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PROXIMAL POINT METHODS FOR MONOTONE OPERATORS IN BANACH SPACES

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Abstract. Some fundamental properties of resolvents of monotone operators in Banach spaces are investigated. Using them, we study the asymptotic behavior of the sequences generated by two modifications of the proximal point algorithm for monotone operators satisfying a range condition defined in Banach spaces.

1. Introduction

The following is a well-known theorem due to Rockafellar [34] on the *proximal point algorithm* for maximal monotone operators in Hilbert spaces:

Theorem 1.1. ([34]). *Let H be a Hilbert space, $A \subset H \times H$ a maximal monotone operator, J_r the resolvent of A defined by $J_r = (I + rA)^{-1}$ for all $r > 0$, and $\{x_n\}$ a sequence defined by $x_1 = x \in H$ and*

$$x_{n+1} = J_{r_n} x_n \tag{1.1}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of positive real numbers such that $\{r_n\}$ is bounded away from zero. Then the following hold:

(1) *$\{x_n\}$ is bounded if and only if $A^{-1}0$ is nonempty;*
(2) *if $A^{-1}0$ is nonempty, then $\{x_n\}$ converges weakly to an element of $A^{-1}0$.*

Later, Reich [29] generalized Theorem 1.1 for m-accretive operators in Banach spaces. It is known that the resolvent J_r of an accretive operator A in a Banach space is a single-valued firmly nonexpansive mapping in the sense of Bruck [10]; see

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also Bruck & Reich [11]. Güler [15] showed that the sequence $\{x_n\}$ in Theorem 1.1 does not converge strongly in general.

Motivated by Mann's type [26, 29] and Halpern's type [16, 35] iterative methods for nonexpansive mappings, Kamimura & Takahashi [19] introduced the following modifications of the proximal point algorithm in a Hilbert space H:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n \tag{1.2}$$

and

$$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) J_{r_n} x_n \tag{1.3}$$

for all $n \in \mathbb{N}$, where $x_1 \in H$, $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{r_n\}$ is a sequence of $(0, \infty)$. Then they established weak and strong convergence theorems for sequences generated by (1.2) and (1.3), respectively; see also Eckstein & Bertsekas [14] on (1.2) and Xu [42] on (1.3). Subsequently, Kamimura & Takahashi [20] also generalized their convergence theorems in Hilbert spaces for an accretive operator $A \subset E \times E$ satisfying

$$D(A) \subset C \subset \bigcap_{r>0} R(I + rA) \tag{1.4}$$

for some nonempty closed convex subset C of a Banach space E; see also Kamimura & Takahashi [21] and Takahashi [37]. Later, motivated by Censor & Reich's definition of convex combination [12] based on Bregman distances, Kamimura, Kohsaka, & Takahashi [18] and Kohsaka & Takahashi [23] obtained the following theorems, which are generalizations of Kamimura & Takahashi's theorems [19]; see also Kamimura [17] and Takahashi [38]:

Theorem 1.2. ([18]). *Let E be a uniformly convex and uniformly smooth Banach space, $A \subset E \times E^*$ a maximal monotone operator such that $A^{-1}0$ is nonempty, Q_r the resolvent of A defined by $Q_r = (J + rA)^{-1} J$ for all $r > 0$, and $\{x_n\}$ a sequence defined by $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1} (\alpha_n J x_n + (1 - \alpha_n) J Q_{r_n} x_n) \tag{1.5}$$

for all $n \in \mathbb{N}$, where J denotes the duality mapping from E into E^*, $\{\alpha_n\}$ is a sequence of $[0, 1]$ such that $\limsup_n \alpha_n < 1$, and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\liminf_n r_n > 0$. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to the strong limit of $\{\Pi_{A^{-1}0}(x_n)\}$, where $\Pi_{A^{-1}0}$ denotes the generalized projection from E onto $A^{-1}0$.

Theorem 1.3. ([23]). *Let E be a smooth and uniformly convex Banach space, $A \subset E \times E^*$ a maximal monotone operator such that $A^{-1}0$ is nonempty, Q_r the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$, and $\{x_n\}$ a sequence defined by $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1}\left(\alpha_n Jx + (1 - \alpha_n) JQ_{r_n} x_n\right) \tag{1.6}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0,1]$ and $\{r_n\}$ is a sequence of $(0,\infty)$ such that

$$\lim_{n\to 0} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \textit{ and } \lim_{n\to\infty} r_n = \infty. \tag{1.7}$$

Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0}(x)$.

Recently, the second and third authors [24, 25] introduced the classes of *firmly nonexpansive-type mappings* and *nonspreading mappings* in Banach spaces. Let C be a nonempty closed convex subset of a smooth Banach space E, J the duality mapping from E into E^*, and T a mapping from C into itself. Then T is said to be of *firmly nonexpansive type* if

$$\langle Tx - Ty, JTx - JTy \rangle \le \langle Tx - Ty, Jx - Jy \rangle \tag{1.8}$$

for all $x, y \in C$. It should be noted that such a mapping T belongs to the class of D-*firm operators* introduced by Bauschke, Borwein, & Combettes [8], where D denotes a Bregman distance. It is known that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \le \phi(Tx, y) + \phi(Ty, x) \tag{1.9}$$

for all $x, y \in C$, where ϕ is defined by (2.6); see [8, 24]. The mapping T is also said to be *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x) \tag{1.10}$$

for all $x, y \in C$. We know that every firmly nonexpansive-type mapping is nonspreading. We also know that if E is a smooth, strictly convex, and reflexive Banach space and $A \subset E \times E^*$ is a maximal monotone operator, then for each $r > 0$, the resolvent Q_r of A defined by $Q_r = (J + rA)^{-1}J$ is a firmly nonexpansive-type mapping from E onto the domain $D(A)$ of A and $F(Q_r) = A^{-1}0$ holds, where $F(Q_r)$ denotes the set of fixed points of Q_r; see [24, 25]. They showed fixed point theorems for nonspreading mappings and convergence theorems for firmly nonexpansive-type mappings in Banach spaces.

The purpose of the present paper is to obtain further generalizations of Theorems 1.1, 1.2, and 1.3 for a monotone operator $A \subset E \times E^*$ satisfying

$$D(A) \subset C \subset \bigcap_{r>0} J^{-1}R(J + rA) \tag{1.11}$$

for some nonempty closed convex subset C of a smooth, strictly convex, and reflexive Banach space E. It should be noted that we do not assume the maximality of A. We also show the equivalence between the existence of a zero point of A and the boundedness of the sequence $\{Q_{r_n}x_n\}$. The proofs of our results are based on the techniques developed in [24, 25].

The present paper is organized as follows: In Section 2, we give some definitions and state some known results. In Section 3, we prove some fundamental lemmas for resolvents of monotone operators (Lemmas 3.1, 3.5, and 3.6). In Section 4, we prove two existence theorems on proximal point methods for monotone operators in Banach spaces (Theorems 4.1 and 4.2). In Section 5, we obtain generalizations of Theorems 1.2 and 1.3 (Theorems 5.1 and 5.2). In Section 6, using the results obtained in Sections 4 and 5, we deduce some results. In Section 7, we apply our results to the problem of finding minimizers of convex functions in Banach spaces and the problem of finding fixed points of nonexpansive mappings in Hilbert spaces.

2. Preliminaries

Throughout the present paper, every linear space is real. We denote the set of positive integers and the set of real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let E be a Banach space and E^* the dual of E. The value of $x^* \in E^*$ at $x \in E$ is denoted by $\langle x, x^* \rangle$. Strong convergence and weak convergence of $\{x_n\}$ to $x \in E$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \tag{2.1}$$

for all $x \in E$. A subset $A \subset E \times E^*$ is sometimes identified with the mapping $\widehat{A}\colon E \to 2^{E^*}$ defined by

$$\widehat{A}x = \{x^* \in E^* : (x, x^*) \in A\} \tag{2.2}$$

for all $x \in E$. Such a subset $A \subset E \times E^*$ is called an operator. An operator $A \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if $A = B$ whenever $B \subset E \times E^*$ is a monotone operator and $A \subset B$. It is well-known that if $A \subset E \times E^*$ is maximal monotone, then $A^{-1}0$ is closed and convex. The *domain* and the *range* of an operator A are defined by $D(A) = \{x \in E : Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in E} Ax$, respectively. By Rockafellar's theorem [31, 32], if $f\colon E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then the *subdifferential* $\partial f \subset E \times E^*$ of f defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y) \quad (\forall y \in E)\} \tag{2.3}$$

for all $x \in E$ is a maximal monotone operator. In this case,

$$(\partial f)^{-1}(0) = \left\{ u \in E : f(u) = \min_{y \in E} f(y) \right\}. \tag{2.4}$$

We also know that $J = \partial(\|\cdot\|^2/2)$.

Let E be a Banach space and $S(E)$ the unit sphere of E. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is also said to be *uniformly convex* if for all $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that $\|(x+y)/2\| \leq 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. The space E is said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.5}$$

exists for all $x, y \in S(E)$. The norm of E is also said to be *uniformly Gâteaux differentiable* if for all $y \in S(E)$, the limit (2.5) converges uniformly in $x \in S(E)$. It is also said to be *uniformly smooth* if the limit (2.5) converges uniformly in $x, y \in S(E)$. We know the following; see, for instance, Cioranescu [13], Reich [30], and Takahashi [39, 40]:

(1) If E is smooth, then J is single-valued;

(2) if E is reflexive, then J is onto;

(3) if E is strictly convex, then J is one-to-one;

(4) if E is strictly convex, then J is strictly monotone, that is, $x = y$ whenever $\langle x - y, Jx - Jy \rangle = 0$.

Let E be a smooth Banach space. Throughout the present paper, let ϕ and V be the mappings defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \tag{2.6}$$

for all $(x, y) \in E \times E$ and

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.7}$$

for all $(x, x^*) \in E \times E^*$. Note that $\phi(x, y) = V(x, Jy)$ for all $x, y \in E$. We know that

(2.8) $$\phi(a,d)+\phi(b,c)-\phi(a,c)-\phi(b,d)=2\langle a-b, Jc-Jd\rangle$$

for all $a,b,c,d\in E$. In particular,

(2.9) $$\phi(a,b)=\phi(a,c)+\phi(c,b)+2\langle a-c, Jc-Jb\rangle$$

for all $a,b,c\in E$. We know the following lemma:

Lemma 2.1. ([22]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of E such that $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_n \phi(x_n,y_n)=0$. Then $\lim_n \|x_n-y_n\|=0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Then for all $x\in E$, there exists a unique $x_0\in C$ (denoted by $\Pi_C x$) such that $\phi(x_0,x)=\min_{y\in C}\phi(y,x)$. The mapping Π_C is called the *generalized projection* from E onto C; see Alber [1], Alber & Reich [2], and Kamimura & Takahashi [22]. We know the following lemmas:

Lemma 2.2. ([1]; see also [22]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $x\in E$, and $x_0\in C$. Then $x_0=\Pi_C x$ if and only if $\langle y-x_0, Jx-Jx_0\rangle\le 0$ for all $y\in C$.*

Lemma 2.3. ([1]; see also [22]). *Let E be a smooth, strictly convex, and reflexive Banach space and C a nonempty closed convex subset of E. Then*

(2.10) $$\phi(u,\Pi_C x)+\phi(\Pi_C x,x)\le\phi(u,x)$$

for all $u\in C$ and $x\in E$.

Lemma 2.4. ([4]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable and let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences of E such that $\lim_n\|x_n-y_n\|=0$. Then $\lim_n\big(\phi(p,x_n)-\phi(p,y_n)\big)=0$ for all $p\in E$.*

Lemma 2.5. ([23]). *Let E be a smooth, strictly convex, and reflexive Banach space. Then*

(2.11) $$V(x,x^*)+2\left\langle J^{-1}x^*-x, y^*\right\rangle\le V(x,x^*+y^*)$$

for all $x\in E$ and $x^*,y^*\in E^*$.

Lemma 2.6. ([27]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, and T a mapping from C into itself such that $F(T)$ is nonempty and $\phi(u,Tx)\le\phi(u,x)$ for all $u\in F(T)$ and $x\in C$. Then $F(T)$ is closed and convex.*

Lemma 2.7. ([5]). *Let E be a smooth and uniformly convex Banach space, $\{s_n\}$ a convergent sequence of real numbers, and $\{u_n\}$ a sequence of E such that*

$$\phi(u_n, u_m) \leq |s_n - s_m| \tag{2.12}$$

for all $m, n \in \mathbb{N}$ with $n < m$. Then $\{u_n\}$ converges strongly.

We also know the following:

Lemma 2.8. ([42, 3]). *Let $\{s_n\}$ be a sequence of $[0,\infty)$, $\{t_n\}$ a sequence of real numbers such that $\limsup_n t_n \leq 0$, $\{u_n\}$ a sequence of $[0,\infty)$ such that $\sum_{n=1}^{\infty} u_n < \infty$, and $\{\alpha_n\}$ a sequence of $[0,1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and*

$$s_{n+1} \leq (1-\alpha_n)\, s_n + \alpha_n t_n + u_n \tag{2.13}$$

for all $n \in \mathbb{N}$. Then $\lim_n s_n = 0$.

3. Some Properties of Resolvents of Monotone Operators

This section is devoted to the study of some properties of resolvents of monotone operators in Banach spaces.

Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, $A \subset E \times E^*$ a monotone operator satisfying (1.11). For each $r > 0$, we can define the resolvent Q_r of A by

$$Q_r x = \{z \in E : Jx \in Jz + rAz\} \tag{3.1}$$

for all $x \in C$. In other words, $Q_r x = (J + rA)^{-1} Jx$ for all $x \in C$. We know that Q_r is a single-valued mapping from C into $D(A)$, $(Jx - JQ_r x)/r \in AQ_r x$ for all $x \in C$ and $r > 0$, and $F(Q_r) = A^{-1}0$, where $F(Q_r)$ is the set of fixed points of Q_r. Motivated by Aoyama & Takahashi [6] and Kohsaka & Takahashi [24], we prove the following lemma. This lemma plays important roles in the present paper:

Lemma 3.1. *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, $A \subset E \times E^*$ a monotone operator satisfying* (1.11), *and Q_r the resolvent of A defined by $Q_r = (J + rA)^{-1} J$ for all $r > 0$. Then*

$$\begin{aligned} r\phi(Q_r x, Q_s y) + s\phi(Q_s y, Q_r x) + s\phi(Q_r x, x) + r\phi(Q_s y, y) \\ \leq r\phi(Q_r x, y) + s\phi(Q_s y, x) \end{aligned} \tag{3.2}$$

for all $x, y \in C$ and $r, s > 0$.

Proof. Put $A_t = (J - JQ_t)/t$ for all $t > 0$. Let $x, y \in C$ and $r, s > 0$ be given. Since $(Q_r x, A_r x), (Q_s y, A_s y) \in A$ and A is monotone, we have

$$\langle Q_r x - Q_s y, A_r x - A_s y \rangle \geq 0, \tag{3.3}$$

which is equivalent to

$$r \langle Q_r x - Q_s y, Jy - JQ_s y \rangle \leq s \langle Q_r x - Q_s y, Jx - JQ_r x \rangle . \tag{3.4}$$

Using (2.8), we obtain

$$\begin{aligned} &r\big(\phi(Q_r x, Q_s y) + \phi(Q_s y, y) - \phi(Q_r x, y) - \phi(Q_s y, Q_s y)\big) \\ &\leq s\big(\phi(Q_r x, Q_r x) + \phi(Q_s y, x) - \phi(Q_r x, x) - \phi(Q_s y, Q_r x)\big). \end{aligned} \tag{3.5}$$

Thus we have the desired result. ∎

In particular, we have the following corollaries:

Corollary 3.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $A \subset H \times H$ a monotone operator satisfying $D(A) \subset C \subset \bigcap_{r>0} R(I + rA)$, and J_r the resolvent of A defined by $J_r = (I + rA)^{-1}$ for all $r > 0$. Then*

$$\begin{aligned} r\|J_r x - J_s y\|^2 + s\|J_s y - J_r x\|^2 + s\|J_r x - x\|^2 + r\|J_s y - y\|^2 \\ \leq r\|J_r x - y\|^2 + s\|J_s y - x\|^2 \end{aligned} \tag{3.6}$$

for all $x, y \in C$ and $r, s > 0$.

Corollary 3.3. ([8]; see also [24]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, $A \subset E \times E^*$ a monotone operator satisfying* (1.11)*, and Q_r the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$. Then for each $r > 0$,*

$$\begin{aligned} \phi(Q_r x, Q_r y) + \phi(Q_r y, Q_r x) + \phi(Q_r x, x) + \phi(Q_r y, y) \\ \leq \phi(Q_r x, y) + \phi(Q_r y, x) \end{aligned} \tag{3.7}$$

for all $x, y \in C$.

Corollary 3.4. (see, for instance, [18]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, $A \subset E \times E^*$ a monotone operator satisfying* (1.11) *and $A^{-1}0$ is nonempty, and Q_r the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$. Then for each $r > 0$,*

$$\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x) \tag{3.8}$$

for all $u \in A^{-1}0$ and $x \in C$.

Using the techniques in [24, 25], we prove the following lemma, which is needed in the proof of Theorem 5.1:

Lemma 3.5. *Let E be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J+rA)^{-1}J$ for all $r > 0$ and $\{r_n\}$ a sequence of $(0,\infty)$ such that $\liminf_n r_n > 0$. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup u$ and $\|x_n - Q_{r_n}x_n\| \to 0$, then u is an element of $A^{-1}0$.*

Proof. Let $s > 0$ and $n \in \mathbb{N}$ be fixed. By Lemma 3.1, we have

$$r_n\phi(Q_{r_n}x_n, Q_s u) + s\phi(Q_s u, Q_{r_n}x_n) \le r_n\phi(Q_{r_n}x_n, u) + s\phi(Q_s u, x_n). \tag{3.9}$$

This implies that

$$\begin{aligned} 0 &\le r_n\big(\phi(Q_{r_n}x_n, u) - \phi(Q_{r_n}x_n, Q_s u)\big) + s\big(\phi(Q_s u, x_n) - \phi(Q_s u, Q_{r_n}x_n)\big) \\ &= r_n\left(2\left\langle Q_{r_n}x_n, JQ_s u - Ju\right\rangle + \|u\|^2 - \|Q_s u\|^2\right) \\ &\quad + s\big(\phi(Q_s u, x_n) - \phi(Q_s u, Q_{r_n}x_n)\big). \end{aligned} \tag{3.10}$$

Thus we have

$$\begin{aligned} 0 \le\ & 2\left\langle Q_{r_n}x_n, JQ_s u - Ju\right\rangle + \|u\|^2 - \|Q_s u\|^2 \\ & + \frac{s}{r_n}\big(\phi(Q_s u, x_n) - \phi(Q_s u, Q_{r_n}x_n)\big). \end{aligned} \tag{3.11}$$

On the other hand, since $x_n \rightharpoonup u$ and $x_n - Q_{r_n}x_n \to 0$, we have $Q_{r_n}x_n \rightharpoonup u$. In particular, $\{x_n\}$ and $\{Q_{r_n}x_n\}$ are bounded. Since $x_n - Q_{r_n}x_n \to 0$ and the norm of E is uniformly Gâteaux differentiable, it also follows from Lemma 2.4 that

$$\lim_{n\to\infty}\big(\phi(Q_s u, x_n) - \phi(Q_s u, Q_{r_n}x_n)\big) = 0. \tag{3.12}$$

By assumption, we know that $\{s/r_n\}$ is bounded. Letting $n \to \infty$ in (3.11), we obtain

$$0 \le 2\left\langle u, JQ_s u - Ju\right\rangle + \|u\|^2 - \|Q_s u\|^2 = -\phi(u, Q_s u). \tag{3.13}$$

This implies that $\phi(u, Q_s u) = 0$. By the strict convexity of E, we obtain $Q_s u = u$. Thus u is an element of $A^{-1}0$. ■

Similarly, we prove the following lemma, which is needed in the proof of Theorems 4.2 and 5.2:

Lemma 3.6. *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{r_n\}$ a sequence of $(0, \infty)$ such that $\lim_n r_n = \infty$. If $\{x_n\}$ is a bounded sequence of C such that $Q_{r_n}x_n \rightharpoonup u$, then u is an element of $A^{-1}0$.*

Proof. Let $s > 0$ and $n \in \mathbb{N}$ be fixed. As in the proof of Lemma 3.5, we can show

$$\begin{aligned} 0 &\leq 2\langle Q_{r_n}x_n, JQ_s u - Ju\rangle + \|u\|^2 - \|Q_s u\|^2 \\ &\quad + \frac{s}{r_n}\left(2\langle Q_s u, JQ_{r_n}x_n - Jx_n\rangle + \|x_n\|^2 - \|Q_{r_n}x_n\|^2\right). \end{aligned} \tag{3.14}$$

Since $\{x_n\}$ and $\{Q_{r_n}x_n\}$ are bounded, $Q_{r_n}x_n \rightharpoonup u$, and $\lim_n r_n = \infty$, letting $n \to \infty$ in (3.14), we obtain

$$0 \leq 2\langle u, JQ_s u - Ju\rangle + \|u\|^2 - \|Q_s u\|^2 = -\phi(u, Q_s u). \tag{3.15}$$

Thus u is an element of $A^{-1}0$. ∎

4. Existence Theorems

In this section, we obtain two existence theorems on proximal point methods for monotone operators in Banach spaces.

We first prove the following theorem, which is a generalization of the part (1) of Theorem 1.1 and the result due to Matsushita & Takahashi [28]. The proof is based on the techniques in [24, 25, 36]:

Theorem 4.1. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = Q_{r_n}x_n; \\ x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx_n + (1 - \alpha_n) Jy_n\right) \end{cases} \tag{4.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ such that $\limsup_n \alpha_n < 1$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\sum_{n=1}^{\infty} r_n = \infty$. Then the following are equivalent:

(1) *$\{y_n\}$ is bounded;*
(2) *$A^{-1}0$ is nonempty.*

Proof. By assumption, without loss of generality, we may suppose that there exists $\alpha \in \mathbb{R}$ such that $\alpha_n \leq \alpha < 1$ for all $n \in \mathbb{N}$.

We first show that (2) implies (1). Suppose that $A^{-1}0$ is nonempty and $u \in A^{-1}0$. Then we have from Lemma 2.3 and Corollary 3.4 that

$$\begin{aligned}
\phi(u, x_{n+1}) &\leq \phi\left(u, J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n)\, Jy_n\right)\right) \\
&= V(u, \alpha_n Jx_n + (1-\alpha_n)\, Jy_n) \\
&\leq \alpha_n V(u, Jx_n) + (1-\alpha_n)\, V(u, Jy_n) \\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\, \phi(u, y_n) \qquad (4.2) \\
&\leq \alpha_n \phi(u, x_n) + (1-\alpha_n)\left(\phi(u, x_n) - \phi(y_n, x_n)\right) \\
&\leq \phi(u, x_n) - (1-\alpha_n)\, \phi(y_n, x_n) \\
&\leq \phi(u, x_n)
\end{aligned}$$

for all $n \in \mathbb{N}$. Thus $\lim_n \phi(u, x_n)$ exists and $\{x_n\}$ is bounded. It also follows from $\phi(u, y_n) \leq \phi(u, x_n)$ that $\{y_n\}$ is bounded.

We next show that (1) implies (2). Suppose that $\{y_n\}$ is bounded. Let $s > 0$, $n \in \mathbb{N}$, $y \in C$, and $k \in \{1, 2, \dots, n\}$ be given. Then, by Lemma 3.1 and (2.9), we have

$$\begin{aligned}
&r_k \phi(y_k, Q_s y) + s\phi(Q_s y, y_k) \\
&\leq r_k \phi(y_k, y) + s\phi(Q_s y, x_k) \qquad (4.3) \\
&= r_k \left(\phi(y_k, Q_s y) + \phi(Q_s y, y) + 2\left\langle y_k - Q_s y, JQ_s y - Jy\right\rangle\right) + s\phi(Q_s y, x_k).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 \leq\ & s\left(\phi(Q_s y, x_k) - \phi(Q_s y, y_k)\right) \\
&+ r_k \phi(Q_s y, y) + 2r_k \left\langle y_k - Q_s y, JQ_s y - Jy\right\rangle . \qquad (4.4)
\end{aligned}$$

Hence we have

$$\begin{aligned}
0 \leq\ & s\big((1-\alpha_k)\phi(Q_s y, x_k) - (1-\alpha_k)\phi(Q_s y, y_k)\big) \\
&+ r_k(1-\alpha_k)\phi(Q_s y, y) + 2r_k(1-\alpha_k)\left\langle y_k - Q_s y, JQ_s y - Jy\right\rangle . \qquad (4.5)
\end{aligned}$$

On the other hand, as well as (4.2), we can see that

$$\phi(Q_s y, x_{k+1}) \leq \alpha_k \phi(Q_s y, x_k) + (1-\alpha_k)\phi(Q_s y, y_k), \qquad (4.6)$$

which implies that

$$-(1-\alpha_k)\phi(Q_s y, y_k) \leq \alpha_k \phi(Q_s y, x_k) - \phi(Q_s y, x_{k+1}). \tag{4.7}$$

By (4.5) and (4.7), we have

$$\begin{aligned} 0 \leq\ & s\big(\phi(Q_s y, x_k) - \phi(Q_s y, x_{k+1})\big) \\ & + r_k(1-\alpha_k)\phi(Q_s y, y) + 2r_k(1-\alpha_k)\langle y_k - Q_s y, JQ_s y - Jy\rangle . \end{aligned} \tag{4.8}$$

Summing these inequalities with respect to $k = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} 0 &\leq s\big(\phi(Q_s y, x_1) - \phi(Q_s y, x_{n+1})\big) \\ &\quad + \sum_{k=1}^{n} r_k(1-\alpha_k)\phi(Q_s y, y) + 2\sum_{k=1}^{n} r_k(1-\alpha_k)\langle y_k - Q_s y, JQ_s y - Jy\rangle \\ &= s\big(\phi(Q_s y, x_1) - \phi(Q_s y, x_{n+1})\big) \\ &\quad + \mu_n \phi(Q_s y, y) + 2\left\langle \sum_{k=1}^{n} r_k(1-\alpha_k) y_k - \mu_n Q_s y, JQ_s y - Jy \right\rangle , \end{aligned} \tag{4.9}$$

where $\mu_n = \sum_{k=1}^{n} r_k(1-\alpha_k)$. Thus we have

$$\begin{aligned} 0 \leq\ & \frac{s}{\mu_n}\big(\phi(Q_s y, x_1) - \phi(Q_s y, x_{n+1})\big) \\ & + \phi(Q_s y, y) + 2\langle z_n - Q_s y, JQ_s y - Jy\rangle , \end{aligned} \tag{4.10}$$

where $z_n = (1/\mu_n)\sum_{k=1}^{n} r_k(1-\alpha_k) y_k$. Since $\{z_n\}$ is a bounded sequence of C, we have a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ which converges weakly to $u \in C$. By assumption, we have $\mu_n \geq (1-\alpha)\sum_{k=1}^{n} r_k \to \infty$. By (4.10), we have

$$0 \leq \phi(Q_s y, y) + 2\langle u - Q_s y, JQ_s y - Jy\rangle = \phi(u, y) - \phi(u, Q_s y). \tag{4.11}$$

This implies that

$$\phi(u, Q_s y) \leq \phi(u, y) \tag{4.12}$$

for all $y \in C$. Putting $y = u$ in the inequality above, we obtain $\phi(u, Q_s u) = 0$. Thus $Q_s u = u$. Therefore u is an element of $A^{-1}0$. This completes the proof. ■

Using Lemma 3.6, we next prove the following existence theorem:

Theorem 4.2. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = Q_{r_n} x_n; \\ x_{n+1} = \Pi_C J^{-1} \left(\alpha_n Jx + (1 - \alpha_n) Jy_n\right) \end{cases} \tag{4.13}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\lim_n r_n = \infty$. Then the following are equivalent:

(1) $\{y_n\}$ *is bounded;*

(2) $A^{-1}0$ *is nonempty.*

Proof. We first prove that (2) implies (1). Suppose that $A^{-1}0$ is nonempty and $u \in A^{-1}0$. Then we have from $\phi(u, y_n) \le \phi(u, x_n)$ that

$$\begin{aligned} \phi(u, x_{n+1}) &\le \alpha_n \phi(u, x) + (1 - \alpha_n)\, \phi(u, y_n) \\ &\le \alpha_n \phi(u, x) + (1 - \alpha_n)\, \phi(u, x_n) \end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N}$. By induction, we can show that $\phi(u, x_n) \le \phi(u, x)$ for all $n \in \mathbb{N}$. This shows that $\{x_n\}$ is bounded and hence $\{y_n\}$ is bounded.

We next prove that (1) implies (2). Put $z_n = J^{-1}\left(\alpha_n Jx + (1 - \alpha_n)\, y_n\right)$ for all $n \in \mathbb{N}$. Suppose that $\{y_n\}$ is bounded. Then $\{z_n\}$ is obviously bounded. Since $\phi(p, x_{n+1}) = \phi(p, \Pi_C z_n) \le \phi(p, z_n)$ for all $p \in C$ and $n \in \mathbb{N}$, $\{x_n\}$ is also bounded. Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup u$. By $\lim_i r_{n_i} = \infty$ and Lemma 3.6, u is an element of $A^{-1}0$. This completes the proof. ■

5. Convergence Theorems

In this section, we obtain two convergence theorems on proximal point methods for monotone operators in Banach spaces.

We first prove the following weak convergence theorem, which is a generalization of Theorem 1.2. It should be noted that the uniform smoothness in Theorem 1.2 is replaced by the uniform Gâteaux differentiability:

Theorem 5.1. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A*

defined by $Q_r = (J + rA)^{-1}J$ *for all* $r > 0$ *and* $\{x_n\}$ *a sequence of* C *defined by* $x_1 = x \in C$ *and*

$$\begin{cases} y_n = Q_{r_n} x_n; \\ x_{n+1} = \Pi_C J^{-1} \left(\alpha_n J x_n + (1 - \alpha_n) J y_n\right) \end{cases} \tag{5.1}$$

for all $n \in \mathbb{N}$*, where* $\{\alpha_n\}$ *is a sequence of* $[0, 1]$ *such that* $\limsup_n \alpha_n < 1$ *and* $\{r_n\}$ *is a sequence of* $(0, \infty)$ *such that* $\liminf_n r_n > 0$*. If* $A^{-1}0$ *is nonempty, then the following hold:*

(1) $\{x_n\}$ *is bounded and every weak subsequential limit of* $\{x_n\}$ *is an element of* $A^{-1}0$*;*

(2) *if* J *is weakly sequentially continuous, then* $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to the strong limit of* $\{\Pi_{A^{-1}0}(x_n)\}$*.*

Proof. It should be noted that Lemma 2.6 and Corollary 3.4 ensure that $A^{-1}0$ is closed and convex. Thus $\Pi_{A^{-1}0}$ is well-defined.

We first show the part (1). As in the proof of Theorem 4.1, we know that $\{x_n\}$ and $\{y_n\}$ are bounded,

$$\phi(u, x_{n+1}) \le \phi(u, x_n), \tag{5.2}$$

and

$$(1 - \alpha_n)\, \phi(y_n, x_n) \le \phi(u, x_n) - \phi(u, x_{n+1}) \tag{5.3}$$

for all $u \in A^{-1}0$ and $n \in \mathbb{N}$. Since $\limsup_n \alpha_n < 1$, we have $\lim_n \phi(y_n, x_n) = 0$. Then Lemma 2.1 implies that

$$\lim_{n\to\infty} \|x_n - Q_{r_n} x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{5.4}$$

Since $\liminf_n r_n > 0$, Lemma 3.5 ensures that every weak subsequential limit of $\{x_n\}$ lies in $A^{-1}0$.

We next show the part (2). Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$. It follows from the part (1) that $u \in A^{-1}0$. Let us denote $\Pi_{A^{-1}0}$ by Π. It follows from (5.2) and the definition of Π that

$$\phi(\Pi x_{n+1}, x_{n+1}) \le \phi(\Pi x_n, x_{n+1}) \le \phi(\Pi x_n, x_n) \tag{5.5}$$

for all $n \in \mathbb{N}$. Thus $\lim_n \phi(\Pi x_n, x_n)$ exists. If $m, n \in \mathbb{N}$ and $n < m$, then it follows from Lemma 2.3 that

$$\begin{aligned} \phi(\Pi x_n, \Pi x_m) &\le \phi(\Pi x_n, x_m) - \phi(\Pi x_m, x_m) \\ &\le \phi(\Pi x_n, x_n) - \phi(\Pi x_m, x_m). \end{aligned} \tag{5.6}$$

By Lemma 2.7, we know that $\{\Pi x_n\}$ converges strongly to $v \in A^{-1}0$. On the other hand, by Lemma 2.2 we have

$$\langle u - \Pi x_n, Jx_n - J\Pi x_n\rangle \le 0 \tag{5.7}$$

for all $n \in \mathbb{N}$. It follows from (5.7) and the weak sequential continuity of J that $\langle u - v, Ju - Jv\rangle \le 0$. This implies that $\langle u - v, Ju - Jv\rangle = 0$. By the strict convexity of E, we have $u = v$. Therefore, $\{x_n\}$ converges weakly to $v = \lim_n \Pi x_n$. By (5.4), $\{y_n\}$ is also weakly convergent to v. This completes the proof. ■

We next prove the following strong convergence theorem, which is a generalization of Theorem 1.3:

Theorem 5.2. *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = Q_{r_n} x_n; \\ x_{n+1} = \Pi_C J^{-1}\left(\alpha_n Jx + (1 - \alpha_n) Jy_n\right) \end{cases} \tag{5.8}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that

$$\lim_{n\to 0} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \text{ and } \lim_{n\to\infty} r_n = \infty. \tag{5.9}$$

If $A^{-1}0$ is nonempty, then $\{x_n\}$ and $\{y_n\}$ converge strongly to $\Pi_{A^{-1}0}(x)$.

Proof. Let us denote $\Pi_{A^{-1}0}$ by Π and put $z_n = J^{-1}\left(\alpha_n Jx + (1 - \alpha_n) Jy_n\right)$ for all $n \in \mathbb{N}$. Since $A^{-1}0$ is nonempty, by Theorem 4.2, $\{y_n\}$ is bounded. By the definition of $\{z_n\}$, we have $Jz_n - Jy_n = \alpha_n(Jx - Jy_n)$ for all $n \in \mathbb{N}$. Thus we have from $\lim_n \alpha_n = 0$ that $Jz_n - Jy_n \to 0$. Since E is uniformly convex, the duality mapping J^{-1} from E^* into E is uniformly norm-to-norm continuous on each bounded subset of E^*; see Takahashi [39, 40]. Thus we obtain

$$\|z_n - y_n\| = \|J^{-1}Jz_n - J^{-1}Jy_n\| \to 0. \tag{5.10}$$

We show that

$$\limsup_{n\to\infty} \langle z_n - \Pi x, Jx - J\Pi x\rangle \le 0. \tag{5.11}$$

In view of (5.10), to show (5.11), it is sufficient to show that

$$\limsup_{n\to\infty} \langle y_n - \Pi x, Jx - J\Pi x\rangle \le 0. \tag{5.12}$$

Since $\{y_n\}$ is bounded, without loss of generality, we have a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\lim_{i\to\infty} \langle y_{n_i} - \Pi x, Jx - J\Pi x\rangle = \limsup_{n\to\infty} \langle y_n - \Pi x, Jx - J\Pi x\rangle \tag{5.13}$$

and $y_{n_i} \rightharpoonup u$. Then, Lemma 3.6 implies that $u \in A^{-1}0$. Thus we have from (5.13) and Lemma 2.2 that

$$\limsup_{n\to\infty} \langle y_n - \Pi x, Jx - J\Pi x\rangle = \langle u - \Pi x, Jx - J\Pi x\rangle \le 0. \tag{5.14}$$

On the other hand, we have from Corollary 3.4 and the definition of Π that

$$\begin{aligned} \phi(\Pi x, y_{n+1}) &\le \phi(\Pi x, x_{n+1}) \\ &\le \phi\left(\Pi x, J^{-1}\left(\alpha_n Jx + (1-\alpha_n) Jy_n\right)\right) \\ &= V\left(\Pi x, \alpha_n Jx + (1-\alpha_n) Jy_n\right) \end{aligned} \tag{5.15}$$

for all $n \in \mathbb{N}$. By Lemma 2.5 and Corollary 3.4, we have

$$\begin{aligned} &V\left(\Pi x, \alpha_n Jx + (1-\alpha_n) Jy_n\right) \\ &\le V\big(\Pi x, \alpha_n Jx + (1-\alpha_n) Jy_n - \alpha_n (Jx - J\Pi x)\big) \\ &\quad - 2\left\langle J^{-1}\left(\alpha_n Jx + (1-\alpha_n) Jy_n\right) - \Pi x, -\alpha_n (Jx - J\Pi x)\right\rangle \\ &= V\left(\Pi x, (1-\alpha_n) Jy_n + \alpha_n J\Pi x\right) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \\ &\le (1-\alpha_n) V(\Pi x, Jy_n) + \alpha_n V(\Pi x, J\Pi x) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \\ &= (1-\alpha_n)\phi(\Pi x, y_n) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \\ &\le (1-\alpha_n)\phi(\Pi x, x_n) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \end{aligned} \tag{5.16}$$

for all $n \in \mathbb{N}$. By (5.15) and (5.16), we have

$$\phi(\Pi x, y_{n+1}) \le (1-\alpha_n)\phi(\Pi x, y_n) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \tag{5.17}$$

and

$$\phi(\Pi x, x_{n+1}) \le (1-\alpha_n)\phi(\Pi x, x_n) + 2\alpha_n \langle z_n - \Pi x, Jx - J\Pi x\rangle \tag{5.18}$$

for all $n \in \mathbb{N}$. By (5.11), (5.17), (5.18), $\sum_{n=1}^{\infty} \alpha_n = \infty$, and Lemma 2.8, we obtain

$$\lim_{n\to\infty} \phi(\varPi x, y_n) = \lim_{n\to\infty} \phi(\varPi x, x_n) = 0. \tag{5.19}$$

By Lemma 2.1, we obtain $\lim_n \|\varPi x - y_n\| = \lim_n \|\varPi x - x_n\| = 0$. This completes the proof. ■

6. Deduced Results

Using Theorems 4.1 and 5.1, we obtain the following two corollaries:

Corollary 6.1. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of E, and $A \subset E \times E^*$ a monotone operator satisfying* (1.11). *Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

$$x_{n+1} = Q_{r_n} x_n \tag{6.1}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\sum_{n=1}^{\infty} r_n = \infty$. Then the following hold:

(1) *$\{x_n\}$ is bounded if and only if $A^{-1}0$ is nonempty;*

(2) *if $A^{-1}0$ is nonempty and $\liminf_n r_n > 0$, then every weak subsequential limit of $\{x_n\}$ is an element of $A^{-1}0$. Further, if J is weakly sequentially continuous, then $\{x_n\}$ and $\{y_n\}$ converge weakly to the strong limit of $\{\varPi_{A^{-1}0}(x_n)\}$.*

Proof. Putting $\alpha_n = 0$ in Theorems 4.1 and 5.1, we obtain the desired result. ■

Corollary 6.2. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable and $A \subset E \times E^*$ a maximal monotone operator. Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of E defined by $x_1 = x \in E$ and*

$$\begin{cases} y_n = Q_{r_n} x_n; \\ x_{n+1} = J^{-1}\left(\alpha_n J x_n + (1 - \alpha_n) J y_n\right) \end{cases} \tag{6.2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\limsup_n \alpha_n < 1$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\sum_{n=1}^{\infty} r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $A^{-1}0$ is nonempty;*

(2) *if $A^{-1}0$ is nonempty and $\liminf_n r_n > 0$, then every weak subsequential limit of $\{x_n\}$ is an element of $A^{-1}0$. Further, if J is weakly sequentially continuous, then $\{x_n\}$ and $\{y_n\}$ converge weakly to the strong limit of $\{\varPi_{A^{-1}0}(x_n)\}$.*

Proof. Since $A \subset E \times E^*$ is a maximal monotone operator, we have $R(J + rA) = E^*$ for all $r > 0$; see [7, 9, 33, 40]. Thus, putting $C = E$ in Theorems 4.1 and 5.1, we obtain the desired result. ∎

Using Theorems 4.2 and 5.2, we similarly obtain the following:

Corollary 6.3. *Let E be a smooth and uniformly convex Banach space and $A \subset E \times E^*$ a maximal monotone operator. Let Q_r be the resolvent of A defined by $Q_r = (J + rA)^{-1}J$ for all $r > 0$ and $\{x_n\}$ a sequence of E defined by $x_1 = x \in E$ and*

$$\begin{cases} y_n = Q_{r_n} x_n; \\ x_{n+1} = J^{-1}\left(\alpha_n Jx + (1-\alpha_n) Jy_n\right) \end{cases} \tag{6.3}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0,1]$ and $\{r_n\}$ is a sequence of $(0,\infty)$ such that $\lim_n r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $A^{-1}0$ is nonempty;*

(2) *if $A^{-1}0$ is nonempty, $\lim_n \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ and $\{y_n\}$ converge strongly to $\Pi_{A^{-1}0}(x)$.*

7. Applications

We first study the problem of minimizing a proper lower semicontinuous convex function in a Banach space.

Corollary 7.1. *Let E be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, $f\colon E \to (-\infty, \infty]$ a proper lower semicontinuous convex function, and $\{x_n\}$ a sequence of E defined by $x_1 = x \in E$ and*

$$\begin{cases} y_n = \arg\min\left\{ f(y) + \dfrac{1}{2r_n}\phi(y, x_n) : y \in E \right\}; \\ x_{n+1} = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) Jy_n\right) \end{cases} \tag{7.1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0,1]$ such that $\limsup_n \alpha_n < 1$ and $\{r_n\}$ is a sequence of $(0,\infty)$ such that $\sum_{n=1}^{\infty} r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $(\partial f)^{-1}(0)$ is nonempty;*

(2) *if $(\partial f)^{-1}(0)$ is nonempty and $\liminf_n r_n > 0$, then every weak subsequential limit of $\{x_n\}$ is an element of $(\partial f)^{-1}(0)$. Further, if J is weakly sequentially continuous, then $\{x_n\}$ and $\{y_n\}$ converge weakly to the strong limit of $\{\Pi_{(\partial f)^{-1}(0)}(x_n)\}$.*

Proof. Let A be an operator defined by $A = \partial f$ and $Q_r = (J + rA)^{-1}J$ for all $r > 0$. Then A is maximal monotone and

$$Q_r z = \arg\min \left\{ f(y) + \frac{1}{2r}\phi(y, z) : y \in E \right\} \tag{7.2}$$

for all $z \in E$ and $r > 0$. Thus, by Corollary 6.2, we obtain the desired result. ■

Using Corollary 6.3, we can similarly obtain the following:

Corollary 7.2. *Let E be a smooth and uniformly convex Banach space and $f\colon E \to (-\infty, \infty]$ a proper lower semicontinuous convex function, and $\{x_n\}$ a sequence of E defined by $x_1 = x \in E$ and*

$$\begin{cases} y_n = \arg\min \left\{ f(y) + \dfrac{1}{2r_n}\phi(y, x_n) : y \in E \right\}; \\ x_{n+1} = J^{-1}\left(\alpha_n Jx + (1 - \alpha_n) Jy_n\right) \end{cases} \tag{7.3}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\lim_n r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $(\partial f)^{-1}(0)$ is nonempty;*

(2) *if $(\partial f)^{-1}(0)$ is nonempty, $\lim_n \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ and $\{y_n\}$ converge strongly to $\Pi_{(\partial f)^{-1}(0)}(x)$.*

We finally study the problem of finding a fixed point of a nonexpansive mapping in a Hilbert space.

Corollary 7.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, T a nonexpansive mapping from C into itself, and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \dfrac{1}{1 + r_n}x_n + \dfrac{r_n}{1 + r_n}Ty_n; \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n \end{cases} \tag{7.4}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0, 1]$ such that $\limsup_n \alpha_n < 1$ and $\{r_n\}$ is a sequence of $(0, \infty)$ such that $\sum_{n=1}^{\infty} r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $F(T)$ is nonempty;*

(2) *if $F(T)$ is nonempty and $\liminf_n r_n > 0$, then $\{x_n\}$ and $\{y_n\}$ converge weakly to the strong limit of $\{P_{F(T)}(x_n)\}$, where $P_{F(T)}$ denotes the metric projection from H onto $F(T)$.*

Proof. Let A be a mapping defined by $A = I - T$. It is well-known that A is a monotone operator from C into H and $D(A) = C \subset \bigcap_{r>0} R(I + rA)$ holds; see Takahashi [39, 40, 41] for more details. Let $J_r = (I + rA)^{-1}$ for all $r > 0$. We know that for each $z \in H$ and $r > 0$, $J_r z$ is the unique point of C such that

(7.5)
$$J_r z = \frac{1}{1+r} z + \frac{r}{1+r} T J_r z.$$

It also holds that $A^{-1}0 = F(T)$. Since H is a Hilbert space, the duality mapping J from H into itself is the identity operator I on H. Thus, by Theorems 4.1 and 5.1, we obtain the desired result. ∎

Using Theorems 4.2 and 5.2, we can similarly show the following:

Corollary 7.4. *Let H be a Hilbert space, C a nonempty closed convex subset of H, T a nonexpansive mapping from C into itself, and $\{x_n\}$ a sequence of C defined by $x_1 = x \in C$ and*

(7.6)
$$\begin{cases} y_n = \dfrac{1}{1+r_n} x_n + \dfrac{r_n}{1+r_n} T y_n; \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)\, y_n \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence of $[0,1]$ and $\{r_n\}$ is a sequence of $(0,\infty)$ such that $\lim_n r_n = \infty$. Then the following hold:

(1) *$\{y_n\}$ is bounded if and only if $F(T)$ is nonempty;*

(2) *if $F(T)$ is nonempty, $\lim_n \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{F(T)}(x)$, where $P_{F(T)}$ denotes the metric projection from H onto $F(T)$.*

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