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TAYLOR EXPANSION FOR AN ANALYTIC HYPERSURFACE IN $\mathbb{R}^N$

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Abstract. Here we obtain a Taylor's expansion of the function $\rho(x, r) = \frac{|B(x,r)|}{|B(x,r)\cap\Omega|}$ for r small and $x \in \partial\Omega$ where the boundary of domain Ω is assumed to be analytic. The coefficients are expressed as recurrence relation and it is proved yhat the series is odd.

1. INTRODUCTION

Related to series of work on stationary isothermic surfaces and hot spots in [4-7], R. Magnanini and S. Sakaguchi obtained a simple geometric/measure theoretic characterization for stationary isothermic hypersurfaces. Precisely, let Γ be an oriented hypersurface in $\mathbb{R}^N$ and for $x \in \Gamma,\ r > 0$, define functions

$$\rho(x, r) = \frac{|B(x, r) \cap \Omega|}{|B(x, r)|} \tag{1.1}$$

$$\sigma(x, r) = \frac{|\partial B(x, r) \cap \Omega|}{|\partial B(x, r)|} \tag{1.2}$$

where Ω is the domain on one of the two sides of the surface Γ and $|.|$ is the Lebesgue measure. It was proved in [3] that Γ is a stationary isothermic surface if and only if $\rho(x, r)$(or $\sigma(x, r)$) is constant (depending only on r) at all points $x \in \Gamma$ and for all $0 < r < \infty$. Observe that the condition that $\rho(x, r)$ is constant is equivalent to $\sigma(x, r)$ is constant (for a.e. r).

Restricting r to vary in $(0, r_0)$ for r_0 small, assuming $\rho(x, r)$ is constant depending only on r, gave astonishing geometric results for Γ:

(i) firstly, this condition implied that Γ must be smooth;

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(ii) for dimensions $N \geq 2$, if Γ is compact, then it is necessarily a sphere in $\mathbb{R}^N$;

(iii) for dimension $N=3$, Γ is either a sphere or a spherical cylinder or a minimal surface.

Proofs of the above results and other related results can be found in [3]. The classifications (ii) and (iii) above were arrived at by writing down the Taylor's expansion for the functions $\rho(x,r)$ and $\sigma(x,r)$ in r and comparing them at two different points.

The quantity $meas(B(x,r) \cap \Omega)$ has been studied in different contexts. For example, related to the isoperimetric inequality in [10], a problem of Cimino in [9] and Besicovitch conjecture in [2], to mention a few of them.

The problem of Cimino is to classify all minimal surfaces S in $\mathbb{R}^3$ with the property that $\rho(x,r) = \frac{1}{2}$ for every $x \in S$, $r > 0$ and $B(x,r)$ is a ball in $\mathbb{R}^3$ with the center x on S. G. Cimino [1] was first to characterize the mean curvature $H(x)$ at a point x on a surface as the limit

$$H(x) = \lim_{r \to 0} \frac{(1/2 - \sigma(x,r))}{r}. \tag{1.3}$$

Nitsche [8], pp.60 is a good reference where one can find above definition as well as Taylor's expansion for $\sigma(0,r)$ upto order 3. The Taylor expansion given in [8] involved the mean curvature function and its derivatives. In [9], Nitsche used it to prove that the plane and the right helicoid are the only (minimal) surfaces in $\mathbb{R}^3$ such that $\rho \equiv 1/2$.

Similar expansion in lower order terms was also used by Preiss and Kowalski in [2] to classify manifolds which satisfy "local Besicovitch property" i.e., to classify smooth n dimensional submanifolds $M \subset \mathbb{R}^k$ ($n \leq k$) with the property

$$Vol(B(x,r) \cap M) = \alpha_n r^n \text{ for each } x \in M \text{ each sufficiently small } r > 0. \tag{1.4}$$

Here, $B(x,r)$ is the k-dimensional open ball in $\mathbb{R}^k$ with center x and radius $r > 0$ and α_n is the n-dimensional volume of a unit ball in $\mathbb{R}^n$. See [2] for more details. I wish to thank Prof. Gursky for informing me this reference.

Due to its varied applications in different fields, as seen from [9, 2] and [3], it is important to have the expression for the complete Taylor expansion for $\sigma(x,r)$. Here, the coefficients of $\sigma(x,r)$ are given by recurrence relations. In special situations of [9, 3, 2], these coefficients can be expressed as polynomials with principal curvatures of the hypersurface as variables. While using mathematica or maple it can be verified that the even terms of the series vanish, I give below a simple proof of the same using induction argument, proving that the Taylor series expansion for $\sigma(x,r)$ is odd.

2. Taylor's Expansion

Taylor's expansion for the density function $\rho(x,r)$ and $\sigma(x,r)$ upto order 3 have already been obtained in [3]. We proceed as described in [3] to write down the complete series. For the sake of completeness and to fix the notations we will repeat some of the proofs already mentioned therein. We begin by the following general lemma(Lemma 5.1 in [3]):

Lemma 2.1. *Let the function*

$$\theta(r) = \sum_{n=1}^{\infty} \theta_n r^n \tag{2.1}$$

be analytic in a neighbourhood of $r = 0$*. Then the function* $A_k(r) = f_k(\theta(r))$ *where*

$$f_k(\theta) := \int_0^{\theta} \cos^k s\, ds;\ k = 0, 1, 2, \ldots, \tag{2.2}$$

is analytic in a neighbourhood of $r = 0$ *and we have that*

$$A_k(r) = \sum_{n=1}^{\infty} A_k(n) r^n, \tag{2.3}$$

where

$$A_k(n) = \sum_{m=1}^{n} \frac{f_k^{(m)}}{m!} \alpha_{m,n},\ n \in \mathbb{N} \tag{2.4}$$

and

$$\alpha_{1,n} = \theta_n,\ n \in \mathbb{N} \tag{2.5}$$

$$\alpha_{m,n} = \sum_{i_1+\cdots+i_m=n} \theta_{i_1} \cdots \theta_{i_m},\ 1 \leq m \leq n,\ n \in \mathbb{N}. \tag{2.6}$$

Proof. Since $f_k(0) = 0$ we have Taylor's expansion

$$f_k(\theta) = \sum_{m=1}^{\infty} \frac{f_k^{(m)}(0)}{m!} \theta^m. \tag{2.7}$$

Using

$$\begin{aligned}
\theta(r)^m &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \theta_{i_1} \cdots \theta_{i_m} r^{i_1+\cdots+i_m} \\
&= \sum_{n=m}^{\infty} \Big(\sum_{i_1+\cdots i_m=n} \theta_{i_1} \cdots \theta_{i_m} \Big) r^n \\
&= \sum_{n=m}^{\infty} \alpha_{m,n} r^n
\end{aligned}$$

and inserting $\theta = \theta(r)$ in (2.7) we get

$$\begin{aligned} A_k(r) &= \sum_{m=1}^{\infty} \frac{f_k^{(m)}}{m!} (\sum_{n=m}^{\infty} \alpha_{m,n} r^n) \\ &= \sum_{n=1}^{\infty} (\sum_{m=1}^{n} \frac{f_k^{(m)}}{m!} \alpha_{m,n}) r^n. \end{aligned}$$ ∎

Note that we do not need to assume that Γ is oriented, since for $p \in \Gamma$ and r sufficiently small, the sphere (in $\mathbb{R}^N$) centered at p and with radius r will intersect the surface in a closed curve which will divide the sphere into two parts. We need to make choice of normal in a local neighbourhood. Moreover, since the computations are local, we may without loss of generality assume that the hypersurface Γ is the boundary $\partial\Omega$ of a domain Ω in $\mathbb{R}^N$. Also, to be able to write the entire series, we need to assume that the $\partial\Omega$ can be represented locally as a graph of an analytic function.

The Taylor's expansion for the functions $\rho(x, r)$ and $\sigma(x, r)$ in r, for r close to 0 will be obtained by writing the surface $\partial\Omega$ in terms of two different local parameterizations. First, for $x \in \partial\Omega$, let $T_x(\partial\Omega)$ denote that tangent space at x and ν denote the unit normal vector to $\partial\Omega$ at x. For fixed $v \in T_x(\partial\Omega)$ with $|v| = 1$, let $\pi_x(v, \nu)$ denote the plane through x spanned by the vectors v and ν. Also, assume that for $r > 0$ sufficiently small, each point $z \in \Omega \cap B(x, r)$ can be parameterized in spherical coordinates as

$$\left.\begin{aligned} z &= x + \rho\cos\phi v + \rho \sin\phi\nu \\ v &\in T_x(\partial\Omega) \cap S^{N-2},\ \theta(\rho, v) \le \phi \le \frac{\pi}{2},\ 0 \le \rho \le r \end{aligned}\right\} \tag{2.8}$$

where for fixed $v \in T_x(\partial\Omega) \cap S^{N-2}$, $\theta(\rho, v)$ parameterizes the curve $\partial\Omega \cap \pi_x(v, \nu)$ in polar coordinates. In the following, for sake of simplicity of notation, we will omit dependence of θ on x. Also, $\theta_r^{(n)}(r, v)$ denotes the n-th partial derivative with respect to the variable r.

For the second parameterization, we assume that $\partial\Omega$ is the graph of an analytic function $\varphi : B \subset \mathbb{R}^{N-1} \to \mathbb{R}$ in a neighbourhood of point $x \in \partial\Omega$. Thus, without loss of generality, supposing that x is origin and denoting the elements of $\mathbb{R}^{N-1}$ by $y = (y_1, \ldots, y_{N-1})$, we have the tangent plane $T_x\partial\Omega$ at X coincides with the hyperplane $y_N = 0$. We may further assume $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. In this neighbourhood, $\partial\Omega$ can be represented by the equation $y_N = \varphi(y)$. For the multi-index $i = (i_1, \ldots, i_{N-1})$, denoting

$$\begin{aligned} |i| &= i_1 + i_2 + \ldots + i_{N-1};\ i! = i_1! \cdots i_{N-1}!, \\ y^i &:= y_1^{i_1} \cdots y_{N-1}^{i_{N-1}},\ \text{for } y \in \mathbb{R}^{N-1} \\ D^i\varphi &= \partial_{y_1}^{i_1} \cdots \partial_{y_{N-1}}^{i_{N-1}} \varphi, \end{aligned}$$

the Taylor's expansion of φ around the origin is given by

$$\varphi(y) = \sum_{n=2}^{\infty} P_n(y) \text{ where } P_n(y) = \sum_{|i|=n} \frac{D^i\varphi(0)}{i!} y^i,\ n = 0, 1, 2, \dots. \tag{2.9}$$

Let ω_N denote the volume of unit sphere in $\mathbb{R}^N$. We have

Theorem 2.2. *Suppose that $\partial\Omega$ is analytic in a neighbourhood of a point $x \in \partial\Omega$ and let the function $\theta(\rho, v)$ parameterize $\partial\Omega$ as specified in (2.8) in neighbourhood of x. Then the functions $\sigma(x, r)$ and $\rho(x, r)$ defined in (1.2) and (1.1) respectively, admit the following Taylor series expansions:*

$$\sigma(x, r) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_n(x) r^n \tag{2.10}$$

and

$$\rho(x, r) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{N}{N+n} \sigma_n(x) r^n \tag{2.11}$$

where

$$\sigma_n(x) = -\frac{1}{\omega_N} \int_{S^{N-2}} \sum_{m=1}^{n} \frac{f_{N-2}^{(m)}(0)}{m!} \alpha_{m,n} dS_v, \tag{2.12}$$

with

$$\theta(r, v) = \theta_1(v) r + \theta_2(v) r^2 + \theta_3(v) r^3 + \dots \tag{2.13}$$

and $\alpha_{m,n}$'s as described in (2.5)-(2.6).

Moreover, using the non parametric representation $(y, \varphi(y))$ with $y = (y_1, \dots, y_{N-1}) \in B(0, R) \subset \mathbb{R}^{N-1}$ of $\partial\Omega$ in neighbourhood of x as described above, we can compute

$$\alpha_{1,1}(v) = P_2(v),\ \alpha_{1,2}(v) = P_3(v),\ \alpha_{1,3} = P_4(v) - \frac{23}{6} P_2(v)^3 \tag{2.14}$$

and for $n \geq 2$,

$$\begin{aligned} \alpha_{1,2n}(v) &= P_{2n+1}(v) - \sum_{k=1}^{n-1} \frac{f_1^{(2k+1)}(0)}{(2k+1)!}\ \alpha_{2k+1,2n}(v) \\ &\quad + \sum_{i=1}^{n-1} P_{2n-2i+1}(v) \sum_{k=1}^{i} \frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i}(v) \\ &\quad + \sum_{i=1}^{n-1} P_{2n-2i}(v) \sum_{k=1}^{i} \frac{f_{2n-2i}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i+1}(v) \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
\alpha_{1,2n+1}(v) &= P_{2n+2}(v) - \sum_{k=1}^{n} \frac{f_1^{(2k+1)}(0)}{(2k+1)!}\ \alpha_{2k+1,2n+1}(v) \\
&\quad + \sum_{i=1}^{n} P_{2n-2i+2}(v) \sum_{k=1}^{i} \frac{f_{2n-2i+2}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i}(v) \\
&\quad + \sum_{i=1}^{n-1} P_{2n-2i+1}(v) \sum_{k=1}^{i} \frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i+1}(v).
\end{aligned} \tag{2.16}
$$

and

$$
\alpha_{m,n}(v) = \sum_{i_1+\cdots+i_m=n} \alpha_{1,i_1} \cdots \alpha_{1,i_m},\ 1 \le m \le n,\ n \in N. \tag{2.17}
$$

Proof. For the polar coordinates (2.8), since the Jacobian of change of variables is $\rho^{N-1} \cos^{N-2} \phi$ we have

$$
|\Omega \cap B(x,r)| = \int_{S^{N-2}} \int_0^r \rho^{N-1} \int_{\theta(r,v)}^{\pi/2} \cos^{N-2} \phi \, d\phi \, d\rho dS_v \tag{2.18}
$$

where dS_v denotes the surface element on the sphere S^{N-2}. Differentiating (2.18) with respect to r and dividing by $\omega_N r^{N-1}$ we get

$$
\begin{aligned}
\sigma(x,r) &= \frac{1}{\omega_N} \int_{S^{N-2}} \int_{\theta(r,v)}^{\pi/2} \cos^{N-2} \phi \, d\phi dS_v \\
&= \frac{1}{2} - \frac{1}{\omega_N} \int_{S^{N-2}} \int_0^{\theta(r,v)} \cos^{N-2} \phi \, d\phi dS_v \\
&= \frac{1}{2} - \frac{1}{\omega_N} \int_{S^{N-2}} f_{N-2}(\theta) \, dS_v
\end{aligned}
$$

in notations of Lemma 2.1. It follows that from Lemma 2.1 that

$$
\sigma(x,r) = \frac{1}{2} + \sum_{n=1}^{\infty} \sigma_n r^n \ \text{ and} \tag{2.19}
$$

$$
\rho(x,r) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{N}{N+n} \sigma_n r^n \tag{2.20}
$$

where

$$
\sigma_n = - \sum_{m=1}^{n} \frac{1}{\omega_N} \int_{S^{N-2}} \frac{f_{N-2}^{(m)}(0)}{m!} \alpha_{m,n} \, dS_v \tag{2.21}
$$

with $\alpha_{m,n}$ as defined in (2.5) and (2.6). Note that here, we have $\theta(r,v) = \theta_1(v)r + \theta_2(v)r^2 + \theta_3(v)r^3 + \ldots$.

To determine the coefficients σ_n in terms of the more accessible geometric functions, we will use the second parameterization. Note that for r sufficiently small, $r\sin\theta(r,v) = \varphi(r\cos\theta(r,v)v)$ in the neighbourhood N. Thus, we have

$$\sin\theta(r,v) = \sum_{n=2}^{\infty} \cos^n\theta(r,v) P_n(v) r^{n-1}. \tag{2.22}$$

Write $\sin\theta(r,v) = \sum\limits_{k=0}^{\infty}(-1)^k \frac{\theta^{2k+1}}{k!}$ and $\cos^n\theta(r,v) = (f_n(\theta))' = \sum\limits_{m=1}^{\infty} \frac{f_n^{(m)}(0)}{m!}\theta^{m-1}$. Using notations of Lemma 2.1, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k \frac{\theta^{2k+1}}{k!} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k \frac{\alpha_{2k+1,2k+1+m}}{(2k+1)!} r^{m+2k+1} \\
&= \sum_{m=1}^{\infty}\left(\sum_{k=1}^{m}(-1)^{k-1}\frac{\alpha_{2k-1,2m}}{(2k-1)!}\right) r^{2m} \\
&\quad + \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}(-1)^{k}\frac{\alpha_{2k+1,2m+1}}{(2k+1)!}\right) r^{2m+1}
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty}\cos^n\theta(r,v)P_n(v)r^{n-1} \\
&= \sum_{n=2}^{\infty} P_n(v) r^{n-1} \sum_{m=1}^{\infty}\frac{f_n^{(m)}(0)}{(m-1)!}\theta^{m-1} \\
&= \sum_{k=0}^{\infty}\sum_{m=1}^{\infty}\sum_{n=2}^{\infty}\frac{f_n^{(m)}(0)}{(m-1)!}\alpha_{m-1,m-1+k}P_n(v)r^{n-1}r^{m-1+k} \\
&= \sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{k=0}^{m-j}\alpha_{m-j-k,m-j}P_{2+j}\frac{f_{2+j}^{(m-j-k+1)}}{(m-j-k)!}\right) r^m.
\end{aligned} \tag{2.24}$$

Note that, by definition $\alpha_{0,k} = 1$ for any $k \geq 0$. Equating the coefficients of r^{2m} we get

$$\sum_{k=1}^{m}(-1)^{k-1}\frac{\alpha_{2k-1,2m}}{(2k-1)!} = \sum_{j=0}^{2m-1}\sum_{k=0}^{j}\alpha_{j,k}P_{2+j}\frac{f_{2+j}^{(k+1)}}{k!} \tag{2.25}$$

and comparing coefficients of r^{2m+1} we have

$$\sum_{k=0}^{m}(-1)^k\frac{\alpha_{2k+1,2m+1}}{(2k+1)!}=\sum_{j=0}^{2m}\sum_{k=0}^{j}\alpha_{j,k}P_{2+j}\frac{f_{2+j}^{(k+1)}}{k!} \tag{2.26}$$

Thus we get

$$\alpha_{1,1}(v)=P_2(v),\ \alpha_{1,2}(v)=P_3(v),\ \alpha_{1,3}=P_4(v)-\frac{23}{6}P_2(v)^3. \tag{2.27}$$

Since $f_k^{(2j)}(0)=0$ for every $j=0,1,\ldots,$, from (2.25) and (2.26) we can write for $n\geq 2$:

$$\begin{aligned}\alpha_{1,2n}(v) &= P_{2n+1}(v)-\sum_{k=1}^{n-1}\frac{f_1^{(2k+1)}(0)}{(2k+1)!}\ \alpha_{2k+1,2n}(v)\\ &\quad+\sum_{i=1}^{n-1}P_{2n-2i+1}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i}(v)\\ &\quad+\sum_{i=1}^{n-1}P_{2n-2i}(v)\sum_{k=1}^{i}\frac{f_{2n-2i}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i+1}(v)\end{aligned}$$

and

$$\begin{aligned}\alpha_{1,2n+1}(v) &= P_{2n+2}(v)-\sum_{k=1}^{n}\frac{f_1^{(2k+1)}(0)}{(2k+1)!}\ \alpha_{2k+1,2n+1}(v)\\ &\quad+\sum_{i=1}^{n}P_{2n-2i+2}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+2}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i}(v)\\ &\quad+\sum_{i=1}^{n-1}P_{2n-2i+1}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\ \alpha_{2k,2i+1}(v).\end{aligned}$$

This completes the proof. ■

The following lemma was proved in [3] (Lemma 5.4):

Lemma 2.3. *Let $i=(i_1,\ldots,i_{N-1})$ be a multi-index. We have that the moments*

$$\int_{S^{N-2}} v^i DS_v = 0 \text{ if atleast one entry of } i \text{ is odd;} \tag{2.28}$$

otherwise,

$$\left.\begin{aligned} \frac{1}{\omega_{N-1}}\int\limits_{S^{N-2}} v^{2i}\,dS_v &= 1\text{for } N=2 \\ \frac{1}{\omega_{N-1}}\int\limits_{S^{N-2}} v^{2i}\,dS_v &= \frac{(N-3)!!(2i)!}{(2|i|+N-3)!!2^{|i|}i!}\text{for } N\geq 3 \end{aligned}\right\} \tag{2.29}$$

where $n!! = \prod_{k=0}^{[\frac{n-1}{2}]}(n-2k)$.

In view of (2.28)-(2.29) and (2.12), the fact that the series is odd will follow once we prove

Lemma 2.4. *Let* $\alpha_{m,n}$ *be defined as in (2.15)-(2.16)-(2.17). Then*

(a) $\alpha_{m,n}$, $1\leq m<n$ *is an odd polynomial in* v *if either* m *is odd or* n *is odd i.e.,* $\alpha_{2k-1,2n}$ *and* $\alpha_{2k,2n+1}$, $1\leq k\leq n$ *are odd polynomials in* v*;*

(b) $\alpha_{m,n}$ *is an even polynomial in* v *if both* m *and* n *are even or both are odd i.e.,* $\alpha_{2k,2n}$ *and* $\alpha_{2k+1,2n+1}$, $1\leq k\leq n$ *are even polynomials in* v.

Proof. We shall prove (a) and (b) by induction on n. Recall,

$$P_n(v) := \sum_{|i|=n}\frac{D^i\varphi(0)}{i!}v^i \tag{2.30}$$

and the relations

$$\begin{aligned} \alpha_{1,2n}(v) &= P_{2n+1}(v) - \sum_{k=1}^{n-1}\frac{f_1^{(2k+1)}(0)}{(2k+1)!}\;\alpha_{2k+1,2n}(v) \\ &\quad + \sum_{i=1}^{n-1}P_{2n-2i+1}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\;\alpha_{2k,2i}(v) \\ &\quad + \sum_{i=1}^{n-1}P_{2n-2i}(v)\sum_{k=1}^{i}\frac{f_{2n-2i}^{(2k+1)}(0)}{(2k)!}\;\alpha_{2k,2i+1}(v) \end{aligned} \tag{2.31}$$

and

$$\begin{aligned} \alpha_{1,2n+1}(v) &= P_{2n+2}(v) - \sum_{k=1}^{n}\frac{f_1^{(2k+1)}(0)}{(2k+1)!}\;\alpha_{2k+1,2n+1}(v) \\ &\quad + \sum_{i=1}^{n}P_{2n-2i+2}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+2}^{(2k+1)}(0)}{(2k)!}\;\alpha_{2k,2i}(v) \\ &\quad + \sum_{i=1}^{n-1}P_{2n-2i+1}(v)\sum_{k=1}^{i}\frac{f_{2n-2i+1}^{(2k+1)}(0)}{(2k)!}\;\alpha_{2k,2i+1}(v). \end{aligned} \tag{2.32}$$

We also observe that (2.17) can be written as

$$\alpha_{m,n}(v) = \sum_{i_1+\cdots+i_m=n} \alpha_{1,i_1}(v)\cdots\alpha_{1,i_m}(v), \quad 1 \le m \le n,\ n \in \mathbb{N}. \tag{2.33}$$

Comparing the coefficients, we get

$$\alpha_{1,1} = P_2(v) = \sum_{|i|=2} \frac{D^i\varphi(0)}{i!} v^i \tag{2.34}$$

which is an even polynomial in v and

$$\alpha_{1,2} = P_3(v) = \sum_{|i|=3} \frac{D^i\varphi(0)}{i!} v^i \tag{2.35}$$

which is an odd polynomial in v. Thus (2.33) yields:

$$\alpha_{2,3}(v) = 2\alpha_{1,1}(v)\alpha_{1,2}(v) = 2P_2(v)P_3(v),$$

that is (a) is verified for $n = 3$. Also, (2.33) implies that

$$\alpha_{2,2}(v) = \alpha_{1,1}(v)^2 = P_2(v)^2 \quad \text{and} \quad \alpha_{3,3}(v) = \alpha_{1,1}(v)^3 = P_2(v)^3,$$

and (2.16) gives

$$\alpha_{1,3}(v) = P_4(v) + \left[\frac{f_2^{(3)}(0)}{2!} - \frac{f_1^{(3)}(0)}{3!}\right] P_2(v)^3,$$

that is (b) is verified for $n = 3$.

Now, assume that (a) and (b) hold for all $1 \le n \le m$, we will show that they hold for $n = m + 1$. It is clear from (2.15) and (2.16) that our assumption implies that $\alpha_{1,2m+2}(v)$ is odd and $\alpha_{1,2m+3}(v)$ is even in v.

Claim: $\alpha_{2k-1,2m+2}(v)$ is odd polynomial in v for $1 \le k \le m + 1$.
Let $\mathcal{S}_{m,n}$ denote the set of all partitions of n into m parts i.e.,

$$\mathcal{S}_{m,n} := \{(i_1, \ldots, i_m) \in \mathbb{N}^m : i_1 + \ldots + i_m = n\}. \tag{2.36}$$

Then, from (2.33)we can write,

$$\alpha_{2k-1,2m+2}(v) = \sum_{(i_1,\ldots,i_{2k-1})\in\mathcal{S}_{2k-1,2m+2}} \alpha_{1,i_1}(v)\cdots\alpha_{1,i_{2k-1}}(v)$$

Now observe that for $2 \le k \le m$, a partition $(i_1, \ldots, i_{2k-1}) \in \mathcal{S}_{2k-1,2m+2}$ can be obtained from an $(j_1, \ldots, j_{2k-1}) \in \mathcal{S}_{2k-1,2m}$ in either of the following ways:

(i) $i_l = j_l + 2$ for some l and $j_s = i_s$ for all $s \neq l$;or
(ii) $i_l = j_l + 1$, $i_r = j_r + 1$ with $l \neq r$ and $i_s = j_s$ for all $s \neq l$, r

In case (i), if i_l is an odd number (resp. even number) then so is j_l. Hence by induction, $\alpha_{1,i_l}(v)$ and $\alpha_{1,j_l}(v)$ are both respectively even polynomial or an odd polynomial.

In case (ii), if j_l is an odd number and j_r is an even number, then i_l will be even and i_r will be odd. If both j_l and j_r are even(resp.odd), then both i_l and i_r are odd(resp. even).

In either case (i) or (ii), we must have the product $\alpha_{1,i_1}(v)\cdots\alpha_{1,i_{2k-1}}(v)$ to be an odd polynomial of v for such $(i_1,\ldots,i_{2k-1}) \in \mathcal{S}_{2k-1,2m+2}$ with $2 \leq k \leq m$, and hence $\alpha_{2k-1,2m+2}(v)$ is an odd polynomial for $2 \leq k \leq m$.

For $k = m+1$, the only possible partition is $i_l = 2$ for some l and $i_r = 1$ for $r \neq l$ and so in this case $\alpha_{2m+1,2m+2}(v) = (2m+1)P_2(v)^{2m}P_3(v)$ which is odd.

The remaining cases can be dealt with similarly. This completes the proof. ∎

In view of Theorem 2.2, Lemma 2.1, Lemma 2.3, Lemma 2.4 and using the fact that $f_j^{2k+1} = (-j)^k$, we have

Proposition 2.5. *With notations of Theorem 2.2,*

$$\sigma(x,r) = \frac{1}{2} + \sum_{n=0}^{\infty} \sigma_{2n+1} r^{2n+1} \tag{2.37}$$

where

$$\begin{aligned}
&\omega_N \sigma_{2n+1} \\
&= -\int_{S^{N-2}} \left\{ \sum_k = 0^n \frac{f_{N-2}^{2k+1}(0)}{(2k+1)!} \alpha_{2k+1,2n+1}(v) \right\} dS_v \\
&= \int_{S^{N-2}} \left\{ (N-2)P_{2n+2} - \frac{[-(N-2)]^n + (-1)^n(N-2)}{(2n+1)!} P_2^{2n+1} \right. \\
&\quad + \sum_{i=1}^{n} P_{2n-2i+2} \sum_{k=1}^{i} \frac{[-(2n-2i+2)]^k}{(2k)!} \alpha_{2k,2i}(v) \\
&\quad + \sum_{i=1}^{n-1} P_{2n-2i+1} \sum_{k=1}^{i} \frac{[-(2n-2i+1)]^k}{(2k)!} \alpha_{2k,2i+1}(v) \\
&\quad \left. - \sum_{k=1}^{n-1} \frac{[-(N-2)]^k + (-1)^k(N-2)}{(2k+1)!} \alpha_{2k+1,2n+1}(v) \right\} dS_v.
\end{aligned} \tag{2.38}$$

To see relations with curvature invariants, recall the parametrisation $(x, \varphi(x))$ of the hypersurface described before the proof of Theorem 2.2. We had assumed $x = 0$, $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Moreover, after change of variables we may assume that the Hessian matrix of φ at 0 is the diagonal matrix $diag(\kappa_1, \ldots, \kappa_{N-1})$ where the κ_j's denote the principal curvatures of $\partial\Omega$ at x. Thus

$$\Delta\varphi(0) = -(N-1)H(x) \tag{2.39}$$

where $H(x)$ denotes the mean curvature at point x. Also, recall that

$$P_2(v) := \sum_{|i|=2} \frac{D^i\varphi(0)}{i!} v^i = \frac{1}{2} < Hess\varphi(0)v, v > .$$

Hence

$$\omega_N \sigma_1(x) = -\int_{S^{N-2}} P_2(v)\, dS_v = \frac{\omega_{N-1}}{2} H(x) \tag{2.40}$$

which gives us the definition (1.3). Observing that $\alpha_{2n+1,2n+1} = P_2^{2n+1}$ it can be seen that

$$\begin{aligned} &\frac{[-(N-2)]^n + (-1)^n(N-2)}{(2n+1)!} P_2^{2n+1} \\ &= \frac{\omega_{N-1}[-(N-2)]^n + (-1)^n(N-2)}{(2N+8n-2)^{(2n+1)}} \sum_{|i|=2n+1} \frac{(2i)!}{(i!)^2} \kappa^i \end{aligned} \tag{2.41}$$

where we use the multi index notation $i = (i_1, \ldots, i_{N-1})$, $\kappa := (\kappa_1, \ldots, \kappa_{N-1})$ and $\kappa^i = \kappa_1^{i_1} \cdots \kappa_{N-1}^{i_{N-1}}$. It can be further verified that

$$\int_{S^{N-2}} P_{2n+2}(v)\, dS_v = \frac{\omega_{N-1}}{(2N+4n+2)^{(n+1)}(n+1)!} \Delta^{(n+1)}\varphi(x). \tag{2.42}$$

Note that we can compute $\Delta^{(n+1)}\varphi(x)$ by successively differentiating the mean curvature equation

$$\begin{aligned} &(1 + |\nabla\varphi(x)|^2)\Delta\varphi(x) \\ &- < Hess.\varphi(x)\nabla\varphi(x), \nabla\varphi(x) >= (N-1)H(x)(1 + |\nabla\varphi(x)|^2)^{3/2}. \end{aligned} \tag{2.43}$$

In particular, for $n = 2$ and dimension $N = 3$, σ_3 coincides with the expression given in Nitsche ([8] pg.61). Thus, in special situations the above series can be used to derive geometric conditions on the hypersurface.

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