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q-EXTENSIONS OF SOME RELATIONSHIPS BETWEEN THE BERNOULLI AND EULER POLYNOMIALS

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Abstract. The main object of this paper is to give q-extensions of several explicit relationships of H. M. Srivastava and Á. Pintér [*Appl. Math. Lett.* **17** (2004), 375-380] between the Bernoulii and Euler polynomials. We also derive several other formulas in series of Carlitz's q-Stirling numbers of the second kind.

1. Introduction and Definitions

Throughout this paper, we make use of the following notations. First of all, $\mathbb{C}$ denotes the set of *complex* numbers and

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad (\mathbb{N} := \{1, 2, 3, \cdots\})$$

denotes the set of *nonnegative* integers.

For $q \in \mathbb{C} \quad (|q| < 1)$, the q-shifted factorial $(\lambda; q)_\mu$ is defined by (see, for details, [2] and [15]; see also [33, p. 346 *et seq.*])

$$(\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1 - \lambda q^j}{1 - \lambda q^{\mu + j}} \right) \qquad (q, \lambda, \mu \in \mathbb{C};\ |q| < 1), \tag{1.1}$$

so that

$$(\lambda; q)_n = \begin{cases} 1 & (n = 0) \\ (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{n-1}) & (n \in \mathbb{N}), \end{cases} \tag{1.2}$$

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$$(\lambda; q)_\infty = \prod_{j=0}^{\infty} (1 - \lambda q^j) \tag{1.3}$$

and

$$\lim_{q \to 1} \left\{ \frac{(q^\lambda; q)_n}{(q^\mu; q)_n} \right\} = \frac{(\lambda)_n}{(\mu)_n} \qquad (n \in \mathbb{N}_0;\ \mu \notin \mathbb{Z}_0 := \{0, -1, -2, \cdots\}), \tag{1.4}$$

where $(\lambda)_\nu$ denotes the Pochammer symbol (or the *shifted* factorial) defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu = \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & (\nu=n \in \mathbb{N};\ \lambda \in \mathbb{C}). \end{cases} \tag{1.5}$$

The q-number $[\lambda]_q$, the q-number factorial $[\lambda]_q!$ and the q-number shifted factorial $([\lambda]_q)_n$ are defined by

$$[0]_q = 0 \quad \text{and} \quad [\lambda]_q = \frac{1 - q^\lambda}{1 - q} \qquad (q \neq 1;\ \lambda \in \mathbb{C} \setminus \{0\}), \tag{1.6}$$

$$[0]_q! = 1 \quad \text{and} \quad [n]_q! = [1]_q [2]_q [3]_q \cdots [n]_q \qquad (n \in \mathbb{N}) \tag{1.7}$$

and

$$([\lambda]_q)_n = [\lambda]_q [\lambda + 1]_q \cdots [\lambda + n - 1]_q \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}), \tag{1.8}$$

respectively. Clearly, we have the following limit cases:

$$\lim_{q \to 1} \{[\lambda]_q\} = \lambda, \quad \lim_{q \to 1} \{[n]_q!\} = n! \qquad \text{and} \qquad \lim_{q \to 1} \{([\lambda]_q)_n\} = (\lambda)_n, \tag{1.9}$$

where the Pochhammer symbol $(\lambda)_n$ is given by (1.5).

Over seven decades ago, Carlitz extended the classical Bernoulli and Euler polynomials and numbers (see, for example, [36]) and introduced the q-Bernoulli and the q-Euler polynomials as well as the q-Bernoulli and the q-Euler numbers (see [3, 4] and [5]). There are numerous recent investigations on this subject by, among many other authors, Cenki *et al.* ([6, 7] and [8]), Choi *et al.* ([10] and [11]), Kim *et al.* ([16-22] and [23]), Ozden and Simsek [26], Ryoo *et al.* [27], Simsek ([28, 29] and [30]) and Srivastava *et al.* [35].

We first recall here the definitions of the q-Bernoulli and the q-Euler polynomials of higher order as follows (see [3-5, 10] and [11]).

Definition 1. (q-Bernoulli Polynomials of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ of order α in q^x are defined by means of the following generating function:

$$(-t)^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!} q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} B_{n;q}^{(\alpha)}(x) \frac{t^n}{n!}. \tag{1.10}$$

Obviously, we have (see Definitions 5 and 6 below)

$$\lim_{q\to 1}\left\{B_{n;q}^{(\alpha)}(x)\right\} = B_n^{(\alpha)}(x) \quad \text{and} \quad \lim_{q\to 1}\left\{B_{n;q}^{(\alpha)}\right\} = B_n^{(\alpha)}. \tag{1.11}$$

We also write

$$B_{n;q}(x) := B_{n;q}^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{1.12}$$

for the *ordinary* q-Bernoulli polynomials $B_{n;q}(x)$.

Definition 2. (q-Bernoulli Numbers of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Bernoulli numbers $B_{n;q}^{(\alpha)}$ of order α are defined by

$$B_{n;q}^{(\alpha)} := B_{n;q}^{(\alpha)}(0). \tag{1.13}$$

We also write

$$B_{n;q} := B_{n;q}(0) \qquad (n \in \mathbb{N}_0) \tag{1.14}$$

for the *ordinary* q-Bernoulli numbers.

Definition 3. (q-Euler Polynomials of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$ of order α in q^x are defined by means of the following generating function:

$$2^\alpha \sum_{n=0}^{\infty} \frac{([\alpha]_q)_n}{[n]_q!}(-1)^n q^{n+x} e^{[n+x]_q t} = \sum_{n=0}^{\infty} E_{n;q}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.15}$$

Obviously, we have (see Definitions 5 and 6 below)

$$\lim_{q\to 1}\left\{E_{n;q}^{(\alpha)}(x)\right\} = E_n^{(\alpha)}(x) \quad \text{and} \quad \lim_{q\to 1}\left\{E_{n;q}^{(\alpha)}\right\} = E_n^{(\alpha)}. \tag{1.16}$$

We also write

$$E_{n;q}(x) := E_{n;q}^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{1.17}$$

for the *ordinary* q-Euler polynomials $E_{n;q}(x)$.

Definition 4. (q-Euler Numbers of Order α). For $q, \alpha \in \mathbb{C}$ $(|q| < 1)$, the q-Euler numbers $E_{n;q}^{(\alpha)}(x)$ of order α are defined by

$$E_{n;q}^{(\alpha)} := 2^n E_{n;q}^{(\alpha)}\left(\frac{\alpha}{2}\right). \tag{1.18}$$

We also write

$$E_{n;q} := 2^n E_{n;q}\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0) \tag{1.19}$$

for the *ordinary* q-Euler numbers $E_{n;q}$.

Definition 5. (Bernoulli and Euler Polynomials of Order α). The classical Bernoulli polynomials $B_n^{(\alpha)}(x)$ and the classical Euler polynomials $E_n^{(\alpha)}(x)$ of order α in x are defined by means of the following generating functions (see, for details, [1], [13], [25] and [32]; see also the recent works by Garg *et al.* [14] and Lin *et al.* [24]):

$$\left(\frac{z}{e^z-1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z|<2\pi;\ 1^{\alpha}:=1) \tag{1.20}$$

and

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{z^n}{n!} \qquad (|z|<\pi;\ 1^{\alpha}:=1). \tag{1.21}$$

Clearly, we have

$$B_n(x) := B_n^{(1)}(x) \quad \text{and} \quad E_n(x) := E_n^{(1)}(x) \qquad (n \in \mathbb{N}_0) \tag{1.22}$$

for the *ordinary* Bernoulli polynomials $B_n(x)$ in x and the *ordinary* Euler polynomials $E_n(x)$ in x, respectively.

Definition 6. (Bernoulli and Euler Numbers of Order α). The classical Bernoulli numbers $B_n^{(\alpha)}$ and the classical Euler numbers $E_n^{(\alpha)}$ of order α are defined by

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0) \qquad \text{and} \qquad E_n^{(\alpha)} := 2^n E_n\left(\frac{\alpha}{2}\right), \tag{1.23}$$

respectively.

We next recall the following elegant results of Srivastava and Pintér [34] given by Theorem A.

Theorem A. (Srivastava and Pintér [34, p. 379, Theorem 1; p. 380, Theorem 2]). *Each of the following relationships holds true*:

$$\begin{aligned} &B_n^{(\alpha)}(x+y) \\ &= \sum_{k=0}^{n}\binom{n}{k}\left(B_k^{(\alpha)}(y)+\frac{k}{2}B_{k-1}^{(\alpha-1)}(y)\right)E_{n-k}(x) \quad (n\in\mathbb{N}_0;\ \alpha\in\mathbb{C}) \end{aligned} \tag{1.24}$$

and

$$E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k}\left(E_{k+1}^{(\alpha-1)}(y)-E_{k+1}^{(\alpha)}(y)\right)B_{n-k}(x) \quad (n\in\mathbb{N}_0;\ \alpha\in\mathbb{C}). \tag{1.25}$$

An interesting special case occurs when we set $\alpha = 1$ in the assertion (1.24) of Theorem A and then let $y \to 0$. Noting that

$$B_n^{(0)}(x) = x^n \quad \text{and} \quad B_1 = -\frac{1}{2}, \tag{1.26}$$

we are thus led to Cheon's main result stated here as Theorem B.

Theorem B. (Cheon [9, p. 368, Theorem 3]). *The following relationship holds true*:

$$B_n(x) = \sum_{\substack{k=0 \\ (k\neq 1)}}^{n} \binom{n}{k} B_k E_{n-k}(x) \qquad (n\in\mathbb{N}_0). \tag{1.27}$$

In the present paper, we investigate q-extensions of Theorem A and Theorem B, which are based esentially upon series rearrangement techniques and several lemmas which we prove in the next section. Some formulas involving Carlitz's q-Stirling numbers of the second kind are also considered.

The paper is organized as follows: In Section 2, we give some lemmas and other necessary preliminaries. In Section 3, we study the aforementioned q-extensions of Theorem A and Theorem B. Finally, in Section 4, we provide other related results involving series of Carlitz's q-Stirling numbers of the second kind.

2. A Set of Lemmas and Other Preliminaries

In this section, we provide some basic formulas and results for the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ and the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$ of order α, which will be needed to prove our main results (Theorem 1 and Theorem 2).

From the generating function (1.10) and (1.15), it is not difficult to deduce Lemma 1 and Lemma 2 below. The proofs are fairly straightforward and will be omitted here.

Lemma 1. (Difference Equations). *Each of the following difference equations holds true for the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ and the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$:*

$$q^{\alpha-1}B_{n;q}^{(\alpha)}(x+1)-B_{n;q}^{(\alpha)}(x)=nB_{n-1;q}^{(\alpha-1)}(x)\qquad (n\in\mathbb{N}\setminus\{1\}) \tag{2.1}$$

and

$$q^{\alpha-1}E_{n;q}^{(\alpha)}(x+1)+E_{n;q}^{(\alpha)}(x)=2E_{n;q}^{(\alpha-1)}(x)\qquad (n\in\mathbb{N}\setminus\{1\}), \tag{2.2}$$

respectively.

Lemma 2. (Addition Theorems). *Each of the following addition theorems holds true for the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ and the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$:*

$$B_{n;q}^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_{k;q}^{(\alpha)}(x)q^{(k-\alpha+1)y}[y]_q^{n-k} \tag{2.3}$$

and

$$E_{n;q}^{(\alpha)}(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_{k;q}^{(\alpha)}(x)q^{(k+1)y}[y]_q^{n-k}, \tag{2.4}$$

respectively.

Upon setting $y=1$ in (2.3) and (2.4), we get

$$B_{n;q}^{(\alpha)}(x+1)=\sum_{k=0}^{n}\binom{n}{k}q^{k-\alpha+1}B_{k;q}^{(\alpha)}(x) \tag{2.5}$$

and

$$E_{n;q}^{(\alpha)}(x+1)=\sum_{k=0}^{n}\binom{n}{k}q^{k+1}E_{k;q}^{(\alpha)}(x), \tag{2.6}$$

respectively. By combining (2.1) and (2.5), we can obtain the following formula:

$$B_{n;q}^{(\alpha-1)}(x)=\frac{1}{n+1}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}q^{k}B_{k;q}^{(\alpha)}(x)-B_{n+1;q}^{(\alpha)}(x)\right]. \tag{2.7}$$

Similarly, by combining (2.2) and (2.6), we can obtain the following formula:

$$E_{n;q}^{(\alpha-1)}(x)=\frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}q^{k+\alpha}E_{k;q}^{(\alpha)}(x)+E_{n;q}^{(\alpha)}(x)\right]. \tag{2.8}$$

Putting $\alpha=1$ in (2.7) and (2.8), and noting that

$$B_{n;q}^{(0)}(x)=E_{n;q}^{(0)}(x)=q^{x}[x]_q^{n},$$

we arrive at the following expansions:

$$q^x[x]_q^n = \frac{1}{n+1}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}q^k B_{k;q}(x) - B_{n+1;q}(x)\right] \tag{2.9}$$

and

$$q^x[x]_q^n = \frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}q^{k+1} E_{k;q}(x) + E_{n;q}(x)\right]. \tag{2.10}$$

Obviously, these last results (2.9) and (2.10) provide q-extensions of the following familiar expansions (see [25, p. 26] and [34, p. 378, Eq. (29)]):

$$x^n = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}B_k(x) \tag{2.11}$$

and

$$x^n = \frac{1}{2}\left[\sum_{k=0}^{n}\binom{n}{k}E_k(x) + E_n(x)\right], \tag{2.12}$$

respectively.

We next define the polynomials $\mathfrak{B}_{n;q;y}^{(\alpha)}(x)$ and $\mathfrak{E}_{n;q;y}^{(\alpha)}(x)$ in q^x as follows:

$$\mathfrak{B}_{n;q;y}^{(\alpha)}(x+1) = \sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha+1)y}B_{k;q}^{(\alpha)}(x) \tag{2.13}$$

and

$$\mathfrak{E}_{n;q;y}^{(\alpha)}(x+1) = \sum_{k=0}^{n}\binom{n}{k}q^{(k+1)y}E_{k;q}^{(\alpha)}(x), \tag{2.14}$$

which, in conjunction with (2.5) and (2.6), yield the following relationships:

$$\mathfrak{B}_{n;q;1}^{(\alpha)}(x) = B_{n;q}^{(\alpha)}(x) \quad \text{and} \quad \mathfrak{E}_{n;q;1}^{(\alpha)}(x) = E_{n;q}^{(\alpha)}(x), \tag{2.15}$$

respectively. We also write

$$\mathfrak{B}_{n;q;y}(x) := B_{n;q;y}^{1}(x) \quad \text{and} \quad \mathfrak{E}_{n;q;y}(x) := E_{n;q;y}^{1}(x). \tag{2.16}$$

Both (2.5) and (2.13) provide q-extensions of the following well-known formula:

$$B_n^{(\alpha)}(x+1) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(\alpha)}(x). \tag{2.17}$$

On the other hand, both (2.6) and (2.14) are q-extensions of the well-known formula:

$$E_n^{(\alpha)}(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k^{(\alpha)}(x). \tag{2.18}$$

The following special values of $\mathfrak{B}_{n;q;y}^{(\alpha)}(x)$ and $\mathfrak{E}_{n;q;y}^{(\alpha)}(x)$ are easily derivable from (2.13) and (2.14):

$$\mathfrak{B}_{n;q;y}^{(0)}(x)=\mathfrak{E}_{n;q;y}^{(0)}(x)=q^{x+y-1}(1+q^y[x-1]_q)^n. \tag{2.19}$$

$$\begin{aligned}&\mathfrak{B}_{0;q;y}^{(\ell)}(x)=q^{x+y-1}\delta_{\ell,0}\quad(\ell\in\mathbb{N}_0)\\ &\text{and}\\ &\mathfrak{B}_{n;q;y}^{(\ell)}(x)=0\quad(n\in\{0,1,2,\cdots,\ell-1\})\end{aligned} \tag{2.20}$$

and

$$\mathfrak{E}_{0;q;y}^{(\alpha)}(x)=\frac{2^\alpha q^{x+y-1}}{(-q;q)_\alpha}, \tag{2.21}$$

where $\delta_{m,n}$ denotes the Kronecker symbol.

Lemma 3. (Recurrence Relationships). *The polynomials* $\mathfrak{B}_{n;q;y}^{(\alpha)}(x)$ *and* $\mathfrak{E}_{n;q;y}^{(\alpha)}(x)$ *in* q^x *satisfy the following difference relationships*:

$$q^{\alpha-1}\mathfrak{B}_{n;q;y}^{(\alpha)}(x+1)-\mathfrak{B}_{n;q;y}^{(\alpha)}(x)=n\mathfrak{B}_{n-1;q;y}^{(\alpha-1)}(x)\qquad(n\in\mathbb{N}\setminus\{1\}) \tag{2.22}$$

and

$$q^{\alpha-1}\mathfrak{E}_{n;q;y}^{(\alpha)}(x+1)+\mathfrak{E}_{n;q;y}^{(\alpha)}(x)=2\mathfrak{E}_{n;q;y}^{(\alpha-1)}(x)\qquad(n\in\mathbb{N}_0). \tag{2.23}$$

Proof. By making use of (2.1) and (2.13), we find that

$$\begin{aligned}&q^{\alpha-1}\mathfrak{B}_{n;q;y}^{(\alpha)}(x+1)-\mathfrak{B}_{n;q;y}^{(\alpha)}(x)\\ &\quad=\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha+1)y}\left[q^{\alpha-1}B_{k;q}^{(\alpha)}(x)-B_{k;q}^{(\alpha)}(x-1)\right]\\ &\quad=\sum_{k=0}^{n}k\binom{n}{k}q^{(k-\alpha+1)y}B_{k-1;q}^{(\alpha-1)}(x-1)\\ &\quad=n\sum_{k=0}^{n-1}\binom{n-1}{k}q^{(k-\alpha+1)y}B_{k;q}^{(\alpha-1)}(x-1)\\ &\quad=n\mathfrak{B}_{n-1;q;y}^{(\alpha-1)}(x),\end{aligned} \tag{2.24}$$

which proves the assertion (2.22) of Lemma 3. Similarly, by applying (2.2) and (2.14), we can prove the assertion (2.23) of Lemma 3. The proof of Lemma 3 is thus completed. ■

In their limit cases when $q \to 1$, (2.1) and (2.2), as well as (2.22) and (2.23), would obviously reduce to the difference formulas for the corresponding ordinary Bernoulli and Euler polynomials of order α. Thus, in their present q-cases, the formulas (2.1) and (2.2), and the formulas (2.22) and (2.23), are analogous to the following well-known difference formulas:

$$B_n^{(\alpha)}(x+1) - B_n^{(\alpha)}(x) = nB_n^{(\alpha-1)}(x) \qquad (n \in \mathbb{N} \setminus \{1\}) \tag{2.25}$$

and

$$E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x) = 2E_n^{(\alpha-1)}(x) \qquad (n \in \mathbb{N}_0), \tag{2.26}$$

respectively.

3. q-Extensions of Theorem A and Theorem B

In this section, we first present some appropriate q-extensions of Theorem A and Theorem B.

Theorem 1. *Each of the following relationships holds true*:

$$\begin{aligned} B_{n;q}^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} \Bigg[& \frac{1}{2}\left(q^{(k-\alpha)x} B_{k;q}^{(\alpha)}(y) + q^{n-k-x-\alpha+2} \mathfrak{B}_{k;q;x}^{(\alpha)}(y) \right) \\ & + \frac{1}{2}\, k q^{n-k-x-\alpha+2} \mathfrak{B}_{k-1;q;x}^{(\alpha-1)}(y) \Bigg] E_{n-k;q}(x) \qquad (n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} E_{n;q}^{(\alpha)}(x+y) = & \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \\ & \left[q^{n-k-x-\alpha+1} \left(2\mathfrak{E}_{k+1;q;x}^{(\alpha-1)}(y) - \mathfrak{E}_{k+1;q;x}^{(\alpha)}(y) \right) - q^{(k+1)x} E_{k+1;q}^{(\alpha)}(y) \right] \\ & B_{n-k;q}(x) + \frac{2^\alpha q^y (q^{n+1}-1)}{(n+1)(-q;q)_\alpha} B_{n+1;q}(x) \quad (n \in \mathbb{N}_0;\ \alpha \in \mathbb{C}) \end{aligned} \tag{3.2}$$

for the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ and the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$, respectively.

Proof. In our proof of the relationship (3.1), we apply (2.3) (with x and y interchanged) and make suitable substitutions from (2.10). We thus find that

$$\begin{aligned}
&B_{n;q}^{(\alpha)}(x+y)\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha+1)x}B_{k;q}^{(\alpha)}(y)[x]_q^{n-k}\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)\left[\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}E_{j;q}(x)+E_{n-k;q}(x)\right]\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)E_{n-k;q}(x)\\
&\qquad+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)\sum_{j=0}^{n-k}\binom{n-k}{j}q^{j+1}E_{j;q}(x)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)E_{n-k;q}(x)\\
&\qquad+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}q^{j+1}E_{j;q}(x)\sum_{k=0}^{n-j}\binom{n-j}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)E_{n-k;q}(x)\\
&\qquad+\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}q^{n-k+1}E_{n-k;q}(x)\sum_{j=0}^{k}\binom{k}{j}q^{(j-\alpha)x}B_{j;q}^{(\alpha)}(y)\\
&\quad=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\left(q^{(k-\alpha)x}B_{k;q}^{(\alpha)}(y)+q^{n-k-x+1}\mathfrak{B}_{k;q;x}^{(\alpha)}(y+1)\right)E_{n-k;q}(x).
\end{aligned}\tag{3.3}$$

In the above process leading eventually to (3.3), we have inverted the order of summation and applied the following elementary combinatorial identity:

$$\binom{\mu}{\lambda}\binom{\lambda}{\nu}=\binom{\mu}{\nu}\binom{\mu-\nu}{\mu-\lambda}\qquad(\lambda,\mu,\nu\in\mathbb{C}).\tag{3.4}$$

Finally, in light of the recurrence relationship (2.22) asserted by Lemma 3, we obtain the q-relationship as asserted by Theorem 1.

In a similar manner, we can prove the q-relationship (3.2). This completes our proof of Theorem 1. ■

Remark 1. Taking $\alpha = 1$ in (3.1) and noting (2.15) and (2.16), we obtain

$$\begin{aligned} B_{n;q}(x+y) &= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{1}{2} \left(q^{(k-1)x} B_{k;q}(y) + q^{n-k-x+1} B_{k;q;x}(y) \right) \right. \\ &\quad \left. + \frac{1}{2}\, k q^{n-k+y} (1 + q^{x}[y-1]_q)^{k-1} \right] E_{n-k;q}(x), \end{aligned} \tag{3.5}$$

which is a q-extension of the following known result (see [34, p. 379, Equation (37)]):

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} \left(B_k(y) + \frac{1}{2}\, k y^{k-1} \right) E_{n-k}(x). \tag{3.6}$$

Theorem 2. *The following relationship holds true*:

$$\begin{aligned} B_{n;q}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left[\frac{1}{2} \left(q^{(k-1)x} B_{k;q} + q^{n-k-x+1} B_{k;q;x}(0) \right) \right. \\ &\quad \left. + \frac{1}{2}\, k q^{n-k} (1 - q^{x-1})^{k-1} \right] E_{n-k;q}(x). \end{aligned} \tag{3.7}$$

Proof. Letting $y = 0$ in (3.5), we can easily deduce the relationship (3.7) asserted by Theorem 2. This completes the proof of Theorem 2. ∎

Remark 2. By setting $\alpha = 1$ in the assertion (3.2) of Theorem 1, we get

$$\begin{aligned} E_{n;q}(x+y) &= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \left(2 q^{n-k+y-1} (1 + q^{x}[y-1]_q)^{k+1} \right. \\ &\quad \left. - q^{n-k-x} E_{k+1;q;x}(y) - q^{(k+1)x} E_{k+1;q}(y) \right) B_{n-k;q}(x) \\ &\quad + \frac{2q^{y}(q^{n+1}-1)}{(n+1)(q+1)} B_{n+1;q}(x), \end{aligned} \tag{3.8}$$

which is a q-extension of the following known result (see [34, p. 380, Equation (39)])

$$E_n(x+y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left(y^{k+1} - E_{k+1}(y) \right) B_{n-k}(x). \tag{3.9}$$

If, in the q-result (3.9), we further put $y = 0$, we have

$$\begin{aligned} E_{n;q}(x) &= \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} \left(2 q^{n-k-1} (1 - q^{x-1})^{k+1} - q^{n-k-x} E_{k+1;q;x}(0) \right. \\ &\quad \left. - q^{(k+1)x} E_{k+1;q}(0) \right) B_{n-k;q}(x) + \frac{2(q^{n+1}-1)}{(n+1)(q+1)} B_{n+1;q}(x), \end{aligned} \tag{3.10}$$

which is a q-extension of the another known result (see [34, p. 380, Equation (40)])

$$E_n(x) = -\sum_{k=0}^{n} \frac{2}{k+1}\binom{n}{k} E_{k+1}(0)B_{n-k}(x). \tag{3.11}$$

4. Formulas Involving the q-Stirling Numbers of the Second Kind

In this section, we propose to derive several formulas for the q-Bernoulli polynomials $B_{n;q}^{(\alpha)}(x)$ and the q-Euler polynomials $E_{n;q}^{(\alpha)}(x)$ of order α in series of the q-Stirling numbers of the second kind, which are defined below.

The q-binomial coefficient $\begin{bmatrix}\lambda\\ n\end{bmatrix}_q$ defined by

$$\begin{bmatrix}\lambda\\ 0\end{bmatrix}_q = 1 \quad \text{and} \quad \begin{bmatrix}\lambda\\ n\end{bmatrix}_q = \frac{[\lambda]_q[\lambda-1]_q\cdots[\lambda-n+1]_q}{[n]_q!} \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}), \tag{4.1}$$

so that

$$\begin{bmatrix}n\\ k\end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} \qquad (n,k \in \mathbb{N}_0), \tag{4.2}$$

satisfies each of the following relationships:

$$\begin{bmatrix}n\\ k\end{bmatrix}_q = \begin{bmatrix}n\\ n-k\end{bmatrix}_q \ (n,k \in \mathbb{N}_0; 0 \leqq k \leqq n) \ \text{and} \ \begin{bmatrix}n\\ k\end{bmatrix}_q = 0 \ (n,k \in \mathbb{N}_0;\ n<k) \tag{4.3}$$

$$\begin{bmatrix}\lambda\\ n\end{bmatrix}_q = \begin{bmatrix}\lambda-1\\ n-1\end{bmatrix}_q + q^n \begin{bmatrix}\lambda-1\\ n\end{bmatrix}_q \qquad (n \in \mathbb{N};\ \lambda \in \mathbb{C}). \tag{4.4}$$

The familiar Stirling numbers $S(n,k)$ of the second kind are defined by means of the following expansion (see [12, p. 207, Theorem B]):

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k), \tag{4.5}$$

so that

$$S(n,0) = \delta_{n,0}, \qquad S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,n-1) = \binom{n}{2}. \tag{4.6}$$

Analogous to the definition (4.5), the q-Stirling numbers $S_q(n,k)$ of the second kind were defined by Carlitz as follows (see [3, p. 989, Equation (3.1)]):

$$[x]_q^n = \sum_{k=0}^{n} S_q(n,k)[k]_q! \begin{bmatrix}x\\ k\end{bmatrix}_q q^{\binom{k}{2}}. \tag{4.7}$$

These q-Stirling numbers $S_q(n,k)$ of the second kind are known to satisfy each of the following relationships (see [3, p. 990, Equations (3.2) and (3.5)]):

$$S_q(n+1,k) = S_q(n,k-1) + [k]_q S_q(n,k) \tag{4.8}$$

and

$$\begin{aligned} S_q(n,k) &= \frac{q^{-\binom{k}{2}}}{[k]_q} \sum_{j=0}^{k} (-)^j \, q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [k-j]_q^n \\ &= \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^{k-j} \, q^{\frac{1}{2}j(j-2k+1)} \begin{bmatrix} k \\ j \end{bmatrix}_q [j]_q^n \\ &= (q-1)^{k-n} \sum_{j=k}^{n} (-1)^{n-j} \binom{n}{j} \begin{bmatrix} j \\ k \end{bmatrix}_q . \end{aligned} \tag{4.9}$$

Obviously, we have [*cf.* Equation (4.6)]

$$S_q(n,0) = \delta_{n,0}, \quad S_q(n,1) = S_q(n,n) = 1 \ \text{ and } \ S_q(n,n-1) = \frac{n-[n]_q}{1-q}. \tag{4.10}$$

Now, by applying (2.3) and (2.4), and making appropriate substitutions from (4.7) as in the proof of Theorem 1, we can obtain Theorem 3 below.

Theorem 3. *Each of the following relationships holds true for the q-Stirling numbers $S_q(n,k)$ of the second kind*:

$$\begin{aligned} B_{n;q}^{(\alpha)}(x+y) &= \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} \\ &\quad \cdot q^{(j-\alpha+1)x+\binom{k}{2}} B_{j;q}^{(\alpha)}(y) S_q(n-j,k) \quad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0) \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} E_{n;q}^{(\alpha)}(x+y) &= \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} \\ &\quad \cdot q^{(j+1)x+\binom{k}{2}} E_{j;q}^{(\alpha)}(y) S_q(n-j,k) \qquad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0). \end{aligned} \tag{4.12}$$

Upon setting $y = 0$ in (4.11) and $y = \frac{\alpha}{2}$ in (4.12), we obtain the following corollary.

Corollary 1. *Each of the following explicit representations*:

$$\begin{aligned} B_{n;q}^{(\alpha)}(x) &= \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} \\ &\quad \cdot q^{(j-\alpha+1)x+\binom{k}{2}} B_j^{(\alpha)} S_q(n-j,k) \qquad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0) \end{aligned} \tag{4.13}$$

and

$$E_{n;q}^{(\alpha)}(x) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x-\frac{\alpha}{2} \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} \frac{E_j^{(\alpha)}}{2^j} q^{(j+1)(x-\frac{\alpha}{2})+\binom{k}{2}} S_q(n-j,k) \quad (\alpha \in \mathbb{C};\ n \in \mathbb{N}_0) \tag{4.13}$$

holds true in terms of the q-Stirling numbers $S_q(n,k)$ of the second kind.

Remark 3. Upon setting $\alpha = 1$ in the assertions (4.13) and (4.14) of Corollary 1, we are led fairly easily to Corollary 2 below.

Corollary 2. *Each of the following explicit representations*:

$$B_{n;q}(x) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} q^{jx+\binom{k}{2}} B_{j;q} S_q(n-j,k) \tag{4.15}$$

and

$$E_{n;q}(x) = \sum_{k=0}^{n} [k]_q! \begin{bmatrix} x-\frac{1}{2} \\ k \end{bmatrix}_q \sum_{j=0}^{n-k} \binom{n}{j} \frac{E_{j;q}}{2^j} q^{(j+1)(x-\frac{1}{2})+\binom{k}{2}} S_q(n-j,k) \tag{4.16}$$

holds true in terms of the q-Stirling numbers $S_q(n,k)$ of the second kind.

Finally, in their limit case when $q \to 1$, these last results (4.15) and (4.16) would reduce to the following (presumably known) formulas for the classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$, respectively:

$$B_n(x) = \sum_{k=0}^{n} k! \binom{x}{k} \sum_{j=0}^{n-k} \binom{n}{j} B_j S(n-j,k) \tag{4.17}$$

and

$$E_n(x) = \sum_{k=0}^{n} k! \binom{x-\frac{1}{2}}{k} \sum_{j=0}^{n-k} \binom{n}{j} \frac{E_j}{2^j} S(n-j,k). \tag{4.18}$$

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