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ON APPROXIMATION OF INVERSE PROBLEMS FOR ABSTRACT HYPERBOLIC EQUATIONS

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. This paper is devoted to the numerical analysis of inverse problems for abstract hyperbolic differential equations. The presentation exploits a general approximation scheme and is based on C_0-cosine and C_0-semigroup theory within a functional analysis approach. We consider both discretizations in space as well as in time. The discretization in time is considered under the Krein-Fattorini conditions.

1. Introduction

Let $B(E)$ denote the Banach algebra of all linear bounded operators on a complex Banach space E. The set of all linear closed densely defined operators in E will be denoted by $\mathcal{C}(E)$.

Let us examine the inverse problem in E consisting of the search for a function $u(\cdot) \in C^2([0;T];E)$ and an element $d \in E$ from the equations

(1.1a) $$u''(t) = Au(t) + \Phi(t)d, \quad 0 \le t \le T,$$

(1.1b) $$u(0) = u^0,\, u'(0) = u^1,$$

(1.1c) $$u(T) = u^T,$$

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where $A \in \mathcal{C}(E)$, $\Phi(\cdot) \in C^2([0;T];B(E))$ and the elements u^0, u^1, $u^T \in E$ are given. The cases of parabolic and elliptic equations were considered in [10, 14]. Here we assume that the abstract differential equation in (1.1a) is of the hyperbolic type. This means that the operator A generates a C_0-cosine operator-function $C(\cdot, A)$. Recall that a C_0-cosine operator-function is used to represent a solution of the abstract Cauchy problem

$$\begin{cases} u''(t) = Au(t) + f(t), \quad 0 \le t \le T, \\ u(0) = u^0, \ u'(0) = u^1. \end{cases} \tag{1.2}$$

Definition 1.1. A function $u(\cdot)$ is called a *classical solution* of problem (1.2) if $u(\cdot)$ is twice continuously differentiable, $u(t) \in D(A)$ for all $t \in [0, T]$, and $u(\cdot)$ satisfies the relations in (1.2).

We denote by $\sigma(B)$ the spectrum of the operator B, by $\rho(B)$ the resolvent set of B.

Proposition 1.1. [6, 18]. *The operator A generates a C_0-cosine operator-function if and only if there are constants M and ω such that for each λ with $Re\,\lambda > \omega$ the value λ^2 is contained in the resolvent set $\rho(A)$ of the operator A and for the same value λ the following estimate holds :*

$$\left\| \frac{d^n}{d\lambda^n} \left(\lambda R(\lambda^2, A) \right) \right\| \leqslant \frac{M n!}{(\lambda - \omega)^{n+1}}, \quad n = 0, 1, 2, \dots . \tag{1.3}$$

For any strongly continuous C_0-cosine operator-function $C(\cdot, A)$ the following inequality holds

$$\|C(t, A)\| \leqslant M \exp(\omega |t|), \quad t \in \mathbb{R}. \tag{1.4}$$

In this case we will write $A \in C(M, \omega)$. Furthermore, we introduce the Kisynski space [7]

$$E^1 = \{x \in E : C(t, A)x \in C^1(\mathbb{R}; E)\}.$$

with the norm $\|x\|_{E^1} = \|x\| + \sup\limits_{0<t\le 1} \|C'(t, A)x\|$. This is a Banach space with the norm $\|\cdot\|_{E^1}$.

If the operator A generates a C_0-cosine operator-function $C(\cdot, A)$ and $f(\cdot) \in C([0, T]; E)$, then for any classical solution of (1.2)

$$u(t) = C(t, A)u^0 + S(t, A)u^1 + \int_0^t S(t - s, A) f(s)\, ds, \quad t \in [0, T], \tag{1.5}$$

where $S(t, A) := \int\limits_0^t C(s, A) ds$ is the corresponding C_0-sine operator-function. The formula (1.5) is the analog of the variation-of-constants formula for C_0-semigroups.

As in the case of C_0-semigroups of operators, the function $u(\cdot)$ given by (1.5) is not a classical solution, in general, since it may be not twice continuously differentiable.

Remark 1.1. According to (1.5), in general, the problem (1.1) is ill-posed. This happens, for instance, if resolvent $(\lambda I - A)^{-1}$ is compact for some λ. Indeed, in this case the integral operator $\int_0^T S(T-s,A)\Phi(s)\,ds$ is compact and thus the equation

$$\int_0^T S(T-s,A)\Phi(s)\,ds\,d = u(T) - C(T,A)u^0 - S(T,A)u^1$$

in the space E leads to an ill-posed problem. However, if we consider the operator $\int_0^T S(T-s,A)\Phi(s)\,ds$ as the operator from E to $\mathfrak{D}(A)$, where $\mathfrak{D}(A)$ equiped with the norm $\|x\|_{\mathfrak{D}(A)} = \|x\| + \|Ax\|$, then the operator $\int_0^T S(T-s,A)\Phi(s)\,ds : E \to \mathfrak{D}(A)$ has a chance to be not compact. Therefore, in case of $u(T), C(T,A)u^0, S(T,A)u^1 \in D(A)$ one can play with formula (4.3) to get a Fredhom equation of the second kind, which is a well-posed problem.

Definition 1.2. The function $u(\cdot) \in C([0,T);E)$ given by (1.5) is called a *mild solution* of problem (1.2).

Proposition 1.2. $[6]$. *Let the operator A be a generator of a C_0-cosine operator-function $C(\cdot,A)$, and let either*

(i) $f(\cdot), Af(\cdot) \in C([0,T);E)$ *and* $f(t) \in D(A)$ *for* $t \in [0,T]$

or

(ii) $f(\cdot) \in C^1([0,T];E)$.

Then the function $u(\cdot)$ given by (1.5) *with $u^0 \in D(A)$ and $u^1 \in E^1$ is a classical solution of problem* (1.2) *on* $[0,T]$.

If we differentiate both sides of (1.5), we get

$$u'(t) = S(t,A)Au^0 + C(t,A)u^1 + \int_0^t C(t-s,A)f(s)ds.$$

Integrating by parts we obtain an alternative form for the first derivative

$$u'(t) = S(t,A)(Au^0 + f(0)) + C(t,A)u^1 + \int_0^t S(t-s,A)f'(s)ds. \tag{1.6}$$

We have to note here that one cannot expect maximal regularity for the problem (1.2), see [4], so in order to get a classical solution the differentiability of $f(\cdot)$ is almost necessary condition. Let us write $v(t) = u'(t)$, $v^0 = u^1$, $v^1 = Au_0 + f(0)$, $f_1(t) = f'(t)$. Then last formula in (1.6) can be written as formula (1.5):

$$v(t) = C(t,A)v^0 + S(t,A)v^1 + \int\limits_0^t S(t-s,A)f_1(s)ds.$$

Proposition 1.2 yields the conditions under which the function $v(\cdot)$ is a classical solution (in particular is twice continuously differentiable) of the problem

$$\begin{cases} v''(t) = Av(t) + f_1(t), \quad 0 \le t \le T, \\ v(0) = v^0,\ v'(0) = v^1. \end{cases}$$

These conditions are that $v^0 \in D(A)$, $v^1 \in E^1$, $f_1(\cdot) \in C^1([0,T];E)$, i. e. $u^0, u^1 \in D(A)$, $Au^0 + f(0) \in E^1$, $f(\cdot) \in C^2([0,T];E)$. It follows from these conditions that $v(\cdot) \in C^2([0,T];E)$, i. e. $u(\cdot) \in C^3([0,T];E)$.

Following the same procedure it is possible to find some sufficient conditions under which the solution of the Cauchy problem becomes as smooth as we like. Set $w(t) = v'(t)$. Then, one can write

$$w(t) = C(t,A)w^0 + S(t,A)w^1 + \int\limits_0^t S(t-s,A)f_2(s)ds, \tag{1.7}$$

where $w^0 = v^1$, $w^1 = Av_0 + f_1(0)$, $f_2(t) = f_1'(t)$.

If $w^0 \in D(A)$, $w^1 \in E^1$ and $f_2(\cdot) \in C^1([0,T];E)$, then $w(\cdot) \in C^2([0,T];E)$, i. e. $u(\cdot) \in C^4([0,T];E)$. This leads us to the next proposition:

Proposition 1.3. *Assume that the operator $A \in C(M,\omega)$ and $u^0, u^1 \in D(A^2)$. Suppose also that the following conditions hold*

(i) $f(\cdot) \in C^3([0,T];E)$,

(ii) $Au^0 + f(0) \in D(A)$, $Au^1 + f'(0) \in E^1$.

Then the function $u(\cdot)$ from (1.5) belongs to $C^4([0,T];E)$. Conversely. Assume that the function $u(\cdot)$ defined by (1.5) belongs to $C^4([0,T];E)$, i.e. $u(\cdot) \in C^4([0,T];E)$, and $f(\cdot) \in C^3([0,T];E)$ with $f(0) = 0$. Then $f'(0) \in E^1$ and so $AS(t,A)f'(0) \in C([0,T];E)$.

Proof. We prove just second part of Proposition. As it can be seen from (1.7) we have

$$u'''(t) = A^2 S(t,A)u^0 + C(t,A)(Au^1 + f'(0)) + \int_0^t C(t-s,A)f''(s)ds$$
$$= S(t,A)A^2u^0 + C(t,A)(Au^1 + f'(0)) + S(t,A)f''(0) + \int_0^t S(t-s,A)f'''(s)ds.$$

Now,

$$u''''(t) = C(t,A)A^2u^0 + S(t,A)A^2u^1 + AS(t,A)f'(0) + C(t,A)f''(0) \tag{1.8}$$
$$+ \int_0^t C(t-s,A)f'''(s)ds,$$

hence the function $AS(t,A)f'(0) \in C([0,T];E)$.

Let consider the homogenous uniformly well-posed Cauchy problem

$$u''(t) = Au(t), \quad t \in I\!R; \qquad u(0) = u^0, \quad u'(0) = u^1. \tag{1.9}$$

Define the matrix operator $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} : E^1 \times E \to E^1 \times E$ acting on the element $(x,y) \in E^1 \times E$ according to the formula $\mathcal{A}(x,y) = (y, Ax)$. This operator has the domain $D(\mathcal{A}) := D(A) \times E^1$.

Let the uniformly well-posed problem (1.9) have the form

$$u''(t) = \mathfrak{B}^2u(t), \quad t \in I\!R; \quad u(0) = u^0, \quad u'(0) = u^1, \tag{1.10}$$

where $\mathfrak{B} \in \mathcal{C}(E)$. Then

Definition 1.3. We say that a solution $u(\cdot)$ of problem (1.10) satisfies Condition (K) if

$$u'(\cdot) \in C\Big([0,T];\mathfrak{D}(\mathfrak{B})\Big).$$

Proposition 1.4. [23]. *Problem* (1.10) *has a unique solution satisfying Condition* (K) *iff the following Cauchy problem:*

$$\begin{pmatrix} u \\ v \end{pmatrix}'(t) = \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}(t), \quad t \in I\!R, \quad \begin{pmatrix} u \\ v \end{pmatrix}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \tag{1.11}$$

is uniformly well posed on the space $E \times E$.

The following Condition (F) is analog to Condition (K), which allows to simplify the study of problem (1.9) by using C_0-semigroups.

Definition 1.4. We say that a C_0-cosine operator-valued function $C(\cdot,A)$ satisfies *Condition* (F) if the following conditions hold:

(i) there exists $\mathfrak{B} \in \mathcal{C}(E)$ such that $\mathfrak{B}^2 = A$, and $\mathfrak{B}$ commutes with any operator from $B(E)$ commuting with A;

(ii) the operator $S(t,A)$ maps E into $D(\mathfrak{B})$ for any $t \in I\!R$;

(iii) the function $\mathfrak{B}S(t,A)x$ is continuous in $t \in I\!R$ for every fixed $x \in E$.

Proposition 1.5. [6]. *Under Condition (F), for each $t \in I\!R$, we have $\mathfrak{B}S(t,A) \in B(E)$ and $\mathfrak{D}(\mathfrak{B}) \subseteq E^1$.*

Proposition 1.6. [6]. *Pairs of a Banach space E and a C_0-cosine operator-function $C(\cdot, A)$ (also uniformly bounded) such that Condition (F) does not hold do exist.*

We have to note that if $0 \in \rho(A)$, then conditions (K) and (F) are equivalent.

Proposition 1.7. [20]. *Let E be a Hilbert space, and let the operator A be self-adjoint and negative-definite. Then $A \in \mathcal{C}(M;\omega)$, condition (F) is satisfied and the corresponding space E^1 coincides with $\mathfrak{D}((-A)^{1/2})$.*

Theorem 1.1. [19]. *Let A and $\mathfrak{B}$ be operators satisfying condition (i) of Definition 1.4, and let $0 \in \rho(\mathfrak{B})$. The following conditions are equivalent:*

- (i) *the C_0-cosine operator-function $C(\cdot, A)$ satisfies Condition (F);*
- (ii) *the operator $\mathfrak{B}$ generates a C_0-group $\exp(\cdot\mathfrak{B})$ on E;*
- (iii) *the operator $\begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}$ with the domain $D(A) \times D(\mathfrak{B})$ generates a C_0-group on $E \times E$;*
- (iv) *the operator $\mathcal{A} := \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$ with the domain $D(A) \times D(\mathfrak{B})$ generates a C_0-group $\exp(\cdot\mathcal{A})$ on $\mathfrak{D}(\mathfrak{B}) \times E$, where $\mathfrak{D}(\mathfrak{B})$ is the Banach space of elements $D(\mathfrak{B})$ endowed with the graph norm;*
- (v) *the embedding $D(\mathfrak{B}) \subseteq E^1$ holds;*
- (vi) $\mathfrak{D}(\mathfrak{B}) = E^1$.

Proposition 1.8. [19]. *Under the conditions of Theorem 1.1, for $t \in I\!R$, we have*

- (i) $\exp(t\mathfrak{B}) = C(t,A) + \mathfrak{B}S(t,A)$, $C(t,A) = \Big(\exp(t\mathfrak{B}) + \exp(-t\mathfrak{B})\Big)/2$;
- (ii) $\exp(t\mathcal{A}) = \begin{pmatrix} \mathfrak{B}^{-1} & 0 \\ 0 & I \end{pmatrix} \exp\left(t \begin{pmatrix} 0 & \mathfrak{B} \\ \mathfrak{B} & 0 \end{pmatrix}\right) \begin{pmatrix} \mathfrak{B} & 0 \\ 0 & I \end{pmatrix}$.

The analog of Proposition 1.2 is given in

Theorem 1.2. [8]. *Let the operator $\mathfrak{B} = \sqrt{A}$ in problem* (1.2) *have a bounded inverse $\mathfrak{B}^{-1} \in B(E)$ and be a generator of a C_0-group. Assume also that the function $f(\cdot)$ have one of the following properties*:

- (i) $f(\cdot) \in C^1([0,T);E)$;
- (ii) $\mathfrak{B}f(\cdot) \in C([0,T);E)$.

Then for any $u^0 \in D(A)$ *and* $u^1 \in D(\mathfrak{B})$, *there exists a unique classical solution of problem* (1.2) *given by formula* (1.5) *in the form*

$$
\begin{aligned}
u(t) &= \frac{1}{2}\Big(\exp(t\mathfrak{B})+\exp(-t\mathfrak{B})\Big)u^0+\frac{1}{2}\Big(\exp(t\mathfrak{B})-\exp(-t\mathfrak{B})\Big)\mathfrak{B}^{-1}u^1 \\
&\quad +\frac{1}{2}\int_0^t\Big(\exp((t-s)\mathfrak{B})-\exp(-(t-s)\mathfrak{B})\Big)\mathfrak{B}^{-1}f(s)ds, \quad t\in[0,T].
\end{aligned} \tag{1.12}
$$

2. A General Approximation Scheme

A general approximation scheme, due to [21], [22], can be described in the following way. Let E_n and E be Banach spaces and $\{p_n\}$ be a sequence of linear bounded operators $p_n : E \to E_n, p_n \in B(E, E_n), n \in I\!N = \{1, 2, \cdots\}$, with the property:

$$\|p_n x\|_{E_n} \to \|x\|_E \text{ as } n \to \infty \text{ for any } x \in E.$$

Definition 2.1. The sequence of elements $\{x_n\}, x_n \in E_n, n \in I\!N$, is said to be $\mathcal{P}$-convergent to $x \in E$ iff $\|x_n - p_n x\|_{E_n} \to 0$ as $n \to \infty$ and we write this $x_n \xrightarrow{\mathcal{P}} x$.

Definition 2.2. The sequence of elements $\{x_n\}$, $x_n \in E_n$, $n \in I\!N$, is said to be $\mathcal{P}$-compact if *for any subset of interges* $I\!N' \subseteq I\!N$ *there exist a subset of interges* $I\!N'' \subseteq I\!N'$ *and* $x \in E$ *such that* $x_n \xrightarrow{\mathcal{P}} x$, *as* $n \to \infty$ *in* $I\!N''$.

Definition 2.3. The sequence of linear bounded operators $B_n \in B(E_n), n \in I\!N$, is said to be $\mathcal{PP}$-convergent to the bounded linear operator $B \in B(E)$ if for every $x \in E$ and for every sequence $\{x_n\}, x_n \in E_n, n \in I\!N$, such that $x_n \xrightarrow{\mathcal{P}} x$ one has $B_n x_n \xrightarrow{\mathcal{P}} Bx$. We write this as $B_n \xrightarrow{\mathcal{PP}} B$.

For general examples of notions of $\mathcal{P}-$convergence see [21].

Remark 2.1. If we set $E_n = E$ and $p_n = I$ for every $n \in I\!N$, where I is the identity operator on E, then Definition 2.1 leads to the usual pointwise convergence of bounded linear operators which we denote by $B_n \to B$.

In case of operators which have a compact resolvent it is natural to consider approximating operators which "preserve" the property of compactness. Hence,

Definition 2.4. A sequence of operators $\{B_n\}$, $B_n : E_n \to E_n$, $n \in I\!N$, *converges compactly* to an operator $B : E \to E$ if $B_n \xrightarrow{\mathcal{PP}} B$ and the following compactness condition holds:

$$\|x_n\|_{E_n} = O(1) \Longrightarrow \{B_n x_n\} \text{ is } \mathcal{P}\text{-compact.}$$

Definition 2.5. The region of stability $\Delta_s = \Delta_s(\{A_n\})$, $A_n \in \mathcal{C}(B_n)$, is defined as *the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \rho(A_n)$ *for almost all* n *and such that the sequence* $\{\|(\lambda I_n - A_n)^{-1}\|\}_{n\in\mathbb{N}}$ *is bounded for almost all* n. The region of convergence $\Delta_c = \Delta_c(\{A_n\})$, $A_n \in \mathcal{C}(E_n)$, is defined *as the set of all* $\lambda \in \mathbb{C}$ *such that* $\lambda \in \Delta_s(\{A_n\})$ *and such that the sequence of operators* $\{(\lambda I_n - A_n)^{-1}\}_{n\in\mathbb{N}}$ *is* $\mathcal{PP}$*-convergent to some operator* $S(\lambda) \in B(E)$.

Definition 2.6. The region of compact convergence of resolvents, $\Delta_{cc} = \Delta_{cc}(A_n, A)$, where $A_n \in \mathcal{C}(E_n)$ and $A \in \mathcal{C}(E)$ is defined as the set of all $\lambda \in \Delta_c \cap \rho(A)$ such that $(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$ compactly.

In the case of unbounded operators (recall that in general infinitesimal generators are unbounded), we consider the notion of *compatibility*.

Definition 2.7. The sequence of closed linear operators $\{A_n\}, A_n \in \mathcal{C}(E_n), n \in \mathbb{N}$, is said to be compatible with a linear closed operator $A \in \mathcal{C}(E)$ iff for each $x \in D(A)$ there is a sequence $\{x_n\}, x_n \in D(A_n) \subseteq E_n, n \in \mathbb{N}$, such that $x_n \xrightarrow{\mathcal{P}} x$ and $A_n x_n \xrightarrow{\mathcal{P}} Ax$. We write this as (A_n, A) are compatible.

Usually, in practice, the Banach spaces E_n are finite-dimensional, although, in general, e.g. in the case of a closed operator A, we have $\dim E_n \to \infty$ and $\|A_n\|_{B(E_n)} \to \infty$ as $n \to \infty$.

Definition 2.8. A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, $n \in \mathbb{N}$, is said to be stably convergent to an operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ *and* $\|B_n^{-1}\|_{B(E_n)} = O(1)$, $n \to \infty$. We will write this as: $B_n \xrightarrow{\mathcal{PP}} B$ stably.

Definition 2.9. A sequence of operators $\{B_n\}$, $B_n \in B(E_n)$, is called regularly convergent to the operator $B \in B(E)$ iff $B_n \xrightarrow{\mathcal{PP}} B$ *and the following implication holds*:

$$\|x_n\|_{E_n} = O(1) \;\&\; \{B_n x_n\} \text{ is } P\text{-compact} \Longrightarrow \{x_n\} \text{ is } P\text{-compact}.$$

We write this as: $B_n \xrightarrow{\mathcal{PP}} B$ regularly.

Theorem 2.1. [22]. *Let* $C_n, S_n \in B(E_n)$, $C, S \in B(E)$ *and* $\mathcal{R}(S) = E$. *Assume also that* $C_n \xrightarrow{\mathcal{PP}} C$ *compactly and* $S_n \xrightarrow{\mathcal{PP}} S$ *stably. Then* $S_n + C_n \xrightarrow{\mathcal{PP}} S + C$ *converges regularly.*

Theorem 2.2. [22]. *For* $Q_n \in B(E_n)$ *and* $Q \in B(E)$ *the following conditions are equivalent:*

(i) $Q_n \xrightarrow{\mathcal{PP}} Q$ *regularly,* Q_n *are Fredholm operators of index* 0 *and* $\mathcal{N}(Q) = \{0\}$*;*

(ii) $Q_n \xrightarrow{\mathcal{PP}} Q$ *stably and* $\mathcal{R}(Q) = E$*;*

(iii) $Q_n \xrightarrow{\mathcal{PP}} Q$ *stably and regularly;*

(iv) *if one of conditions* (i)–(iii) *holds, then there exist* $Q_n^{-1} \in B(E_n)$*,* $Q^{-1} \in B(E)$*, and* $Q_n^{-1} \xrightarrow{\mathcal{PP}} Q^{-1}$ *regularly and stably.*

Theorem 2.3. [5]. *Let the operators* A *and* A_n *generate* C_0*-semigroups. The following conditions* (A) *and* (B) *are equivalent to condition* (C)*.*

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ *such that the resolvents converge*
$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

(B) *Stability. There are some constants* $M_1 \geq 1$ *and* $\omega_1 \in I\!\!R$ *independent of* n *such that for any* $t \geq 0$

$$\| \exp(tA_n) \| \leq M_1 e^{\omega t} \text{ for all } n \in I\!\!N;$$

(C) *Convergence. For any finite* $T > 0$ *we have*

$$\max_{t \in [0,T])} \| \exp(tA_n) u_n^0 - p_n \exp(tA) u^0 \| \to 0$$

as $n \to \infty$ *for any* $u^0 \in E$*, whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$.

Usually it is assumed that conditions (A) and (B) for the corresponding C_0-semigroup case are satisfied without any loss of generality whatever process of discretization in time is considered. We denote by $T_n(\cdot)$ a family of discrete semigroups $T_n(t) = T_n(\tau_n)^{k_n}$, where $k_n = [\frac{t}{\tau_n}]$, as $\tau_n \to 0$, $n \to \infty$, see [13]. The generator of discrete semigroup is defined by $\breve{A}_n = \frac{1}{\tau_n}(T_n(\tau_n) - I_n) \in B(E_n)$ and hence $T_n(t) = (I_n + \tau_n \breve{A}_n)^{k_n}$, where $t = k_n \tau_n$.

Theorem 2.4. (Theorem ABC-discr, [13]). *The following conditions* (A) *and* (B') *are equivalent to condition* (C')*.*

(A) *Consistency. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(\breve{A}_n)$ *such that the resolvents converge*
$(\lambda I_n - \breve{A}_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$*;*

(B') *Stability. There are some constants* $M \geq 1$ *and* $\omega_1 \in I\!\!R$ *such that*

$$\|T_n(t)\| \leq M \exp(\omega_1 t) \text{ for } t \in \overline{I\!\!R}_+ = [0, \infty), n \in I\!\!N;$$

(C') *Convergence. For any finite* $T > 0$ *one has*

$$\max_{t \in [0,T]} \|T_n(t) u_n^0 - p_n \exp(tA) u^0\| \to 0$$

as $n \to \infty$*, whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$ *for any* $u^0 \in E$*,* $u_n^0 \in E_n$*.*

Theorem 2.5. [13]. *Assume that* $A \in \mathcal{C}(E)$, $A_n \in \mathcal{C}(E_n)$ *and let* A, A_n *generate* C_0*-semigroups. Assume also that conditions* (A) *and* (B) *of Theorem* 2.3 *hold. Then, the implicit difference scheme*

$$\frac{\overline{U}_n(t+\tau_n)-\overline{U}_n(t)}{\tau_n} = A_n\overline{U}_n(t+\tau), \, \overline{U}_n(0)=u_n^0 \tag{2.1}$$

is stable, i.e. $\|(I_n-\tau_nA_n)^{-k_n}\| \le M_1e^{\omega_1 t}$, $t = k_n\tau_n \in \overline{I\!R}_+$, *and gives an approximation to the* $\exp(tA)u_n^0$, *i.e.* $\overline{U}_n(t) \equiv (I_n-\tau_nA_n)^{-k_n}u_n^0 \xrightarrow{\mathcal{P}} \exp(tA)u_n^0$ *uniformly with respect to* $t = k_n\tau_n \in [0,T]$ *as* $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $n \to \infty$, $k_n \to \infty$, $\tau_n \to 0$.

For C_0-cosine operator-functions the following ABC Theorem holds:

Theorem 2.6. [13]. *Let the operators* A *and* A_n *be generators of* C_0*-cosine operator-functions. Then, the following conditions* (A) *and* (B'') *are equivalent to condition* (C'') :

(A) *Compatability. There exists* $\lambda \in \rho(A) \cap \cap_n \rho(A_n)$ *such that the resolvents converge*
$(\lambda I_n - A_n)^{-1} \xrightarrow{\mathcal{PP}} (\lambda I - A)^{-1}$;

$(B")$ *Stability. There are some constants* $M_3 \ge 1$ *and* $\omega_3 \ge 0$ *such that*

$$\|C(t,A_n)\| \le M_3e^{\omega_3 t}, \quad t \ge 0, \quad n \in I\!N;$$

$(C")$ *Convergence. For any finite* $T > 0$ *one has*

$$\max_{t\in[0,T]} \|C(t,A_n)u_n^0 - p_nC(t,A)u^0\| \to 0$$

as $n \to \infty$ *for any* $u^0 \in E$, *whenever* $u_n^0 \xrightarrow{\mathcal{P}} u^0$.

3. Discretizing in Space and Time

The semidiscrete approximation of (1.2) leads to the following Cauchy problems in the Banach spaces E_n:

$$\begin{aligned} u_n''(t) &= A_nu_n(t) + f_n(t), \, t \in [0,T], \\ u_n(0) &= u_n^0, u_n'(0) = u_n^1, \end{aligned} \tag{3.1}$$

with operators A_n, which generate C_0-cosine operator-functions, the operators A_n and A are compatible, $u_n^0 \xrightarrow{\mathcal{P}} u^0$, $u_n^1 \xrightarrow{\mathcal{P}} u^1$ and $f_n(\cdot) \xrightarrow{\mathcal{P}} f(\cdot)$ in an appropriate sense. It is natural to assume that conditions (A) and (B'') of Theorem 2.6 for C_0-cosine operator-functions are satisfied.

The discretization of (3.1) in the time variable has been considered in many papers [1, 11, 17]. One of the simplest difference scheme is

$$\frac{U_n^{k+1} - 2U_n^k + U_n^{k-1}}{\tau_n^2} \tag{3.2}$$
$$= A_n U_n^{k+1} + \varphi_n^k,\ k \in \left\{1, ..., \left[\frac{T}{\tau_n}\right]\right\},\ U_n^0 = u_n^0,\ U_n^1 = u_n^0 + \tau_n u_n^1,$$

where, for instance if $f_n(\cdot) \in C([0,T]; E_n)$, one can set $\varphi_n^k = f_n(k\tau_n), k \in \{1, ..., K\}, K = \left[\frac{T}{\tau_n}\right]$, and in case that $f_n(\cdot) \in L^1([0,T]; E_n)$, one can set

$$\varphi_n^k = \frac{1}{\tau_n} \int_{t_{k-1}}^{t_k} f_n(s)ds,\ t_k = k\tau_n,\ k \in \{1, ..., K\}.$$

The solution to problem (3.2) is given by the formula [16]:

$$U_n^k = C_k^{(n)} U_n^0 + S_k^{(n)} U_n^1 + \tau_n^2 R_n \sum_{j=2}^{k} S_{k+1-j}^{(n)} \varphi_n^{j-1}, \tag{3.3}$$

where $k \geqslant 2$. Indeed, in order to solve the homogeneous equations associated to (3.2), i.e.

$$U_n^{k+1} - 2(I_n - \tau_n^2 A_n)^{-1} U_n^k + (I_n - \tau_n^2 A_n)^{-1} U_n^{k-1} = 0, \tag{3.4}$$

we consider the discrete operator-functions defined by the recurrent relations

$$\begin{aligned} C_{k+1}^{(n)} &= R_n(2C_k^{(n)} - C_{k-1}^{(n)}), \quad C_0^{(n)} = I_n, \quad C_1^{(n)} = 0, \\ S_{k+1}^{(n)} &= R_n(2S_k^{(n)} - S_{k-1}^{(n)}), \quad S_0^{(n)} = 0, \quad S_1^{(n)} = I_n, \end{aligned} \tag{3.5}$$

where $R_n = (I_n - \tau_n^2 A_n)^{-1}$. Then, the solution of (3.4) is given by

$$U_n^k = C_k^{(n)} U_n^0 + S_k^{(n)} U_n^1 = (C_k^{(n)} + S_k^{(n)}) U_n^0 + \tau_n S_k^{(n)} \frac{U_n^1 - U_n^0}{\tau_n}.$$

To operate with representations of discrete families of operators we give the following

Definition 3.1. [12]. The operators A_n of C_0-cosine operator-valued function $C(\cdot, A_n)$ satisfy *the discrete Krein-Fattorini Conditions* if the following conditions hold:

(i) there exist $\mathfrak{B}_n \in \mathcal{C}(E_n)$ such that $\mathfrak{B}_n^2 = A_n$, and $\mathfrak{B}_n$ commutes with any operator from $B(E_n)$ commuting with A_n;

(ii) the operators $\mathfrak{B}_n$ generate C_0-groups such that $\|\exp(\pm t\mathfrak{B}_n)\| \leq M_0 e^{\omega_0|t|}, t \in \mathbb{R}$;

(iii) the operators $-A_n$ are strongly positive, i.e.

$$\|(\lambda I_n - A_n)^{-1}\| \leq \frac{M}{1+|\lambda|}, \quad Re\lambda \geq 0,$$

and $\|\mathfrak{B}_n^{-1}\| \leq C$ as $n \in I\!N$.

We can obtain explicit representations for the functions $C_k^{(n)}, S_k^{(n)}$ in the following way. Let us introduce the operators

$$R_{1,n} = (I_n - \tau_n \mathfrak{B}_n)^{-1}, \quad R_{2,n} = (I_n + \tau_n \mathfrak{B}_n)^{-1},$$

where the operators $\mathfrak{B}_n$ are those in the Krein-Fattorini conditions. These operators satisfy the relations

$$R_{1,n}R_{2,n} = R_n, \quad R_{1,n} - R_{2,n} = 2\tau_n \mathfrak{B}_n R_n, \quad R_{1,n} + R_{2,n} = 2R_n, \tag{3.6}$$

which follow from the well-known Hilbert identity for resolvents. Since under the Krein-Fattorini conditions the operator $\mathfrak{B}_n$ generates a C_0-group one has that $\|R_{j,n}^k\| \leq const(t), k\tau_n = t$ for $j = 1, 2$.

Simple calculations show that the general solution of (3.4) is given as in [16] by the formula

$$U_n^k = R_{1,n}^k x + R_{2,n}^k y, \tag{3.7}$$

where x and y are arbitrary elements of E_n. Note that the representation (3.7) was established also in [2], [3] without Krein-Fattorini conditions, but in our case we need that $\|R_{1,n}^{k_n}\| \leq Me^{\omega t}, \|R_{2,n}^{k_n}\| \leq Me^{\omega t}$ with $k_n \tau_n = t$. Now if we solve the system

$$\begin{cases} x + y = U_n^0 \\ R_{1,n}x + R_{2,n}y = U_n^1 \end{cases}$$

and insert x and y in (3.7), we obtain by some calculations

$$C_k^{(n)} = -R_n \sum_{s=0}^{k-2} R_{1,n}^s R_{2,n}^{k-2-s}, \quad S_k^{(n)} = \sum_{s=0}^{k-1} R_{1,n}^s R_{2,n}^{k-1-s}.$$

From (3.6) we derive

$$R_n \mathfrak{B}_n S_k^{(n)} = \frac{1}{2\tau_n} \left(R_{1,n} - R_{2,n}\right) \sum_{s=0}^{k-1} R_{1,n}^s R_{2,n}^{k-1-s} = \frac{1}{2\tau_n} \left(R_{1,n}^k - R_{2,n}^k\right). \tag{3.8}$$

We note also that

$$R_{1,n}^{k} - R_{1,n}^{k-1} = \tau_n \mathfrak{B}_n R_{1,n}^{k}, \qquad R_{2,n}^{k} - R_{2,n}^{k-1} = -\tau_n \mathfrak{B}_n R_{2,n}^{k}, \tag{3.9}$$

and

$$R_{1,n}^{k} + R_{2,n}^{k} = 2R_n(S_k^{(n)} - S_{k-1}^{(n)}). \tag{3.10}$$

The equality (3.10) can be proved by induction on k. For $k = 1$ and $k = 2$ it can be checked by direct calculations. For $k > 2$,

$$R_{1,n}^{k+1} + R_{2,n}^{k+1} = \left(R_{1,n}^{k} + R_{2,n}^{k}\right)(R_{1,n} + R_{2,n}) - R_{1,n}R_{2,n}\left(R_{1,n}^{k-1} + R_{2,n}^{k-1}\right)$$
$$= 2R_n(S_k^{(n)} - S_{k-1}^{(n)}) \cdot 2R_n - R_n \cdot \left(S_{k-1}^{(n)} - S_{k-2}^{(n)}\right) = 2R_n^2\left(2S_k^{(n)} - 3S_{k-1}^{(n)} + S_{k-2}^{(n)}\right)$$
$$= 2R_n\left(R_n\left(2S_k^{(n)} - S_{k-1}^{(n)}\right) - R_n\left(2S_{k-1}^{(n)} - 2S_{k-2}^{(n)}\right)\right) = 2R_n\left(S_{k+1}^{(n)} - S_k^{(n)}\right).$$

From (3.6) and equality $I - R_{1,n} = -\tau_n \mathfrak{B}_n R_{1,n}$ we have

$$\begin{aligned}
C_k^{(n)} + S_k^{(n)} &= -R_{1,n}R_{2,n}\sum_{s=0}^{k-2} R_{1,n}^{s}R_{2,n}^{k-2-s} + \sum_{s=0}^{k-1} R_{1,n}^{s}R_{2,n}^{k-1-s} \\
&= -\sum_{s=0}^{k-2} R_{1,n}^{s+1}R_{2,n}^{k-1-s} + \sum_{s=0}^{k-1} R_{1,n}^{s}R_{2,n}^{k-1-s} \\
&= \sum_{s=0}^{k-2}\left(R_{1,n}^{s} - R_{1,n}^{s+1}\right)R_{2,n}^{k-1-s} + R_{1,n}^{k-1} \\
&= -\tau_n \mathfrak{B}_n \sum_{s=0}^{k-2} R_{1,n}^{s+1}R_{2,n}^{k-1-s} + R_{1,n}^{k-1} \\
&= -\tau_n \mathfrak{B}_n R_{1,n}R_{2,n}\sum_{s=0}^{k-2} R_{1,n}^{s}R_{2,n}^{k-2-s} + R_{1,n}^{k-1} \\
&= -\tau_n \mathfrak{B}_n R_n S_{k-1}^{(n)} + R_{1,n}^{k-1}.
\end{aligned}$$

Using (3.8) we get

$$\tau_n \mathfrak{B}_n R_n S_{k-1}^{(n)} = \frac{1}{2}\left(R_{1,n}^{k-1} - R_{2,n}^{k-1}\right), \tag{3.11}$$

and consequently

$$C_k^{(n)} + S_k^{(n)} = \frac{1}{2}\left(R_{1,n}^{k-1} + R_{2,n}^{k-1}\right). \tag{3.12}$$

Let consider the inhomogeneous equation (3.2), i.e.

$$U_n^{k+1} - 2(I_n - \tau_n^2 A_n)^{-1} U_n^k + (I_n - \tau_n^2 A_n)^{-1} U_n^{k-1} = \tau_n^2 (I_n - \tau_n^2 A_n)^{-1} \varphi_n^k. \tag{3.13}$$

Using the recurrent relation (3.5) we derive formula (3.3), see [16].

4. Existence of Solutions to the Inverse Problem

Consider the inverse problem (1.1) in the following form: for given elements $u^T, u^0, u^1 \in D(A)$ find a solution $u(\cdot) \in C^2([0,T];E)$ and an element $d \in E$ such that

$$\begin{cases} u''(t) = Au(t) + \Phi(t)d, \quad 0 \le t \le T, \\ u(0) = u^0, u'(0) = u^1, \\ u(T) = u^T. \end{cases} \tag{4.1}$$

Here $A \in C(M;\omega)$. The problem (4.1) is an inverse problem with overdetermination. Details on such kind of description of problems can be found in [15].

Basing on Remark 1.1, we can treat the solution of (4.1) as follows

$$A \int_0^T S(T-s,A)\Phi(s)\,ds\,d = Au(T) - C(T,A)Au^0 - AS(T,A)u^1$$

and then use the identities

$$A \int_0^T S(T-s,A)\Phi(s)\,ds = \int_0^T C(T-s,A)\Phi'(s)\,ds - \Phi(T) + C(T,A)\Phi(0) \tag{4.2}$$

and

$$\begin{aligned} &A \int_0^T S(T-s,A)\Phi(s)\,ds \\ &= \int_0^T S(T-s,A)\Phi''(s)\,ds - S(T,A)\Phi'(0) - \Phi(T) + C(T,A)\Phi(0). \end{aligned} \tag{4.3}$$

Proposition 4.1. *Let $\Phi(\cdot) \in C^1([0,T];B(E))$, the operator $\Phi(T)$ be invertible, i.e. $\Phi(T)^{-1} \in B(E)$. Then the inverse problem (4.1) is equivalent to that of solving*

$$Id - B_1 d = g_1, \tag{4.4}$$

where

$$B_1 := \Phi(T)^{-1} \left(\int_0^T \Big(C(T-s,A)\Phi'(s) - \lambda S(T-s,A)\Phi(s) \Big) ds + C(T,A)\Phi(0) \right)$$

and

$$g_1 := -\Phi(T)^{-1}(A - \lambda I)(u^T - C(T,A)u^0 - S(T,A)u^1) \text{ for } \lambda \in \rho(A).$$

Proposition 4.2. *Let* $\Phi(\cdot) \in C^2([0,T]; B(E))$*, and assume that the operator*

(4.5) $$D = \Phi(T) - C(T,A)\Phi(0)$$

is invertible, i.e. $D^{-1} \in B(E)$*. Then the inverse problem* (4.1) *is equivalent to that of solving*

(4.6) $$Id - B_2 d = g_2,$$

where

$$B_2 := D^{-1}\left(\int_0^T S(T-s,A)\Big(\Phi''(s) - \lambda\Phi(s)\Big)ds + S(T,A)\Phi'(0)\right)$$

and

$$g_2 := -D^{-1}(A - \lambda I)(u^T - C(T,A)u^0 - S(T,A)u^1) \text{ for } \lambda \in \rho(A).$$

Proposition 4.3. [15]. *Let the conditions of Proposition* 4.1 *be satisfied and*

$$\int_0^T (\|\Phi'(s)\| + |\lambda|(T-s)\|\Phi(s)\|)e^{\omega(T-s)}ds + \|\Phi(0)\|e^{\omega T} < \frac{1}{M\|\Phi(T)^{-1}\|}\,.$$

Then a solution $(u(\cdot), d)$ *of the inverse problem* (4.1) *exists and is unique for any input data* u^0*,* $u^T \in D(A)$*,* $u^1 \in E^1$*.*

Proposition 4.4. [15]. *Assume that the conditions of Proposition* 4.2 *and the inequality*

$$\int_0^T (T-s)\|\Phi''(s) - \lambda\Phi(s)\|e^{\omega(T-s)}ds + T\|\Phi'(0)\|e^{\omega T} < \frac{1}{M\|D^{-1}\|}$$

are satisfied. Then a solution $(u(\cdot), d)$ *of the inverse problem* (4.1) *exists and is unique for any input data* u^0*,* $u^T \in D(A)$*,* $u^1 \in E^1$*.*

Proposition 4.5. [15]. *Assume that the operator* A *generates a strongly continuous* C_0*-cosine operator-function* $C(\cdot, A)$ *on the Banach space* E*,* $\Phi(t) \equiv I$ *and* $0 \in \rho(A)$*. Then the inverse problem* (4.1) *is uniquely solvable for any input data* u^0*,* $u^T \in D(A)$*,* $u^1 \in E^1$ *if and only if* $1 \in \rho(C(T,A))$*.*

We now assume that E is the Hilbert space and the operator A is selfadjoint and negative. For any real-valued function $\Phi(\cdot)$, the value $\Phi(t)$ will be identified with the operator of multiplication by the number $\Phi(t)$ in the space E. The characteristic function $\varphi(\cdot)$ on the negative semi-axis is defined by

(4.7) $$\varphi(\lambda) = \frac{1}{\sqrt{-\lambda}}\int_0^T \Phi(s)\sin(\sqrt{-\lambda}(T-s))ds.$$

Note that we might extend the function $\varphi(\cdot)$ from the negative semi-axis to construct an entire function of the complex variable λ. If, in particular, $\Phi(t) \not\equiv 0$, then the zeroes of the function $\varphi(\cdot)$ are isolated.

In what follows, we denote by E_λ the spectral decomposition of unity of the operator A. With this notation, we can write

$$A = \int_0^{+\infty} \lambda dE(\lambda).$$

Theorem 4.1. [15]. *If the operator A is self-adjoint and semibounded from above on the Hilbert space E, $\Phi(\cdot) \in C^1[0,T]$ and $\Phi(\cdot) \not\equiv 0$, then the following statements hold:*

(i) *the inverse problem* (4.1) *with the fixed input data u^0, $u^T \in D(A)$, $u^1 \in E^1$ is solvable if and only if*

$$\int_0^{+\infty} |\varphi(\lambda)|^{-2} d(E_\lambda g, g) < \infty, \tag{4.8}$$

being $g := u^T - C(T,A)u^0 - S(T,A)u^1$;

(ii) *if the inverse problem* (4.1) *is solvable, then its solution is unique if and only if the point spectrum of the operator A contains no zeros of the entire function $\varphi(\cdot)$ defined by* (4.7).

Of special interest is the particular case $\Phi(t) \equiv tI$. In this case we have

$$\varphi(\lambda) = \begin{cases} \dfrac{\sin(\sqrt{-\lambda}T) - \sqrt{-\lambda}T}{\lambda\sqrt{-\lambda}T} & , \ \lambda \neq 0, \\ T^3/6 & , \ \lambda = 0. \end{cases}$$

This function has no zeros on the negative semi-axis and $\varphi(\lambda) \sim -\frac{T}{\lambda}$ as $\lambda \to +\infty$. Hence the convergence of the integral

$$\int_0^{+\infty} |\varphi(\lambda)|^{-2} d(E_\lambda g, g)$$

is equivalent to that of

$$\int_{-\infty}^{+\infty} |\lambda|^2 d(E_\lambda g, g).$$

This integral converges for every element $g \in D(A)$. We thus proved the following

Proposition 4.6. *Assume that the operator A is self-adjoint and negative on the Hilbert space E, $\Phi(t) \equiv tI$, $t \geq 0$. Then the solution $(u(\cdot), d)$ of the inverse problem* (4.1) *exists and is unique for any input data u^0, $u^T \in D(A)$, $u^1 \in E^1$.*

5. Approximating the Solution of the Inverse Problems

Let us consider the semidiscretization of the inverse problem for the second-order equation (4.1): for given elements $u_n^T, u_n^0, u_n^1 \in D(A_n)$ find a solution $u_n(\cdot) \in C^2([0,T]; E_n)$ and an element $d_n \in E_n$ such that

$$\begin{cases} u_n''(t) = A_n u_n(t) + \Phi_n(t) d_n, \quad 0 \le t \le T, \\ u_n(0) = u_n^0, u_n'(0) = u_n^1, \\ u_n(T) = u_n^T, \end{cases} \tag{5.1}$$

where operators $A_n \in C(M;\omega)$, the operators A, A_n are consistent, $u_n^T \xrightarrow{\mathcal{P}} u^T$, $u_n^0 \xrightarrow{\mathcal{P}} u^0, u_n^1 \xrightarrow{\mathcal{P}} u^1$ and $\Phi_n(\cdot) \xrightarrow{\mathcal{PP}} \Phi(\cdot)$ in the sense.

The solution d_n of the problem (5.1) must satisfy the equation

$$I_n d_n - B_{n,2} d_n = g_{n,2}, \tag{5.2}$$

where

$$\begin{aligned} B_{n,2} :&= D_n^{-1} \left(\int_0^T S(T-s, A_n) \Big(\Phi_n''(s) - \lambda \Phi_n(s) \Big) ds + S(T, A_n) \Phi_n'(0) \right), \ \lambda \in \Delta_{cc}, \\ g_{n,2} :&= -D_n^{-1} (A_n - \lambda I_n) \Big(u_n^T - C(T, A_n) u_n^0 - S(T, A_n) u_n^1 \Big), \\ D_n &= \Phi_n(T) - C(T, A_n) \Phi_n(0). \end{aligned}$$

Theorem 5.1. [24]. *Let $A, A_n \in C(M;\omega)$. Then $S(t, A_n) \xrightarrow{\mathcal{PP}} S(t, A)$ compactly for any $t > 0$ iff $\Delta_{cc} \neq \emptyset$.*

Theorem 5.2. *Assume that $\Phi(\cdot) \in C^3([0,T]; B(E))$, $\Phi_n(\cdot) \in C^3([0,T]; B(E_n))$, $D_n^{-1} \xrightarrow{\mathcal{PP}} D^{-1}$, the resolvents $(\lambda I_n - A_n)^{-1}$, $(\lambda I - A)^{-1}$ are compact, $(A), (B'')$ and (1.4) are satisfied, $\Phi_n^{(j)}(t) \xrightarrow{\mathcal{PP}} \Phi^{(j)}(t)$ uniformly in $t \in [0,T]$ for $j \in \overline{1,3}$, and $\Delta_{cc} \neq \emptyset$. Assume also that the problem (4.1) has a unique solution for any $u^T \in D(A)$. Then there are solutions to problems (5.1) for almost all n and they converge to solution of problem (4.1), i.e. $u_n(t) \xrightarrow{\mathcal{P}} u(t)$ uniformly in $t \in [0,T]$ and $d_n \xrightarrow{\mathcal{P}} d$ as $n \in I\!N$, whenever $A_n u_n^0 \xrightarrow{\mathcal{P}} A u^0, A_n u_n^1 \xrightarrow{\mathcal{P}} A u^1, A_n u_n^T \xrightarrow{\mathcal{P}} A u^T$.*

Proof. We, first, show that the solutions of equations (5.2) converge to the solution of equation (4.6). Since $D_n^{-1} \xrightarrow{\mathcal{PP}} D^{-1}$, it is clear that $g_{n,2} \xrightarrow{\mathcal{P}} g_2$. If $B_{n,2} \xrightarrow{\mathcal{PP}} B_2$ compactly, then by Theorems 2.1 and 2.2 it follows that $d_n \xrightarrow{\mathcal{P}} d$ and Theorem 5.2 is proved.

Using Theorem 5.1 one can show that the operators $B_{n,2} \xrightarrow{\mathcal{PP}} B_2$ compactly. To see this recall that operators $B_{n,2}, B_2$ can be split into two parts. The first term

$$D_n^{-1}S(T,A_n)\Phi'_n(0)\xrightarrow{\mathcal{PP}}D^{-1}S(T,A)\Phi'(0)$$

converges compactly and the second term

$$D_n^{-1}(\lambda I_n - A_n)^{-1}(\lambda I_n - A_n)$$
$$\int_0^T S(T-s,A_n)\Phi''_n(s)\,ds\xrightarrow{\mathcal{PP}}D^{-1}(\lambda I - A)^{-1}(\lambda I - A)$$
$$\int_0^T S(T-s,A)\Phi''(s)\,ds,$$

also converges compactly, since $\Delta_{cc}\neq\emptyset$ and

$$(\lambda I_n - A_n)\int_0^T S(T-s,A_n)\Phi''_n(s)\,ds\xrightarrow{\mathcal{PP}}(\lambda I - A)\int_0^T S(T-s,A)\Phi''(s)\,ds.$$

The last statement can be derived from the representation like (4.2). Therefore from Theorems 2.1 and 2.2 it follows that $d_n\xrightarrow{\mathcal{P}}d$. The convergence of solutions $u_n(t)\xrightarrow{\mathcal{P}}u(t)$ uniformly in $t\in[0,T]$ then follows from representation formulae like (1.5).

Consider the discretization of (5.1) in time

$$\begin{aligned}&\frac{U_n^{k+1}-2U_n^k+U_n^{k-1}}{\tau_n^2} = A_nU_n^{k+1}+\Phi_n(k\tau_n)\tilde{d}_n,\\ &k\in\left\{1,...,\left[\frac{T}{\tau_n}\right]\right\},\ U_n^0 = u_n^0,\ U_n^1 = u_n^0+\tau_n u_n^1.\end{aligned}\tag{5.3}$$

According to (3.3) one can write its solution as

$$U_n^k = C_k^{(n)}U_n^0 + S_k^{(n)}U_n^1 + \tau_n^2R_n\sum_{j=2}^{k}S_{k+1-j}^{(n)}\Phi_n^{j-1}\tilde{d}_n,\tag{5.4}$$

where we wrote $\Phi_n^j = \Phi_n(j\tau_n)$. Using (3.8) and (3.9) and summing by parts we have

$$\begin{aligned}&R_nA_n\tau_n^2\sum_{j=2}^{k}S_{k+1-j}^{(n)}\Phi_n^{j-1} = R_n\mathfrak{B}_n^2\tau_n^2\sum_{j=2}^{k}S_{k+1-j}^{(n)}\Phi_n^{j-1}\\ &=\frac{\tau_n\mathfrak{B}_n}{2}\sum_{j=2}^{k}\left(R_{1,n}^{k+1-j}-R_{2,n}^{k+1-j}\right)\Phi_n^{j-1} = \frac{\tau_n\mathfrak{B}_n}{2}\sum_{j=2}^{k}R_{1,n}^{k+1-j}\Phi_n^{j-1}\\ &-\frac{\tau_n\mathfrak{B}_n}{2}\sum_{j=2}^{k}R_{2,n}^{k+1-j}\Phi_n^{j-1}\\ &=\frac{1}{2}\sum_{j=2}^{k}\left(R_{1,n}^{k+1-j}-R_{1,n}^{k-j}\right)\Phi_n^{j-1}+\frac{1}{2}\sum_{j=2}^{k}\left(R_{2,n}^{k+1-j}-R_{2,n}^{k-j}\right)\Phi_n^{j-1}\end{aligned}\tag{5.5}$$

$$= \frac{1}{2}\left(\sum_{j=2}^{k} R_{1,n}^{k+1-j}\left(\Phi_n^{j-1} - \Phi_n^{j-2}\right) - \Phi_n^{k-1} + R_{1,n}^{k-1}\Phi_n^0\right)$$
$$+\frac{1}{2}\left(\sum_{j=2}^{k} R_{2,n}^{k+1-j}\left(\Phi_n^{j-1} - \Phi_n^{j-2}\right) - \Phi_n^{k-1} + R_{2,n}^{k-1}\Phi_n^0\right)$$
$$= \frac{1}{2}\sum_{j=2}^{k}\left(R_{1,n}^{k+1-j} + R_{2,n}^{k+1-j}\right)\left(\Phi_n^{j-1} - \Phi_n^{j-2}\right) - \Phi_n^{k-1} + \frac{1}{2}\left(R_{1,n}^{k-1} + R_{2,n}^{k-1}\right)\Phi_n^0.$$

Using (3.10) and summing by parts we obtain again

$$\begin{aligned} &\frac{1}{2}\sum_{j=2}^{k}\left(R_{1,n}^{k+1-j} + R_{2,n}^{k+1-j}\right)\left(\Phi_n^{j-1} - \Phi_n^{j-2}\right) \\ &= R_n\sum_{j=2}^{k}\left(S_{k+1-j}^{(n)} - S_{k-j}^{(n)}\right)\left(\Phi_n^{j-1} - \Phi_n^{j-2}\right) \\ &= R_n\left(\sum_{j=2}^{k} S_{k+1-j}^{(n)}\left(\Phi_n^{j-1} - 2\Phi_n^{j-2} + \Phi_n^{j-3}\right) + S_{k-1}^{(n)}\left(\Phi_n^0 - \Phi_n^{-1}\right)\right). \end{aligned} \tag{5.6}$$

From (5.5) and (5.6) we get the next identity valid for any solution of (5.3)

$$\begin{aligned} &A_nU_n^k = A_nC_k^{(n)}U_n^0 + A_nS_k^{(n)}U_n^1 \\ &+R_n\sum_{j=2}^{k} S_{k+1-j}^{(n)}\left(\Phi_n^{j-1} - 2\Phi_n^{j-2} + \Phi_n^{j-3}\right)\tilde{d}_n \\ &+\left(R_n\tau_nS_{k-1}^{(n)}\frac{\Phi_n^0 - \Phi_n^{-1}}{\tau_n} - \Phi_n^{k-1} + \frac{1}{2}\left(R_{1,n}^{k-1} + R_{2,n}^{k-1}\right)\Phi_n^0\right)\tilde{d}_n,\ k \geq 2. \end{aligned} \tag{5.7}$$

As in (4.5) define the operator

$$D_{n,k_n} = \Phi_n^{k_n-1} - \left(C_{k_n}^{(n)} + S_{k_n}^{(n)}\right)\Phi_n^0.$$

Then we have the following

Theorem 5.3. *Assume that* $\Phi(\cdot) \in C^4([0,T]; B(E))$, $\Phi_n(\cdot) \in C^4([0,T]; B(E_n))$, $\mathfrak{B}_n^{-1}\xrightarrow{\mathcal{PP}}\mathfrak{B}^{-1}$ *compactly,* $(D_{n,k_n})^{-1}\xrightarrow{\mathcal{PP}}D^{-1}, k_n\tau_n = T$, *the resolvents* $(\lambda I_n - A_n)^{-1}$, $(\lambda I - A)^{-1}$ *are compact,* (B'') *and* (1.4) *are satisfied and* $\Phi_n^{(l)}(t)\xrightarrow{\mathcal{PP}}\Phi^{(l)}(t)$ *uniformly in* $t \in [0,T]$ *for* $l = \overline{1,4}$. *Assume also that the problem* (4.1) *has a unique solution for any* $u^T \in D(A)$ *and the Krein-Fattorini conditions are satisfied. Then there are solutions of the problem* (5.3) *for almost all* n *and they converge to the solution of problem* (4.1), *i.e.*

$$U_n(t)\xrightarrow{\mathcal{P}}u(t) \textit{ uniformly in } t\in[0,T],$$

and $\tilde{d}_n\xrightarrow{\mathcal{P}}d$ *as* $n\in I\!N$, *whenever* $A_n u_n^T\xrightarrow{\mathcal{P}}Au^T$, $A_n u_n^0\xrightarrow{\mathcal{P}}Au^0$, $\mathfrak{B}_n u_n^1\xrightarrow{\mathcal{P}}\mathfrak{B}u^1$.

Proof. First we apply the operator $(A_n-\lambda I_n)$ to (5.4) for $\lambda\in\Delta_{cc}$. Using (5.7) we get equation

$$\begin{aligned}&\left(\Phi_n^{k-1}-\left(C_k^{(n)}+S_k^{(n)}\right)\Phi_n^0\right)\tilde{d}_n-\Bigg[R_n\sum_{j=2}^{k}S_{k+1-j}^{(n)}\left(\Phi_n^{j-1}-2\Phi_n^{j-2}+\Phi_n^{j-3}\right)\\ &-\lambda R_n\tau_n^2\sum_{j=2}^{k}S_{k+1-j}^{(n)}\Phi_n^{j-1}+R_nS_{k-1}^{(n)}\left(\Phi_n^0-\Phi_n^{-1}\right)\Bigg]\tilde{d}_n\\ &=(A_n-\lambda I_n)\left[C_k^{(n)}U_n^0+S_k^{(n)}U_n^1-U_n^k\right].\end{aligned}\tag{5.8}$$

Since

$$(D_{n,k_n})^{-1}=\left(\Phi_n^{k_n-1}-\left(C_{k_n}^{(n)}+S_{k_n}^{(n)}\right)\Phi_n^0\right)^{-1}\xrightarrow{\mathcal{PP}}D^{-1},$$

we can rewrite (5.8) in the form

$$I_n\tilde{d}_n-B_{n,3}\tilde{d}_n=g_{n,3},\tag{5.9}$$

where

$$\begin{aligned}B_{n,3}:=(D_{n,k_n})^{-1}\Bigg[&R_n\sum_{j=2}^{k_n}S_{k_n+1-j}^{(n)}\left(\Phi_n^{j-1}-2\Phi_n^{j-2}+\Phi_n^{j-3}\right)\\ &-\lambda R_n\tau_n^2\sum_{j=2}^{k_n}S_{k_n+1-j}^{(n)}\Phi_n^{j-1}+R_nS_{k_n-1}^{(n)}\left(\Phi_n^0-\Phi_n^{-1}\right)\Bigg]\end{aligned}$$

and

$$g_{n,3}:=(D_{n,k_n})^{-1}(A_n-\lambda I_n)\left(C_{k_n}^{(n)}U_n^0+S_{k_n}^{(n)}U_n^1-u_n^T\right),\quad k_n\tau_n=T.$$

To show that $B_{n,3}\xrightarrow{\mathcal{PP}}B_2$ compactly we split the operators $B_{n,3},B_2$ into two parts. Compact convergence

$$\frac{1}{2}(R_{1,n}^{k-1}-R_{2,n}^{k-1})\mathfrak{B}_n^{-1}\xrightarrow{\mathcal{PP}}\frac{1}{2}(\exp(t\mathfrak{B})-\exp(-t\mathfrak{B}))\mathfrak{B}^{-1},$$

because of (3.11) and Theorem 2.5, implies that

$$(D_{n,k_n})^{-1}R_n\tau_nS_{k_n-1}^{(n)}\frac{\Phi_n^0-\Phi_n^{-1}}{\tau_n}\xrightarrow{\mathcal{PP}}D^{-1}S(T,A)\Phi'(0)$$

compactly. The other parts of the operators $B_{n,3}$ also converge compactly to the corresponding parts of B_2. One can see by the same reasons as in (3.11) and (1.12) that

$$\mathfrak{B}_n\left(B_{n,3} - (D_{n,k_n})^{-1}R_n\tau_n S^{(n)}_{k_n-1}\frac{\Phi^0_n - \Phi^{-1}_n}{\tau_n}\right)\xrightarrow{\mathcal{PP}}\mathfrak{B}\left(B_2 - D^{-1}S(T,A)\Phi'(0)\right)$$

and this implies that $B_{n,3}\xrightarrow{\mathcal{PP}}B_2$ compactly.

The convergence of the finite differences to derivatives follows, e.g., from [9], p. 409. Therefore, from Theorems 2.1 and 2.2 it follows that $\tilde{d}_n\xrightarrow{\mathcal{P}}d$. The convergence of solutions $U_n(t)\xrightarrow{\mathcal{P}}u(t)$ uniformly in $t \in [0,T]$ follows from the representation formulas (5.4) and (1.5).

Remark 5.1. In case of Hilbert space and negative self-adjoint operators A in Theorem 5.3, one can omit the condition that $\mathfrak{B}_{n,3}\xrightarrow{\mathcal{PP}}\mathfrak{B}_2$ compactly and just claim the condition $\Delta_{cc} \neq \emptyset$. Indeed, then one can get the compact convergence of square roots of operators as in [10] and then get the compact convergence $B_{n,3}\xrightarrow{\mathcal{PP}}B_2$ as before.

Remark 5.2. In case of a Banach space in Theorem 5.3 one can also omit the condition of compact convergence $\mathfrak{B}_n^{-1}\xrightarrow{\mathcal{PP}}\mathfrak{B}^{-1}$ and just use the condition $\Delta_{cc} \neq \emptyset$. In this case one should assume that problem (4.1) possesses some extra smoothness condition. More precisely, assume that $u(\cdot) \in C^4([0,T];B(E)), \Phi(\cdot) \in C^3([0,T];B(E))$, $\Phi(0) = 0$ and $u^0, u^1 \in D(A^2)$. Then from Proposition 1.3 follows that $AS(\cdot,A)\Phi'(0) \in C([0,T];E)$. Moreover, if we assume that for the problems (5.1) and (4.1) $U_n^{(4)}(t)\xrightarrow{\mathcal{P}}u^{(4)}(t)$ uniformly in $t \in [0,T]$ for any $\tilde{d}_n\xrightarrow{\mathcal{P}}d$, then from the discrete analog of (1.8) follows that

$$A_nR_n\tau_nS^{(n)}_{k_n-1}\frac{\Phi^0_n - \Phi^{-1}_n}{\tau_n}\tilde{d}_n\xrightarrow{\mathcal{P}}AS(T,A)\Phi'(0)d. \tag{5.10}$$

This means that without loss of generality one can assume that

$$A_nR_n\tau_nS^{(n)}_{k_n-1}\frac{\Phi^0_n - \Phi^{-1}_n}{\tau_n}\xrightarrow{\mathcal{PP}}AS(T,A)\Phi'(0),$$

and then the compact convergence $B_{n,3}\xrightarrow{\mathcal{PP}}B_2$ can be established directly from the convergence $A_nB_{n,3}\xrightarrow{\mathcal{PP}}AB_2$.

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