 r_2}{3}\right)^3. \tag{35}$$

Note that $\varphi_1 = \varphi_2$ or $\varphi_1 = -\varphi_2$. The above equation implies that φ is a nonzero constant, and thus by (32) and (33) we see that both φ_1 and φ_2 are nonzero constants.

Taking the derivative in equation (25) yields

$$\frac{f'L(f)}{f(f-r_1)(f-r_2)} + \frac{g'L(g)}{g(g-r_1)(g-r_2)} = 0, \tag{36}$$

where $L(z)$ is a polynomial defined by

$$L(z) = 3\left(z^2 - \frac{2}{3}(r_1+r_2)z + \frac{r_1 r_2}{3}\right). \tag{37}$$

Note that zeros of f' are poles of g, and zeros of g' are poles of f. By (36), we see that $L(f)$ and $L(g)$ share 0 CM.

We distinguish two cases below:

Case (a). $r_1^2 - r_1 r_2 + r_2^2 \neq 0$.

In this case, the equation $L(z) = 0$ has two distinct roots denoted by a_1 and a_2. Therefore, $L(z) = 3(z-a_1)(z-a_2)$. Eq. (36) implies that f and g share the set $S = \{a_1, a_2\}$ CM, i.e., $f^{-1}(S) = g^{-1}(S)$ counting multiplicities. Note that

$$\begin{aligned}
a_1(a_1-r_1)(a_1-r_2) &= a_1(a_1^2 - (r_1+r_2)a_1 + r_1 r_2)\\
&= a_1\left(\frac{2}{3}(r_1+r_2)a_1 - \frac{r_1 r_2}{3} - (r_1+r_2)a_1 + r_1 r_2\right)\\
&= \frac{2}{3}r_1 r_2 a_1 - \frac{1}{3}(r_1+r_2)a_1^2\\
&= \frac{2}{3}r_1 r_2 a_1 - \frac{1}{3}(r_1+r_2)\left(\frac{2}{3}(r_1+r_2)a_1 - \frac{r_1 r_2}{3}\right).
\end{aligned}$$

We obtain

$$a_1(a_1-r_1)(a_1-r_2) = -\frac{2}{9}(r_1^2 - r_1 r_2 + r_2^2)a_1 + \frac{r_1 r_2(r_1+r_2)}{9}. \tag{38}$$

Similarly we can get

$$a_2(a_2-r_1)(a_2-r_2) = -\frac{2}{9}(r_1^2 - r_1 r_2 + r_2^2)a_2 + \frac{r_1 r_2(r_1+r_2)}{9}. \tag{39}$$

If both $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$ and $\overline{N}(r, f = a_1, g = a_2) \neq S(r)$ hold, then by (25), we have $\varphi = \big(a_1(a_1 - r_1)(a_1 - r_2)\big)^2$ and $\varphi = a_1(a_1 - r_1)(a_1 - r_2)a_2(a_2 - r_1)(a_2 - r_2)$. It follows that $a_1(a_1 - r_1)(a_1 - r_2) = a_2(a_2 - r_1)(a_2 - r_2)$. From (38) and (39), we get $a_1 = a_2$, a contradiction. Similarly, $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$ and $\overline{N}(r, f = a_2, g = a_1) \neq S(r)$ cannot hold simultaneously. Therefore, the condition $\overline{N}(r, f = a_1, g = a_1) \neq S(r)$ implies $\overline{N}(r, f = a_1, g = a_2) = S(r)$ and $\overline{N}(r, f = a_2, g = a_1) = S(r)$. Accordingly, f and g share the value a_1 and a_2 CM*. Hence the following function

$$\alpha = \frac{f - a_1}{g - a_1}\frac{g - a_2}{f - a_2}$$

satisfy $N(r, \alpha) = S(r)$ and $\alpha \not\equiv 0, 1$. From (24) and by Lemma 1, we have $m(r, 1/f - a_2) = S(r)$ and $m(r, 1/g - a_2) = S(r)$. Therefore $m(r, \alpha) = S(r)$. Hence $T(r, \alpha) = S(r)$. It follows from the above equation that

$$g = a_1 + \frac{(a_1 - a_2)(f - a_1)}{(\alpha - 1)f + a_1 - \alpha a_2} = \frac{(\alpha a_1 - a_2)f + (1 - \alpha)a_1 a_2}{(\alpha - 1)f + a_1 - \alpha a_2}.$$

And thus

$$g - r_i = \frac{\big(\alpha a_1 - a_2 - (\alpha - 1)r_i\big)f + (1 - \alpha)a_1 a_2 - r_i(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2}, \ i = 1, 2.$$

By substituting the above equation into (25), we get

$$\begin{aligned}\varphi = f(f - r_1)(f - r_2)&\frac{(\alpha a_1 - a_2)f + (1 - \alpha)a_1 a_2}{(\alpha - 1)f + a_1 - \alpha a_2}\\ &\cdot \frac{\big(\alpha a_1 - a_2 - (\alpha - 1)r_1\big)f + (1 - \alpha)a_1 a_2 - r_1(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2}\\ &\cdot \frac{\big(\alpha a_1 - a_2 - (\alpha - 1)r_2\big)f + (1 - \alpha)a_1 a_2 - r_2(a_1 - \alpha a_2)}{(\alpha - 1)f + a_1 - \alpha a_2}.\end{aligned}$$

By Lemma 2 and the above equation, we deduce that φ is not constant, a contradiction.

Similarly, if $\overline{N}(r, f = a_2, g = a_2) \neq S(r)$, then we can deduce that $f - a_1$ and $g - a_2$ share the value 0 in the sense CM*, and $f - a_2$ and $g - a_1$ share 0 CM*. And thus we can show that the following function

$$\beta = \frac{f - a_1}{g - a_2}\frac{g - a_1}{f - a_2}$$

satisfy $T(r, \beta) = S(r)$ and $\beta \not\equiv 0$. By a similar argument as above, we can also deduce a contradiction. Hence Eq. (23) has no nonconstant meromorphic solution in Case (a).

Case (b). $r_1^2 - r_1 r_2 + r_2^2 = 0$.

In this case, the equation $L(z) = 0$ has a multiple root $r = (r_1 + r_2)/3$, and $L(z) = 3(z-r)^2$. By (36), we see that f and g share the value r CM. Note that r cannot be the exceptional value of f and g. By (25), we get

$$\varphi = \big(r(r-r_1)(r-r_2)\big)^2 = \left(\frac{r_1 r_2}{3}\right)^3 = r^6. \tag{40}$$

Combining with (35), we obtain $\varphi_2 = -\varphi_1$. It follows from (32) that

$$\varphi_1^2 = -\frac{d^3}{r^{6(3k+2)}}. \tag{41}$$

By (24) and (36), we get

$$\frac{f'}{\varphi_1 (L(f))^2} = \frac{g'}{\varphi_2 (L(g))^2}.$$

Since $L(z) = 3(z-r)^2$ and $\varphi_1 = -\varphi_2$, the above equation becomes

$$\frac{f'}{(f-r)^4} = -\frac{g'}{(g-r)^4}.$$

This and (26) yield

$$\frac{(f')^2}{(f-r)^4} = -\frac{f'g'}{(g-r)^4} = -\frac{d}{\varphi^k (g-r)^4}.$$

And thus

$$\frac{f'}{(f-r)^2} = \frac{A}{(g-r)^2}, \tag{42}$$

where A is a constant satisfying $A^2 = -d/\varphi^k$. By the above equation and the first equation in (24), we get

$$\frac{A^3}{(g-r)^6} = \frac{(f')^3}{(f-r)^6} = \frac{\varphi_1 f^2 (f-r_1)^2 (f-r_2)^2}{(f-r)^6}.$$

It follows that

$$\frac{1}{(g-r)^3} = M \frac{f(f-r_1)(f-r_2)}{(f-r)^3}, \tag{43}$$

where M is a constant satisfying $M^2 = \varphi_1 / A^3$. On the other hand, by $\varphi = r^6$ and $z(z-r_1)(z-r_2) = (z-r)^3 + r^3$, Eq. (25) can be rewritten as

$$\frac{1}{(g-r)^3} = -\frac{1}{r^3} \frac{f(f-r_1)(f-r_2)}{(f-r)^3}.$$

Therefore, $M = -1/r^3$. And thus $A^3 = \varphi_1 r^6$. Combining with $M^2 = \varphi_1/A^3$, we get $A = -r^6\varphi_1\varphi^k/d$. By (42) and (43), we get

$$g = r + B\frac{f'(f-r)}{f(f-r_1)(f-r_2)}, \tag{44}$$

where $B = \frac{d}{r^3\varphi_1\varphi^k} = \frac{d}{r^{6k+3}\varphi_1}$.

So, if f is a solution of the equation

$$(f')^3 = \varphi_1 f^2(f-r_1)^2(f-r_2)^2 \tag{45}$$

and g is the function in (44), then (f, g) is a pair of solution of (23). Now we prove that Eq. (45) has no meromorphic solution. Since r_1 and r_2 satisfy $r_1^2 - r_1 r_2 + r_2^2 = 0$ in the present case, we can rewrite (45) as

$$\left(\frac{f'}{cr}\right)^3 + \left(\frac{r-f}{r}\right)^3 = 1,$$

where $r = (r_1 + r_2)/3$, and c is a constant satisfying $c^3 = \varphi_1$. By Lemma 4, there exists a cube-root η of unity, and a nonconstant entire function h such that

$$\frac{f'}{cr} = \frac{1}{2\sqrt{3}}\frac{\sqrt{3}+\wp'(h)}{\wp(h)}, \quad \frac{r-f}{r} = \frac{\eta}{2\sqrt{3}}\frac{\sqrt{3}-\wp'(h)}{\wp(h)}, \tag{46}$$

where $\wp$ is the function satisfying (9). Taking derivative in both side of the second equation in (46), we get

$$\frac{f'}{r} = \frac{-\eta h'}{2\sqrt{3}} \cdot \frac{\wp''(h)\wp(h) + \sqrt{3}\wp'(h) - (\wp'(h))^2}{\wp^2(h)}. \tag{47}$$

By (9), we have $\wp'' = 6\wp^2$. It follows that

$$\begin{aligned}
\frac{f'}{r} &= \frac{-\eta h'}{2\sqrt{3}}\frac{6\wp^3(h) + \sqrt{3}\wp'(h) - (\wp'(h))^2}{\wp^2(h)} \\
&= \frac{-\eta h'}{2\sqrt{3}}\frac{\frac{3}{2}[(\wp'(h))^2 + 1] + \sqrt{3}\wp'(h) - (\wp'(h))^2}{\wp^2(h)} \\
&= \frac{-\eta h'}{4\sqrt{3}}\left(\frac{\wp'(h) + \sqrt{3}}{\wp(h)}\right)^2.
\end{aligned}$$

Combining with the first equation in (46), we get $f' = \frac{c^2 r\sqrt{3}}{3\eta h'}$, which implies that f' has no zero. And thus by (45), we see that $0, r_1$ and r_2 are exceptional values of f. This is impossible. Hence Eq. (23) has no meromorphic solution.

3.2. Solution of equation (21)

The arguments in this section and the next section are very similar to that in Section 3.1, we just state the main steps and omit some of the details. In the present case, by a transformation, we need to study the equation

$$f^{2k+1}(f-r_1)^k(f-r_2)^k f'(g^{2k+1}(g-r_1)^k(g-r_2)^k g' = d \tag{48}$$

only, where d, r_1, r_2 are nonzero constant and $r_1 \neq r_2$. Suppose that f and g are nonconstant meromorphic functions satisfying Eq. (48). Then the multiplicities of "almost all" zeros of f, g are 2, the multiplicities of "almost all" r_1 points and r_2 points of f, g are 4, and "almost all" poles of f, g are simple. Let

$$\phi_1 = \frac{(f')^4}{f^2(f-r_1)^3(f-r_2)^3}, \quad \phi_2 = \frac{(g')^4}{g^2(g-r_1)^3(g-r_2)^3}. \tag{49}$$

Obviously, we have $T(r, \phi_i) = S(r)$ and $\phi_i \not\equiv 0$, $i = 1, 2$. The first equation in (49) can be rewritten as

$$f = \frac{1}{\phi_1}\frac{f'}{f}\left(\frac{f'}{(f-r_1)(f-r_2)}\right)^3.$$

By Lemma 1, we have $m(r, f) = S(r)$. Similarly, we have $m(r, g) = S(r)$. Let

$$\phi = f^2(f-r_1)(f-r_2)g^2(g-r_1)(g-r_2). \tag{50}$$

Like the arguments in Section 4.1, we can get $T(r, \phi) = S(r)$ and $\phi \not\equiv 0$. Combining with (48), we get

$$\phi^k f f' g g' = d. \tag{51}$$

By considering the Laurent expansions in the neighborhood of a zero with multiplicity 2 of f and g, respectively, we can obtain

$$\phi_1 = \phi_2 = \frac{4d^2}{(r_1 r_2)^2 \phi^{2k+1}}. \tag{52}$$

On the other hand, by considering the Laurent expansions in the neighborhood of a r_1 point with multiplicity 4 of f, we can get

$$\phi^{2k+1}\phi_1 = \frac{16d^2}{(r_1-r_2)^2 r_1^2}. \tag{53}$$

From (52) and (53), we get $r_1 = 3r_2$ or $r_1 = -r_2$. By the symmetry of r_1 and r_2, we get $r_1 = -r_2$. Let $r := r_1 = -r_2$. From (49), (50) and (51), we get

$$\phi^{4k+3}\phi_1\phi_2 = d^4. \tag{54}$$

From (52) and (54), we get $\phi = \frac{r^8}{16}$ which is a nonzero constant, and thus ϕ_1 is a nonzero constant, too.

Taking $\phi = \frac{r^8}{16}$ and $r_1 = -r_2 = r$ into (50), we get

$$f^2(f^2 - r^2)g^2(g^2 - r^2) = \frac{r^8}{16}. \tag{55}$$

Let $h = fg$. Then by (55), we get

$$f^2 + g^2 = \frac{h^4 - r^4h^2 - \frac{r^8}{16}}{r^2h^2}.$$

Hence

$$(f + g)^2 = \frac{h^4 + 2r^2h^3 - r^4h^2 - \frac{r^8}{16}}{r^2h^2}, \tag{56}$$

$$(f - g)^2 = \frac{h^4 - 2r^2h^3 - r^4h^2 - \frac{r^8}{16}}{r^2h^2}. \tag{57}$$

Since r is a nonzero number, equation $z^4 + 2r^2z^3 - r^4z^2 - \frac{r^8}{16} = 0$ and $z^4 - 2r^2z^3 - r^4z^2 - \frac{r^8}{16} = 0$ have no multiple roots, and all of the roots of the two equations are pairwise distinct. Thus by (56) and (57), we deduce that h has eight multiple values. By Nevanlinna's second fundamental theorem, we know that a nonconstant meromorphic function has four multiple values at most. Therefore, h must be a constant. Thus both $f + g$ and $f - g$ are constants, which implies that f and g are constants, a contradiction. Hence equation (48) has no nonconstant meromorphic solution.

3.3. Solution of equation (22)

In this case, we need to consider the equation

$$f^{3k+2}(f - r_1)^{2k+1}(f - r_2)^k f' g^{3k+2}(g - r_1)^{2k+1}(g - r_2)^k g' = d \tag{58}$$

only, where d, r_1, r_2 are nonzero constant and $r_1 \neq r_2$. Suppose that f and g are nonconstant meromorphic functions satisfying this equation. Then the multiplicities of "almost all" zeros of f and g are 2, the multiplicities of "almost all" r_1 points of f and g are 3, the multiplicities of "almost all" r_2 points of f and g are 6, "almost all" poles of f and g are simple. Let

$$\psi_1 = \frac{(f')^6}{f^3(f - r_1)^4(f - r_2)^5}, \quad \psi_2 = \frac{(g')^6}{g^3(g - r_1)^4(g - r_2)^5}. \tag{59}$$

It is easily seen that $T(r, \psi_i) = S(r)$ and $\psi_i \not\equiv 0$, $i = 1, 2$. By Lemma 1, we have $m(r, f) = S(r)$ and $m(r, g) = S(r)$. Let

$$\psi = f^3(f-r_1)^2(f-r_2)g^3(g-r_1)^2(g-r_2), \tag{60}$$

Similar to Section 4.1, we have $T(r,\psi)=S(r)$ and $\psi \not\equiv 0$. Taking (60) into (58), we get

$$\psi^k f^2(f-r_1)f'g^2(g-r_1)g' = d. \tag{61}$$

Suppose that z_4 is a zero of f of multiplicity 2. Then it is a simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_4, We can obtain

$$\psi = \left(\frac{r_1 r_2}{2}\right)^6, \tag{62}$$

and

$$\psi_1 = \psi_2 = \frac{(dr_1r_2)^3}{(-2\psi^{k+1})^3}. \tag{63}$$

Therefore, ψ, ψ_1 and ψ_2 are nonzero constant.

Suppose that z_5 is a r_1 point of f of multiplicity 3. Then it is a simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_5, we can get

$$\psi = \left(\frac{r_1(r_1-r_2)}{3}\right)^6, \tag{64}$$

and

$$-\psi^{3k+1} = d^3\left(\frac{r_1(r_1-r_2)}{3}\right)^3 \frac{1}{\psi_2}. \tag{65}$$

Combined with (63), we obtain

$$\left(\frac{r_2}{2}\right)^3 = \left(\frac{r_1-r_2}{3}\right)^3. \tag{66}$$

Suppose that z_6 is a r_2 point of f of multiplicity 6. We see that z_6 is a simple pole of g. By considering the Laurent expansions of f and g in a neighborhood of z_6, we can deduce that

$$\psi = \left(\frac{r_2(r_2-r_1)}{6}\right)^6, \tag{67}$$

and

$$-\psi^{3k+1} = d^3\left(\frac{r_2(r_2-r_1)}{6}\right)^3 \frac{1}{\psi_2}. \tag{68}$$

Combined with (63), we get

$$\left(\frac{r_1}{2}\right)^3 = \left(\frac{r_2 - r_1}{6}\right)^3. \tag{69}$$

From (66) and (69), we get $r_2^3 + 8r_1^3 = 0$. Therefore, we have $r_2 = -2r_1$ or $r_2^2 = 2r_1r_2 - 4r_1^2$. In both cases, we get $r_1 = 0$ from (66), which contradicts the assumption. Hence Eq. (58) has no nonconstant solution. This also completes the proof of Theorem 1.

4. Proof of Theorem 2

Suppose that (f, g) is a pair of nonconstant meromorphic solution of (2). Then all zeros of f and all 1-points of f are poles of g, and poles of f come from zeros, 1-points of g, or from zeros of g'. By Nevanlinna's second fundamental theorem, we have

$$\begin{aligned} T(r,f) &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}(r,f) + S(r,f) \\ &\le \overline{N}(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right) + S(r,f) \\ &\le 5T(r,g) + S(r,f). \end{aligned}$$

Similarly, we have $T(r,g) \le 5T(r,f) + S(r,g)$. Therefore, $S(r,f) = S(r,g) := S(r)$.

Suppose that z_0 is a zero of f with multiplicity n. Then from (2), z_0 is a pole of g with multiplicity p. Since the right-hand side of (2) is a nonzero constant, we have $2(k+1)n - 1 = (3k+2)p + 1$. If $p = 1$, then $2(k+1)n = 3(k+1) + 1$, and thus 1 is divisible by $k+1$. This is impossible. Therefore, $p \ge 2$, and $2(k+1)n \ge 2(3k+2) + 2 = 6(k+1)$. Hence $n \ge 3$. This means that all zeros of f have multiplicities ≥ 3. Similarly, all zeros of g have multiplicities ≥ 3. By a similar argument, we can see that all 1-points of f and g have multiplicities ≥ 6. And all poles of f have multiplicities ≥ 2 if they are not zeros of g'. All poles of g have multiplicities ≥ 2 if they are not zeros of f'. Denote by $N'(r,f)$ the counting function of those poles of f which are zeros or 1-points of g, and by $N''(r,f)$ the counting function of those poles of f which are not zeros and 1-points of g. Then $N(r,f) = N'(r,f) + N''(r,f)$ and $\overline{N}''(r,f) \le \overline{N}(r, 1/g')$. The notations $N'(r,g)$ and $N''(r,g)$ are defined similarly. Using Nevanlinna's second fundamental theorem to the function f and g, respectively, we get

$$\begin{aligned} T(r,f) &\le \overline{N}'(r,f) + \overline{N}''(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) - N_0\left(r,\frac{1}{f'}\right) + S(r), \\ T(r,g) &\le \overline{N}'(r,g) + \overline{N}''(r,g) + \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) - N_0\left(r,\frac{1}{g'}\right) + S(r), \end{aligned}$$

where $N_0(r, 1/f')$ is the counting function of those zeros of f' which are not zeros and 1-points of f. The notation $N_0(r, 1/g')$ is defined similarly. Note that $\overline{N}''(r, f) \leq N_0(r, 1/g')$ and $\overline{N}''(r, g) \leq N_0(r, 1/f')$. By adding the above two inequalities together, we get

$$\begin{aligned} T(r,f)+T(r,g) &\leq \overline{N}'(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right) \\ &\quad + \overline{N}'(r,g)+\overline{N}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g-1}\right)+S(r) \\ &\leq \frac{1}{2}N'(r,f)+\frac{1}{3}N\left(r,\frac{1}{f}\right)+\frac{1}{6}N\left(r,\frac{1}{f-1}\right) \\ &\quad + \frac{1}{2}N'(r,g)+\frac{1}{3}N\left(r,\frac{1}{g}\right)+\frac{1}{6}N\left(r,\frac{1}{g-1}\right)+S(r) \\ &\leq T(r,f)+T(r,g)+S(r). \end{aligned}$$

The above inequalities show that "almost all" poles of f and g have multiplicities 2, "almost all" zeros of f and g have multiplicities 3, and "almost all" 1-points of f and g have multiplicities 6. Let

$$\alpha_1 = \frac{(f')^6}{f^4(f-1)^5}, \quad \alpha_2 = \frac{(g')^6}{g^4(g-1)^5}. \tag{70}$$

Then α_1 and α_2 are small functions of f and g, and $\alpha_1\alpha_1 \not\equiv 0$. The above equations imply $m(r,f) = S(r)$ and $m(r,g) = S(r)$. Let

$$\alpha = f^2(f-1)g^2(g-1). \tag{71}$$

Then α is also a small function of f and g, not identically zero. By considering the Laurent expansion of f at a zero of f with multiplicity 3, we can show that $\frac{\alpha_1}{\alpha_2} = (\frac{3}{2})^6\alpha$. Similarly, by considering the Laurent expansions of g at a zero of g with multiplicity 3, we get $\frac{\alpha_2}{\alpha_1} = (\frac{3}{2})^6\alpha$. Therefore, $\alpha_1 = \alpha_2$ or $\alpha_1 = -\alpha_2$, and

$$\alpha = \left(\frac{2}{3}\right)^6 \frac{\alpha_1}{\alpha_2}.$$

On the other hand, by considering the Laurent expansion of f at a 1-point of f with multiplicity 6, we can obtain $\alpha = (1/3)^6\frac{\alpha_1}{\alpha_2}$, which contradicts the above equation. Hence equation (2) has no nonconstant meromorphic solutions.

5. Proof of Theorem 3

Suppose that (f, g) is a pair of nonconstant meromorphic solution of (3). By an argument similar to that in the proof of Theorem 2, we can prove that "almost all"

1-points and -1 points of f and g have multiplicities 4, and "almost all" poles of f and g have multiplicities 2. Let

$$\beta_1 = \frac{(f')^4}{(f-1)^3(f+1)^3}, \quad \beta_2 = \frac{(g')^4}{(g-1)^3(g+1)^3}. \tag{72}$$

Then $\beta_1\beta_2 \not\equiv 0$. Both β_1 and β_2 are small functions of f and g. Let

$$\beta = (f-1)(f+1)(g-1)(g+1). \tag{73}$$

Then β is also a small function of f and g, and $\beta \not\equiv 0$. By considering the Laurent expansions of f at a 1-point, and a -1-point of f, respectively. We can show that $\beta_1 = \beta_2 = c^2$, and $\beta = 1$. Then f is a solution of the following equation:

$$(f')^4 = c^2(f-1)^3(f+1)^3, \tag{74}$$

and

$$g = -\frac{1}{c}\frac{f(f')^2}{(f-1)^2(f+1)^2}. \tag{75}$$

By Lemma 5, we get $f = 1 + \frac{(\wp - c/2)^2}{c\wp}$, where $\wp$ is the $\wp$-function satisfies (6). Hence the conclusion can be proved easily.

6. Proof of Corollary 1

Let $F = H(f) = (f-a)^n(f-b)^m$ and $G = H(g) = (g-a)^n(g-b)^m$. Then we get $F' = (n+m)(f-c)(f-a)^{n-1}(f-b)^{m-1}f'$ and $G' = (n+m)(g-c)(g-a)^{n-1}(g-b)^{m-1}g'$, where $c = (ma+nb)/(m+n)$. We have

$$N_2(r, F') = 2\overline{N}(r, f) \le 2T(r, f), \ N_2(r, G') = 2\overline{N}(r, g) \le 2T(r, g), \tag{76}$$

$$\begin{aligned} N_2\left(r, \tfrac{1}{F'}\right) &\le 2\overline{N}\left(r, \frac{1}{f-a}\right) + 2\overline{N}\left(r, \frac{1}{f-b}\right) + N_2\left(r, \frac{1}{f-c}\right) + N_0\left(r, \frac{1}{f'}\right) \\ &\le 4T(r, f) + N_0\left(r, \frac{1}{f'}\right) + N_2\left(r, \frac{1}{f-c}\right) + O(1), \end{aligned} \tag{77}$$

and

$$N_2\left(r, \frac{1}{G'}\right) \le 4T(r, g) + N_0\left(r, \frac{1}{g'}\right) + N_2\left(r, \frac{1}{g-c}\right) + O(1), \tag{78}$$

where $N_0(r, 1/f')$ is the counting function of those zeros of f' that are not zeros of $(f-a)(f-b)(f-c)$. Note that F' and G' share the value 1 CM. If $F' \not\equiv G'$ and $F'G' \not\equiv 1$, then by Lemma 3 and inequalities (76), (77) and (78), we have

$$T(r,F') \le 6T(r,f) + N_0\left(r,\frac{1}{f'}\right) + N_2\left(r,\frac{1}{f-c}\right) + 6T(r,g) \tag{79}$$
$$+N_0\left(r,\frac{1}{g'}\right) + N_2\left(r,\frac{1}{g-c}\right) + S(r,f) + S(r,g).$$

Since

$$m\left(r,\frac{1}{F}\right) \le m\left(r,\frac{1}{F'}\right) + m\left(r,\frac{F'}{F}\right) + O(1),$$

we have

$$T(r,F) - N\left(r,\frac{1}{F}\right) \le T(r,F') - N\left(r,\frac{1}{F'}\right) + S(r,f),$$

which implies

$$(n+m)T(r,f) \le T(r,F') + \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{f-b}\right)$$
$$- N\left(r,\frac{1}{f-c}\right) - N_0\left(r,\frac{1}{f'}\right) + S(r,f).$$

From this and (79), we get

$$(n+m-8)T(r,f) \le 9T(r,g) + S(r,f) + S(r,g).$$

Symmetrically, we have

$$(n+m-8)T(r,g) \le 9T(r,f) + S(r,f) + S(r,g).$$

These two inequalities yield

$$(n+m-17)(T(r,f)+T(r,g)) \le S(r,f) + S(r,g),$$

which is impossible when $n+m \ge 18$. Hence $F'G' = 1$ or $F' = G'$.

By Theorem 1, we can rule out the case $F'G' = 1$. Therefore, we have $F' = G'$, and thus there exists a constant C such that $H(f) = H(g) + C$, i.e., $(f-a)^n(f-b)^m = (g-a)^n(g-b)^m + C$. This implies $T(r,f) = T(r,g) + O(1)$. If $C \ne 0$, then by the second fundamental theorem, we get

$$T(r,H(f)) \le \overline{N}(r,H(f)) + \overline{N}\left(r,\frac{1}{H(f)}\right) + \overline{N}\left(r,\frac{1}{H(f)-C}\right) + S(r,H(f)),$$

which yields

$$(n+m)T(r,f) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}\left(r,\frac{1}{f-b}\right)$$
$$+ \overline{N}\left(r,\frac{1}{g-a}\right) + \overline{N}\left(r,\frac{1}{g-b}\right) + S(r,f).$$

From the above inequality, we get $n+m\le 5$, which contradicts the assumption. Therefore, $C=0$. It follows that $(f-a)^n(f-b)^m=(g-a)^n(g-b)^m$, i.e.,

$$\left(\frac{f-a}{g-a}\right)^n=\left(\frac{g-b}{f-b}\right)^n.$$

Since m and n are relatively prime, there exist two integers u and v such that $um+vn=1$. Let $h=[(f-a)/(g-a)]^u[(g-b)/(f-b)]^v$. It follows from the above equation that $(f-a)/(g-a)=h^m$, $(g-b)/(f-b)=h^n$. From these two equations, we see that h is not constant and

$$f=\frac{bh^{m+n}+(a-b)h^m-a}{h^{m+n}-1},\quad g=\frac{ah^{m+n}+(b-a)h^m-b}{h^{m+n}-1},$$

which completes the proof of Corollary 1.

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