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CHARACTERIZATION OF CONVEXITY FOR A PIECEWISE C^2 FUNCTION BY THE LIMITING SECOND-ORDER SUBDIFFERENTIAL

Nguyen Huy Chieu and Jen-Chih Yao*

Abstract. We prove in this paper that a piecewise C^2 function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every $(x, y) \in \mathrm{gph}\partial\varphi$, the limiting second-order subdifferential mapping $\partial^2\varphi(x, y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has the so-called positive semi-definiteness (PSD) - in analogy with the notion of positive semi-definiteness of symmetric real matrices. As a by-product, characterization for strong convexity of φ is established.

1. INTRODUCTION

Due to its important role in mathematical economics, engineering, management science, and optimization theory, convexity of functions and sets has been studied intensively; see [1, 3, 5-7, 9, 11, 13, 14] and the references therein.

First-order characterizations for the convexity of extended real-valued functions via the monotonicity of the Fréchet derivative and the monotonicity of the Fréchet subdifferential mapping or the limiting subdifferential mapping can be found, e.g., in [6, 11, 12] and [7, Theorem 3.56].

The classical second-order characterization of convexity of real-valued functions (see for instance [11, 12]) says that a C^2 function $\varphi : U \to \mathbb{R}$ where U is an open convex subset of $\mathbb{R}^n$ is convex if and only if for every $x \in U$ the Hessian $\nabla^2 f(x)$ is a positive semidefinite matrix. To relax the assumption on the C^2 smoothness of the function under consideration, several authors have characterized the convexity by using various kinds of generalized second-order directional derivatives. The reader is referred to [1, 4, 5, 13, 14] for results in this direction.

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*Corresponding author.

Recently, the authors in [3] have found that to a certain extent convexity of functions can be characterized by second-order subdifferential mappings. Among other things, they obtained some characterizations for convexity of piecewise linear functions and of piecewise C^2 functions of a special type via the limiting second-order subdifferential. The purpose of this paper is to *characterize the convexity of piecewise C^2 functions by the limiting second-order subdifferential.*

We will show that a piecewise C^2 function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for every $(x, y) \in \mathrm{gph}\partial\varphi$, the limiting second-order subdifferential mapping $\partial^2\varphi(x, y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has the so-called positive semi-definiteness (PSD) - in analogy with the notion of positive semi-definiteness of symmetric real matrices. Since strong convexity of functions plays a remarkable role in theory of algorithms [12] and stability theory of optimization problems [2], by using the limiting second-order subdifferential we derive a necessary and sufficient condition for strong convexity of piecewise C^2 functions.

The rest of the paper is organized as follows. Section 2 contains some definitions and results which are needed in the sequel. Section 3 is devoted to the necessary and sufficient condition for convexity of a piecewise C^2 function by its limiting second-order subdifferential. As a by-product, the second-order necessary and sufficient condition for strong convexity of piecewise C^2 functions is given.

2. Preliminaries

We start by recalling some notions related to generalized differentiation. The notions and related results of generalized differentiation can be found in [7].

For a set $\Omega \subset \mathbb{R}^n$ and an extended real-valued function $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$, the symbols $x \xrightarrow{\Omega} \bar{x}$ and $x \xrightarrow{\varphi} \bar{x}$ mean that $x \to \bar{x}$ with $x \in \Omega$ and $x \to \bar{x}$ with $\varphi(x) \to \varphi(\bar{x})$, respectively. Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} F(x) := \Big\{ x^* \in \mathbb{R}^n \;\Big|\; \exists \text{ sequences } x_k \xrightarrow{\Omega} \bar{x} \text{ and } x_k^* \to x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in I\!N \Big\}$$

the *sequential Painlevé-Kuratowski upper limit* of the mapping F as $x \xrightarrow{\Omega} \bar{x}$.

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $\bar{x} \in \mathbb{R}^n$ and let $\varepsilon \geqslant 0$. The $\varepsilon-$*subdifferential* of φ at $\bar{x}$ is the set $\widehat{\partial}_\varepsilon\varphi(\bar{x})$ defined by

$$\widehat{\partial}_\varepsilon\varphi(\bar{x}) = \Big\{ x^* \in \mathbb{R}^n : \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geqslant -\varepsilon \Big\}.$$

We put $\widehat{\partial}_\varepsilon\varphi(\bar{x}) = \emptyset$ if $|\varphi(\bar{x})| = \infty$. When $\varepsilon = 0$ the set $\widehat{\partial}_0\varphi(\bar{x})$, denoted by $\widehat{\partial}\varphi(\bar{x})$, is called the Fréchet subdifferential of φ at $\bar{x}$. The *limiting subdifferential* (or *Mordukhovich subdifferential*) of φ at $\bar{x}$ is given by

$$\partial\varphi(\bar{x}) = \mathop{\mathrm{Lim\,sup}}_{x \xrightarrow{\varphi} \bar{x};\ \varepsilon \downarrow 0} \widehat{\partial}_\varepsilon \varphi(x), \tag{2.1}$$

that is, $x^* \in \partial\varphi(\bar{x})$ if and only if there exist sequences $\varepsilon_k \downarrow 0$, $x_k \xrightarrow{\varphi} \bar{x}$ and $x_k^* \to x^*$ such that $x_k^* \in \widehat{\partial}_{\varepsilon_k}\varphi(x_k)$. Note that $\widehat{\partial}_\varepsilon\varphi(\cdot)$ can be replaced by $\widehat{\partial}\varphi(\cdot)$ in (2.1) when φ is lower semicontinuous around $\bar{x}$.

Given $\Omega \subset \mathbb{R}^n$ with its *indicator function* $\delta(x;\Omega) = 0$ if $x \in \Omega$ and $\delta(x;\Omega) = \infty$ otherwise, the *Fréchet normal cone* and the *limiting normal cone* to Ω at x are defined, respectively, by

$$\widehat{N}(x;\Omega) = \widehat{\partial}\delta(x;\Omega) \text{ and } N(x;\Omega) = \partial\delta(x;\Omega).$$

Obviously, $\widehat{N}(x;\Omega) \subset N(x;\Omega)$ and

$$x^* \in \widehat{N}(x;\Omega) \Leftrightarrow \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0.$$

Let $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping with the *graph*

$$\operatorname{gph} F = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \ : \ y \in F(x)\}.$$

The *limiting coderivative* $D^*F(\bar{x},\bar{y})\colon \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*) = \big\{x^* \in \mathbb{R}^n :\ (x^*, -y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} F)\big\}.$$

We omit $\bar{y} = f(\bar{x})$ in the above coderivative notion if $F = f\colon \mathbb{R}^n \to \mathbb{R}^m$ is single-valued. If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *strictly differentiable* at $\bar{x}$ in the sense that

$$\lim_{x,u \to \bar{x}} \frac{f(x) - f(u) - \langle \nabla f(\bar{x}), x - u\rangle}{\|x - u\|} = 0$$

with the derivative operator $\nabla f(\bar{x}) : \mathbb{R}^m \to \mathbb{R}^m$, being linear contiunous, then $D^*f(\bar{x})(y^*) = \big\{\nabla f(\bar{x})^* y^*\big\}$ for all $y^* \in \mathbb{R}^m$. Therefore, the limiting coderivative is an extension of the *adjoint derivative* operator of the classical derivative to nonsmooth functions and set-valued mappings.

Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an extended real-valued function with a finite value at $\bar{x}$. Given $\bar{y} \in \partial\varphi(\bar{x})$, the mapping $\partial^2\varphi(\bar{x},\bar{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\partial^2\varphi(\bar{x},\bar{y})(u) = (D^*\partial\varphi)(\bar{x},\bar{y})(u), \quad u \in \mathbb{R}^n,$$

is called the *limiting second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$. If φ is twice continuously differentiable at $\bar{x}$ and $\bar{y} \in \partial\varphi(\bar{x})$ (actually, $\bar{y} = \nabla\varphi(\bar{x})$), then

$$\partial^2\varphi(\bar{x},\bar{y})(u) = \left\{\nabla^2\varphi(\bar{x})(u)\right\} \text{ for all } u \in \mathbb{R}^n,$$

which is known as the symmetric Hessian matrix. The reader can find various properties and calculus rules for the limiting second-order subdifferential with a number of applications in [7, 8, 10] and the references therein.

Theorem 2.1. (see [3, Theorem 3.2]). *Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be proper lower semicontinuous. If φ is convex, then*

$$\langle z, u\rangle \geq 0 \quad \textit{for all } u \in \mathbb{R}^n \textit{ and } z \in \partial^2\varphi(x,y)(u) \textit{ with } (x,y) \in \mathrm{gph}\partial\varphi;$$

that is, for every $(x,y) \in \mathrm{gph}\partial\varphi$, the mapping $\partial^2\varphi(x,y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is positive semi-definite (*PSD*).

3. Characterization of Convexity

Recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *piecewise* C^2 if there exist families $\{P_1, ..., P_k\}$ of polyhedral convex sets in $\mathbb{R}^n$ and twice continuously differentiable functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ such that $\mathbb{R}^n = \bigcup\limits_{i=1}^{k} P_i$, $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ for all $i \neq j$, and

$$\varphi(x) = \varphi_i(x) \text{ for any } x \in P_i, \quad i \in \{1, ..., k\}. \tag{3.1}$$

From (3.1) it follows that $\varphi_i(x) = \varphi_j(x)$ whenever $x \in P_i \cap P_j$ and $i, j \in \{1, ..., k\}$.

We need the following two lemmas taken from [3].

Lemma 3.1. *If $I := \{i \in \{1, 2, ..., k\} \mid \ \mathrm{int}P_i \neq \emptyset\}$, then $\bigcup\limits_{i \in I} P_i = \mathbb{R}^n$.*

Lemma 3.2. *Let $[x,y]$ be an interval in $\mathbb{R}^n$ $(x \neq y)$, $0 = \tau_0 < \tau_1 < ... < \tau_{m-1} < \tau_m = 1$ $(m \in I\!N,\ m > 1)$, and $x_i := x + \tau_i(y - x)$ $(i = 0, 1, ..., m)$. Suppose that φ is nonconvex and continuous on $[x,y]$. Then there must exist $i \in \{0, 1, ..., m-2\}$ such that φ is nonconvex on $[x_i, x_{i+2}]$.*

We are now ready to state and prove the main result of this paper.

Theorem 3.3. *Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a piecewise C^2 function. Then φ is convex if and only if*

$$\langle z, u\rangle \geq 0 \ \ \textit{for all } u \in \mathbb{R}^n, z \in \partial^2\varphi(x,y)(u) \textit{ with } (x,y) \in \mathrm{gph}\partial\varphi. \tag{3.2}$$

Proof. The necessary condition is due to Theorem 2.1. It remains to prove the sufficient condition. By Lemma 3.1, we can assume that $\mathrm{int}P_i \neq \emptyset$ for all $i \in \{1, 2, ..., k\}$. Suppose that (3.2) holds but φ is nonconvex. Since φ is twice continuously differentiable on $\mathrm{int}P_i$, $\partial^2\varphi(x, y)(u) = \{\nabla^2\varphi(x)(u)\}$ for all $x \in \mathrm{int}P_i$, $y \in \partial\varphi(x)$ and $u \in \mathbb{R}^n$. Together with (3.2) this implies that $\nabla^2\varphi(x)$ is positive semi-definite on $\mathrm{int}P_i$. By the classical result on characterizing the convexity of C^2 functions, φ is convex on P_i ($i = 1, 2, ..., k$). We consider the following two cases.

Case 1. $k = 2$. Let $P_1 = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \alpha\}$, $P_2 = \{x \in \mathbb{R}^n : \langle a, x\rangle \geq \alpha\}$ ($a \in \mathbb{R}^n \backslash \{0\}$, $\alpha \in \mathbb{R}$), $P_{12} = P_1 \cap P_2$ and

$$\varphi(x) = \begin{cases} \varphi_1(x) & \text{if } x \in P_1, \\ \varphi_2(x) & \text{if } x \in P_2, \end{cases}$$

where $\varphi_1, \varphi_2 \in C^2$ and $\varphi_1(x) = \varphi_2(x)$ for all $x \in P_{12}$. Observe that $\mathbb{R}^n$ is the union of disjoint nonempty sets $\mathrm{int}P_1$, $\mathrm{int}P_2$, P_{12}, and

$$\partial\varphi(x) = \widehat{\partial}\varphi(x) = \begin{cases} \{\nabla\varphi_1(x)\} & \text{if } x \in \mathrm{int}P_1, \\ \{\nabla\varphi_2(x)\} & \text{if } x \in \mathrm{int}P_1. \end{cases} \tag{3.3}$$

Since φ is convex on each one of the convex sets P_1 and P_2 but it is nonconvex on $\mathbb{R}^n = P_1 \cup P_2$, there exist $x_0 \in \mathrm{int}P_1$, $y_0 \in \mathrm{int}P_2$ and $t_1 \in (0, 1)$ such that

$$\varphi(z_1) > (1 - t_1)\varphi(x_0) + t_1\varphi(y_0), \tag{3.4}$$

where $z_1 = (1 - t_1)x_0 + t_1y_0$. We will prove that

$$\varphi(z_0) > (1 - t_0)\varphi(x_0) + t_0\varphi(y_0) \tag{3.5}$$

with $z_0 = (1 - t_0)x_0 + t_0y_0 \in P_{12}$ ($t_0 \in (0, 1)$). If $t_0 = t_1$ then (3.5) follows from (3.4), because $z_1 = z_0$. If $t_0 \in (0, t_1)$ then $z_1 = (1 - \lambda)y_0 + \lambda z_0$ with $\lambda = (1 - t_1)/(1 - t_0) \in (0, 1)$. Since φ is convex on $[z_0, y_0] \subset P_2$, $\varphi(z_1) \leq (1 - \lambda)\varphi(y_0) + \lambda\varphi(z_0)$. Combining this fact with (3.4), we obtain

$$\begin{aligned} \varphi(z_0) &> \lambda^{-1}[(1 - t_1)\varphi(x_0) + t_1\varphi(y_0) - (1 - \lambda)\varphi(y_0)] \\ &= (1 - t_0)\varphi(x_0) + t_0\varphi(y_0), \end{aligned}$$

which gives (3.5). Similarly, (3.5) is also valid if $t_0 \in (t_1, 1)$. Therefore (3.5) holds. Since $x_0 \in P_1$, $y_0 \in P_2$ and $z_0 \in P_{12}$, by (3.5), we have

$$(1 - t_0)\varphi_1(z_0) + t_0\varphi_2(z_0) > (1 - t_0)\varphi_1(x_0) + t_0\varphi_2(y_0)$$

or in other words,

$$(1-t_0)(\varphi_1(z_0)-\varphi_1(x_0))+t_0(\varphi_2(z_0)-\varphi_2(y_0)>0. \tag{3.6}$$

According to the mean value theorem, we have

$$\varphi_1(z_0)-\varphi_1(x_0)=\langle\nabla\varphi_1(a_1),z_0-x_0\rangle \text{ and } \varphi_2(z_0)-\varphi_1(y_0)=\langle\nabla\varphi_2(a_2),z_0-y_0\rangle,$$

for some $a_1\in(x_0,z_0)$ and $a_2\in(z_0,y_0)$. Note that $z_0=(1-t_0)x_0+t_0y_0$ and $t_0\in(0,1)$. By (3.6),

$$\langle\nabla\varphi_1(a_1)-\nabla\varphi_2(a_2),y_0-x_0\rangle>0. \tag{3.7}$$

Our next task is to show

$$\langle\nabla\varphi_1(z_0)-\nabla\varphi_2(z_0),y_0-x_0\rangle>0. \tag{3.8}$$

Assume by contradiction that $\langle\nabla\varphi_1(z_0)-\nabla\varphi_2(z_0),y_0-x_0\rangle\le 0$. Since φ_1 is convex on $[a_1,z_0]$,

$$\langle\nabla\varphi_1(z_0)-\nabla\varphi_1(a_1),z_0-a_1\rangle\ge 0$$

from which we get

$$\langle\nabla\varphi_1(z_0)-\nabla\varphi_1(a_1),y_0-x_0\rangle\ge 0.$$

Similarly, $\langle\nabla\varphi_2(a_2)-\nabla\varphi_2(z_0),y_0-x_0\rangle\ge 0$. Hence

$$\begin{aligned}\langle\nabla\varphi_1(a_1)-\nabla\varphi_2(a_2),y_0-x_0\rangle &\le\langle\nabla\varphi_1(z_0)-\nabla\varphi_2(z_0),y_0-x_0\rangle\\ &\le 0.\end{aligned}$$

This contradicts (3.7) and thus (3.8) is valid.

We claim that $\widehat{\partial}\varphi(z_0)=\emptyset$. Suppose that it is not true. Then there exists $x^*\in\mathbb{R}^n$ satisfying

$$\liminf_{u\to z_0}\frac{\varphi(u)-\varphi(z_0)-\langle x^*,u-x\rangle}{\|u-z_0\|}\ge 0. \tag{3.9}$$

Let $u_j:=z_0-\frac{1}{j}(y_0-x_0)$. Then $u_j\to z_0$ as $j\to\infty$. It is easy to see that $u_j\in P_1$ for all $j\in I\!N$. Together with (3.9) this gives

$$\liminf_{j\to\infty}\frac{\varphi_1(u_j)-\varphi_1(z_0)-\langle x^*,u_j-z_0\rangle}{\|u_j-z_0\|}\ge 0.$$

By the mean value theorem,

$$\liminf_{j\to\infty} \frac{\langle \nabla\varphi_1(\xi_j), -\frac{1}{j}(y_0-x_0)\rangle - \langle x^*, -\frac{1}{j}(y_0-x_0)\rangle}{\frac{1}{j}\|y_0-x_0\|} \geq 0,$$

where $\xi_j \in (u_j, z_0)$. Since $\nabla\varphi_1(\cdot)$ is continuous and $\xi_j \to z_0$ as $j\to\infty$, we have

$$\langle \nabla\varphi_1(z_0), y_0-x_0\rangle \leq \langle x^*, y_0-x_0\rangle.$$

Similarly, by taking $u_j = z_0 + \frac{1}{j}(y_0-x_0)$ we obtain

$$\langle x^*, y_0-x_0\rangle \leq \langle \nabla\varphi_2(z_0), y_0-x_0\rangle.$$

Consequently, $\langle \nabla\varphi_1(z_0) - \nabla\varphi_2(z_0), y_0-x_0\rangle \leq 0$ which contradicts (3.8). Hence $\widehat{\partial}\varphi(z_0) = \emptyset$ and $\nabla\varphi_1(z_0) \neq \nabla\varphi_2(z_0)$ by (3.8). By virtual of (3.5), we can find a positive number γ such that for each $u \in P_{12}\cap(z_0+\gamma\mathbb{B})$ there exist $x_u \in \mathrm{int}P_1$, $y_u \in \mathrm{int}P_2$ satisfying $u = (1-t_0)x_u + t_0 y_u$ and

$$\varphi(u) > (1-t_0)\varphi(x_u) + t_0\varphi(y_u),$$

where $\mathbb{B} := \{x\in\mathbb{R}^n \mid \|x\| < 1\}$. Then as in the proof of the claim $\widehat{\partial}\varphi(z_0) = \emptyset$, we can show that $\widehat{\partial}\varphi(u) = \emptyset$ and $\nabla\varphi_1(u) \neq \nabla\varphi_2(u)$ for all $u \in P_{12}\cap(z_0+\gamma\mathbb{B})$. By the continuity of $\nabla\varphi_1(\cdot)$ and of $\nabla\varphi_2(\cdot)$, together with (3.3) this gives

$$\begin{aligned}\partial\varphi(x) &= \operatorname*{Lim\,sup}_{u\to x} \widehat{\partial}\varphi(u)\\ &= \operatorname*{Lim\,sup}_{u\xrightarrow{\mathrm{int}P_1}x} \widehat{\partial}\varphi(u) \cup \operatorname*{Lim\,sup}_{u\xrightarrow{\mathrm{int}P_2}x} \widehat{\partial}\varphi(u) \cup \operatorname*{Lim\,sup}_{u\xrightarrow{P_{12}}x} \widehat{\partial}\varphi(u)\\ &= \{\nabla\varphi_1(x), \nabla\varphi_2(x)\},\end{aligned}$$

for all $x \in P_{12}\cap(z_0+\gamma\mathbb{B})$. Hence

$$\partial\varphi(x) = \begin{cases}\{\nabla\varphi_1(x)\} & \text{if } x\in\mathrm{int}P_1,\\ \{\nabla\varphi_2(x)\} & \text{if } x\in\mathrm{int}P_2,\\ \{\nabla\varphi_1(x),\nabla\varphi_2(x)\} & \text{if } x\in P_{12}\cap(z_0+\gamma\mathbb{B}).\end{cases} \tag{3.10}$$

For $x \in P_{12}\cap(z_0+\gamma\mathbb{B})$, $y = \nabla\varphi_1(x)$, and $u\in\mathbb{R}^n$, it holds

$$\partial^2\varphi(x,y)(u) = \nabla^2\varphi_1(x)(u) + \mathbb{R}_+ a.$$

Indeed, let $z = \nabla^2\varphi_1(x)(u) + \lambda a$ for some $\lambda \geq 0$. Since $\nabla\varphi_1(\cdot)$, $\nabla\varphi_2(\cdot)$ are continuous and $y = \nabla\varphi_1(x) \neq \nabla\varphi_2(x)$, by (3.10) for all $(x', y') \in \mathrm{gph}\partial\varphi$ near

(x, y) we have $x' \in P_1$ and $y' = \nabla\varphi_1(x')$. Hence

$$\begin{aligned}
&\limsup_{(x',y')\xrightarrow{\text{gph}\partial\varphi}(x,y)} \frac{\langle z, x'-x\rangle - \langle u, y'-y\rangle}{\|x'-x\| + \|y'-y\|} \\
&= \limsup_{x'\xrightarrow{P_1}x} \frac{\langle z, x'-x\rangle - \langle u, \nabla\varphi_1(x') - \nabla\varphi_1(x)\rangle}{\|x'-x\| + \|\nabla\varphi_1(x') - \nabla\varphi_1(x)\|} \\
&= \limsup_{x'\xrightarrow{P_1}x} \frac{\langle z - \nabla^2\varphi_1(\xi_{x'})(u), x'-x\rangle}{\|x'-x\| + \|\nabla\varphi_1(x') - \nabla\varphi_1(x)\|} \\
&= \limsup_{x'\xrightarrow{P_1}x} \frac{\langle \nabla^2\varphi_1(x)(u) - \nabla^2\varphi_1(\xi_{x'})(u), x'-x\rangle + \lambda\langle a, x'-x\rangle}{\|x'-x\| + \|\nabla\varphi_1(x') - \nabla\varphi_1(x)\|} \\
&\le \|u\| \limsup_{x'\xrightarrow{P_1}x} \|\nabla^2\varphi_1(x) - \nabla^2\varphi_1(\xi_{x'})\| = 0,
\end{aligned}$$

where $\xi_{x'} \in (x', x)$. This implies that $z \in \partial^2\varphi(x, y)(u)$ and thus,

$$\nabla^2\varphi_1(x)(u) + \mathbb{R}_+ a \subset \partial^2\varphi(x, y)(u).$$

To prove the reverse inclusion, take any $z \in \partial^2\varphi(x, y)(u)$. Then there exist $(z_i, u_i) \to (z, u)$ and $(x_i, y_i) \to (x, y)$ with $(x_i, y_i) \in \text{gph}\partial\varphi$ such that $(z_i, -u_i) \in \widehat{N}((x_i, y_i); \text{gph}\partial\varphi)$ for all i. Note that $\nabla\varphi_1(\cdot)$, $\nabla\varphi_2(\cdot)$ are continuously differentiable functions satisfying $\nabla\varphi_1(x) \ne \nabla\varphi_2(x)$ for all $x \in P_{12} \cap (z_0 + \gamma\mathbb{B})$. By (3.10), we may assume that $x_i \in P_1 \cap (z_0 + \gamma\mathbb{B})$, $y_i = \nabla\varphi_1(x_i)$ for all i. Hence,

$$\begin{aligned}
&(z_i, -u_i) \in \widehat{N}((x_i, y_i); \text{gph}\partial\varphi) \\
&\Leftrightarrow \limsup_{x'\xrightarrow{P_1}x_i} \frac{\langle z_i, x'-x_i\rangle - \langle u_i, \nabla\varphi_1(x') - \nabla\varphi_1(x_i)\rangle}{\|x'-x_i\| + \|\nabla\varphi_1(x') - \nabla\varphi_1(x_i)\|} \le 0 \\
&\Rightarrow \limsup_{x'\xrightarrow{P_1}x_i} \frac{\langle z_i - \nabla^2\varphi_1(\xi_{x'})(u_i), x'-x_i\rangle}{(1 + \sup\limits_{\xi\in z_0+\gamma\bar{\mathbb{B}}} \|\nabla^2\varphi_1(\xi)\|)\|x'-x_i\|} \le 0 \quad (\text{for some } \xi_{x'} \in (x', x_i)) \\
&\Rightarrow \langle z_i - \nabla^2\varphi_1(x_i)(u_i), x'\rangle \le 0 \quad \text{whenever } \langle a, x'\rangle \le 0.
\end{aligned}$$

Taking $i \to \infty$, we have $\langle z - \nabla^2\varphi_1(x)(u), x'\rangle \le 0$ if $\langle a, x'\rangle \le 0$. By the Farkas lemma, there exists $\lambda \ge 0$ such that $z - \nabla^2\varphi_1(x)(u) = \lambda a$ which proves $\partial^2\varphi(x, y)(u) \subset \nabla^2\varphi_1(x)(u) + \mathbb{R}_+ a$. Therefore, $\partial^2\varphi(x, y)(u) = \nabla^2\varphi_1(x)(u) + \mathbb{R}_+ a$ for all $x \in P_{12} \cap (z_0 + \gamma\mathbb{B})$, $y = \nabla\varphi_1(x)$ and $u \in \mathbb{R}^n$. Let $x \in P_{12} \cap (z_0 + \gamma\mathbb{B})$, $y = \nabla\varphi_1(x)$, $z = -\nabla^2\varphi_1(x)(a) + ta$ ($t \ge 0$) and $u = -a$. We have $z \in \partial^2\varphi(x, y)(u)$ and $\langle z, u\rangle = \langle \nabla^2\varphi_1(x)(a), a\rangle - t\|a\|^2 < 0$ for $t \ge 0$ large enough. This contradicts (3.2).

Remark. As can be seen from the above proof, we also obtain the contradiction if it is only supposed that φ is nonconvex on some ball $\bar{x}+\varepsilon\mathbb{B}$ ($\bar{x}\in\mathbb{R}^n, \varepsilon>0$) and (3.2) is replaced by the condition:

$$\langle z,u\rangle \geq 0 \;\text{ for all } u\in\mathbb{R}^n, z\in\partial^2\varphi(x,y)(u) \text{ with } (x,y)\in \mathrm{gph}\partial\varphi \text{ and } x\in\bar{x}+\varepsilon\mathbb{B}.$$

This remark will be used in the sequel.

Case 2. $k>2$. Since φ is nonconvex on $\mathbb{R}^n=\bigcup_{j=1}^{k} P_j$ but it is convex on each one of the polyhedrals P_j ($j=1,2,...,k$), there exist $x,y\in\mathbb{R}^n$ ($x\neq y$), $0=\tau_0<\tau_1<...<\tau_{m-1}<\tau_m=1$ ($m\in I\!N, m>1$), and $x_i:=x+\tau_i(y-x)$ ($i=0,1,...,m$) such that φ is convex on $[x_i,x_{i+1}]$ ($i=0,1,...,m-1$) but it is nonconvex on $[x,y]$. By Lemma 3.2 we can find $i\in\{0,1,...,m-2\}$ such that φ is nonconvex on $[x_i,x_{i+2}]$. Thus, without loss of generality we can assume that there exists $\bar{x}\in(x,y)$ such that φ is convex on each one of intervals $[x,\bar{x}]$ and $[\bar{x},y]$ but it is nonconvex on $[x,y]$. For each $u\in\mathbb{R}^n$, we put $I(u)=\{i\in\{1,2,...,k\} : u\in P_i\}$. Let $\varepsilon>0$ such that $(\bar{x}+\varepsilon\mathbb{B})\cap P_i=\emptyset$ for all $i\in\{1,2,...,k\}\backslash I(\bar{x})$. We may assume that $x,y\in\mathbb{B}(\bar{x},\varepsilon)$. Since φ is convex on $[x,\bar{x}]$ and on $[\bar{x},y]$ and it is nonconvex on $[x,y]$, $|I(\bar{x})|\geq 2$ and $\bar{x}\notin \mathrm{int}P_i$ for all i. If $|I(\bar{x})|=2$, then we obtain a contradiction by using the above remark. If $|I(\bar{x})|>2$, then $\dim L<n-1$ where $L:=\mathrm{aff}(\bigcap_{i\in I(\bar{x})} P_i)$ denotes the affine hull of $\bigcap_{i\in I(\bar{x})} P_i$. Indeed, without loss of generality we may assume that $\{1,2,3\}\subset I(\bar{x})$. Since $\mathrm{int}P_i\neq\emptyset$, $\mathrm{int}P_j\neq\emptyset$ and $\mathrm{int}P_i\cap\mathrm{int}P_j=\emptyset$ ($\forall i\neq j$), by the separation theorem, there exists a hyperplane L_{ij} separating the sets $\mathrm{int}P_i$ and $\mathrm{int}P_j$ ($1\leq i<j\leq 3$). Since it is impossible that $L_{12}=L_{13}=L_{23}$, $\dim(L_{12}\cap L_{13}\cap L_{23})<n-1$. Noting that $L\subset(L_{12}\cap L_{13}\cap L_{23})$, we have $\dim L<n-1$. In the case where $y\in L$, by invoking the last property we can find $\tilde{y}\in\mathbb{R}^n\setminus L$ as close to y as desired. Let $t\in(0,1)$ be such that $\bar{x}=(1-t)x+ty$. Define $\tilde{x}_{\tilde{y}}$ by the condition $\bar{x}=(1-t)\tilde{x}_{\tilde{y}}+t\tilde{y}$. Clearly, $\tilde{x}_{\tilde{y}}\notin L$ and $\tilde{x}_{\tilde{y}}\to x$ as $\tilde{y}\to y$. Since φ is continuous and nonconvex on $[x,y]$, there exists $\tilde{y}\in\mathbb{R}^n\setminus L$ as close to y as desired such that φ is nonconvex on $[\tilde{x}_{\tilde{y}},\tilde{y}]$. Thus, replacing (x,y) by $(\tilde{x}_{\tilde{y}},\tilde{y})$ if necessary, we can assume that $y\notin L$ and $x\notin L$. (Note that such replacement may destroy the property of φ of being convex on each one of the segments $[x,\bar{x}]$ and $[\bar{x},y]$. But this property will not be employed in the sequel.) Take $\rho>0$ such that $(y+\rho\mathbb{B})\subset(\bar{x}+\varepsilon\mathbb{B})$, $(y+\rho\mathbb{B})\cap L=\emptyset$, $x\notin(y+\rho\mathbb{B})$, and φ is nonconvex on $[x,z]$ for each $z\in(y+\rho\mathbb{B})$. Our aim now is to show that there exists $z\in(y+\rho\mathbb{B})$ such that $[x,z]\cap L=\emptyset$. Suppose that this is not true. Then $[x,z]\cap L\neq\emptyset$ for all $z\in(y+\rho\mathbb{B})$. Choose $y_i\in(y+\rho\mathbb{B})$ ($i=1,2,...,n-1$) such that $\{x-y,y_1-y,...,y_{n-1}-y\}$ is linearly independent. For each $i\in\{1,2,...,n-1\}$, we can take a vector $\bar{x}_i\in[x,y_i]\cap L$ because $[x,z]\cap L\neq\emptyset$ for all $z\in(y+\rho\mathbb{B})$ and $y_i\in(y+\rho\mathbb{B})$ ($i=1,2,...,n-1$). Note that $\bar{x}_i-\bar{x}=\alpha_i(x-y)+\beta_i(y_i-y)$ for some $\alpha_i \in \mathbb{R}, \beta_i \in \mathbb{R}\backslash\{0\}$ and $\{x-y, y_1-y, ..., y_{n-1}-y\}$ is linearly independent. Hence the system $\{\bar{x}_1-\bar{x}, ..., \bar{x}_{n-1}-\bar{x}\}$ is linearly independent from which we get $\dim L \geq n-1$. This contradicts the fact $\dim L < n-1$ derived above and thus there exists $z \in (y+\rho\mathbb{B})$ satisfying $[x,z] \cap L = \emptyset$. Since φ is nonconvex on $[x,z]$, we can find $[x',y'] \subset [x,z]$ and $\bar{x}' \in (x',y')$ such that φ is convex on each of the two intervals $[x',\bar{x}']$ and $[\bar{x}',y']$ and it is nonconvex on $[x',y']$. Observing that $\bar{x}' \in (\bar{x}+\varepsilon\mathbb{B})\backslash[\bigcap_{i\in I(\bar{x})} P_i]$ and $(\bar{x}+\varepsilon\mathbb{B}) \cap P_i = \emptyset$ for all $i \in \{1,2,...,k\}\backslash I(\bar{x})$, we have $|I(\bar{x}')| < |I(\bar{x})|$. Hence if $|I(\bar{x})| > 2$, then there exist $[x',y']$ and $\bar{x}' \in (x',y')$ such that φ is convex on each of the segments $[x',\bar{x}']$ and $[\bar{x}',y']$ but it is nonconvex on $[x',y']$ and $|I(\bar{x}')| < |I(\bar{x})|$. Thus, by repeating this procedure after finitely many times, we can reduce the case where $|I(\bar{x})| = 2$ and obtain a contradiction. The proof is now completed. ■

Recall that a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be *strongly convex* on a convex subset $\Omega \subset \mathrm{dom}\varphi$ if there exists a constant $\rho > 0$ such that

$$\varphi((1-t)x+ty) \leq (1-t)\varphi(x) + t\varphi(y) - \rho t(1-t)\|x-y\|^2$$

for any $x, y \in \Omega$ and $t \in (0,1)$. It is well known (see e.g. [12, Lemma 1, p. 184]) that the above condition is fulfilled if and only if the function $\widetilde{\varphi}(x) := \varphi(x) - \rho\|x\|^2$ is convex on Ω.

We now have the following characterization of strong convexity for piecewise C^2 functions.

Theorem 3.4. *Suppose that* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is a piecewise* C^2 *function. Then* φ *is strongly convex on* $\mathbb{R}^n$ *with the constant* $\rho > 0$ *if and only if for any* $(x,y) \in \mathrm{gph}\partial\varphi$ *the second-order subdifferential mapping* $\partial^2\varphi(x,y) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ *satisfies the condition*

$$\langle z,u\rangle \geq 2\rho\|u\|^2 \quad \textit{for all } u\in\mathbb{R}^n \textit{ and } z\in\partial^2\varphi(x,y)(u) \textit{ with } (x,y)\in\mathrm{gph}\partial\varphi. \tag{3.11}$$

Proof. It is well-known that φ is strongly convex on $\mathbb{R}^n$ with the constant $\rho > 0$ if and only if the function $\widetilde{\varphi} := \varphi + \psi$ where $\psi(x) = -\rho\|x\|^2$ is convex. By [7, Proposition 1.107(ii)],

$$\partial\widetilde{\varphi}(x) = \partial\varphi(x) - 2\rho x \quad \forall x \in \mathbb{R}^n. \tag{3.12}$$

Now, applying the coderivative sum rule with equality [7, Proposition 1.62(ii)] to the case where $F(x) = \partial\varphi(x)$ and $f(x) = -2\rho x$, we have

$$D^*(F+f)(x,y)(u) = D^*F(x,y-f(x))(u) - 2\rho u$$

for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ with $y - f(x) \in F(x)$ and for any $u \in \mathbb{R}^n$. Together with (3.12) this gives

$$\partial^2\widetilde{\varphi}(x,y)(u) = \partial\varphi^2(x,y)(u) - 2\rho u \quad \forall x \in \mathbb{R}^n,\ \forall y \in \partial\varphi(x). \tag{3.13}$$

According to Theorem 3.3, the convexity of $\widetilde{\varphi}$ is equivalent to the PSD of the second-order subdifferential mapping $\partial^2\widetilde{\varphi}(\cdot)$. Hence, by (3.13) we obtain $\langle v - 2\rho u, u\rangle \geq 0$ for any $v \in \partial\varphi^2(x,y)(u)$ which yields (3.11). ■

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Nguyen Huy Chieu
Department of Mathematics,
Vinh University,
Vinh, Nghe An,
Vietnam
E-mail: nghuychieu@vinhuni.edu.vn
nghuychieu@gmail.com.

Jen-Chih Yao
Department of Applied Mathematics,
National Sun Yat-Sen University,
Kaohsiung 804,
Taiwan
E-mail: yaojc@math.nsysu.edu.tw.