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LINEAR ORTHOGONALITY PRESERVERS OF STANDARD OPERATOR ALGEBRAS

Chung-Wen Tsai and Ngai-Ching Wong

Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. In 2003, Araujo and Jarosz showed that every bijective linear map $\theta : A \to B$ between unital standard operator algebras preserving zero products in two ways is a scalar multiple of an inner automorphism. Later in 2007, Zhao and Hou showed that similar results hold if both A, B are unital standard algebras on Hilbert spaces and θ preserves range or domain orthogonality. In particular, such maps are automatically bounded. In this paper, we will study linear orthogonality preservers in a unified way. We will show that every surjective linear map between standard operator algebras preserving range/domain orthogonality carries a standard form, and is thus automatically bounded.

1. Introduction

An algebra A of bounded linear operators on a Banach space M is called *standard* if A contains the algebra $\mathcal{F}(M)$ of all bounded finite rank operators on M. Assume that $\theta : A \to B$ is a bijective linear map between two unital standard operator algebras on Banach spaces M, N, *preserving zero products* in two ways, i.e., $ab = 0$ in A if and only if $\theta(a)\theta(b) = 0$ in B. Araujo and Jarosz [1] showed that in this case there exist a nonzero scalar λ and a bounded invertible linear map $S : M \to N$ such that

$$\theta(a) = \lambda S a S^{-1}, \quad \forall a \in A.$$

It was pointed out in [3] that the same result holds also when A, B are non-unital.

On the other hand, let A, B be unital standard operator algebras on (real or complex) infinite dimensional Hilbert spaces H, K, respectively. Assume that θ :

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$A \to B$ is a surjective additive map *preserving range orthogonality* in two ways, i.e., $a^*b = 0$ in A if and only if $\theta(a)^*\theta(b) = 0$ in B. Zhao and Hou [6] showed that in this case there exist a unitary (or conjugate unitary) operator $U : H \to K$ and a bounded linear (or conjugate linear) invertible operator $V : K \to H$ such that

$$\theta(a) = UaV, \quad \forall a \in A.$$

Zhao and Hou [6] also obtained a similar version for surjective additive maps *preserving domain orthogonality* in two ways, i.e., the ones with $\theta(a)\theta(b)^* = 0$ exactly when $ab^* = 0$.

In this paper, we will give a unified approach, with new proofs, to different linear orthogonality preservers. We will show that every surjective linear map $\theta : A \to B$ between two standard operator algebras on (real or complex) Hilbert spaces preserving range/domain orthogonality in two ways carries a standard form, and is thus automatically bounded as well. The following table summarizes our results.

The structures θ preserves	The form of θ carries
$ab = 0 \Leftrightarrow \theta(a)\theta(b) = 0$	λSaS^{-1}
$a^*b = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0$	UaT
$ab^* = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0$	SaV
$a^*b = 0 \Leftrightarrow \theta(a)\theta(b)^* = 0$	Sa^tV
$ab^* = 0 \Leftrightarrow \theta(a)^*\theta(b) = 0$	Ua^tT
$a^*b = ab^* = 0 \Leftrightarrow \theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$	λUaV or λUa^tV

θ : a surjective linear map between standard operator algebras
λ : a nonzero scalar
S, T : bounded invertible linear operators
U, V : unitary operators

We note that we need to assume A is unital or A contains all trace class operators in the second to the fifth cases. Without this assumption, θ can be unbounded. For example, let H be an infinite dimensional Hilbert space and let T be any unbounded bijective linear operator on H. Then $x \otimes y \mapsto x \otimes Ty$ (resp. $x \otimes y \mapsto Tx \otimes y$) defines an unbounded bijective range (resp. domain) orthogonality preserving linear map from $\mathcal{F}(H)$ onto $\mathcal{F}(H)$. However, in the first case of zero product preservers and in the last case of doubly orthogonality preservers, this assumption can be dropped.

2. Preliminaries

In the following, A and B are standard operator algebras on (real or complex) Hilbert spaces H, K, respectively, and θ is a surjective linear map from A onto B.

As pointed out in [6], if θ preserves any kind of orthogonality in two ways, then θ is injective. For example, if $\theta(x) = 0$ and θ preserves zero products in two ways, then $\theta(x)\theta(y) = 0$ implies $xy = 0$ for all y in A. Thus $x = 0$ as well.

Recall that by the Fundamental Theorem of Affine Geometry any bijective linear map $\theta : \mathcal{F}(H) \to \mathcal{F}(K)$ sending exactly rank one operators onto rank one operators must be in one of the following forms.

(1) $\theta(x \otimes y) = Sx \otimes Ry$, where $S, R\colon H \to K$ are invertible linear maps.
(2) $\theta(x \otimes y) = Ry \otimes Sx$, where $S, R\colon H \to K$ are invertible conjugate linear maps.

Here, $x \otimes y(z) = \langle z, y\rangle x$ is the rank at most one operator, and $\langle \cdot, \cdot \rangle$ is the inner product of the Hilbert space H or K. Note that for any scalar α we have $\alpha(x\otimes y) = (\alpha x) \otimes y = x \otimes (\bar{\alpha} y)$. See, e.g., [4, 6].

Fix an orthonormal basis $\{e_j\}$ of a Hilbert space H. For all $x = \sum \langle x, e_j\rangle e_j$ in H, we set $\overline{x} = \sum \langle e_j, x\rangle e_j$. Let T be a bounded linear operator on H. The transpose operator T^t of T with respect to $\{e_j\}$ is the bounded linear operator satisfying the condition

$$\langle Te_i, e_j\rangle = \langle e_i, T^*e_j\rangle = \langle T^t e_j, e_i\rangle, \quad \forall i, j.$$

The transpose operator is well-defined and $\|T\| = \|T^*\| = \|T^t\|$. Here T^* is the adjoint operator of T. Note that the definition of $\overline{x}$ and T^t depend on the choice of the orthonormal basis. However, they are unique up to unitarily equivalence.

Some properties of the transpose operators are given below. For all $x, y \in H$ we have

(1) $\langle \overline{x}, \overline{y}\rangle = \langle y, x\rangle$.
(2) $(x \otimes y)^t = \overline{y} \otimes \overline{x}$.
(3) $(T^t)^* = (T^*)^t$.
(4) $T^t x = \overline{T^*\overline{x}}$.

3. Results

We first give, with a new proof, a modified version of the result of Zhao and Hou in [6] about linear range orthogonality preservers mentioned in the introduction. Note that we can allow the algebras not being unital, provided instead that they contain trace class operators.

Theorem 1. *Let A, B be standard operator algebras on Hilbert spaces H, K, respectively. Suppose A is unital, or A contains all trace class operators on H. Assume that $\theta : A \to B$ is a surjective linear map such that $a^*b = 0$ if and only if $\theta(a)^*\theta(b) = 0$. Then θ is bounded, and there exist a bounded invertible linear operator $T : K \to H$ and a unitary operator $U : H \to K$ such that*

$$\theta(a) = UaT, \quad \forall a \in A.$$

Proof. Note that θ is indeed bijective. Put

$$a^{\dashv} = \{c \in A : c^*a = 0\}, \quad \text{for all nonzero } a \text{ in } A.$$

For any a and b in A, it is clear that $a^{\dashv} \subseteq b^{\dashv}$ if and only if the closure of the range space of a contains that of b. We define a partial order on A by $a \leq b$ if and only if $a^{\dashv} \subseteq b^{\dashv}$. In this partial order, a is a maximum if and only if a is a rank one operator. By the two way range orthogonality preserving assumption, we see that both θ and θ^{-1} preserve this partial order, and thus send the maxima onto the maxima. In other words, θ and θ^{-1} send rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist invertible linear or conjugate linear maps $S : H \to K$ and $R : K \to H$ such that either

$$\theta(x \otimes y) = Sx \otimes Ry, \quad \forall x, y \in H,$$

or

$$\theta(x \otimes y) = Sy \otimes Rx, \quad \forall x, y \in H.$$

However, the second case does not give us a range orthogonality preserver, and thus be ruled out.

Observe that

$$\begin{aligned} & \langle x_1, x_2 \rangle = 0 \\ \text{implies} \quad & (x_2 \otimes y_2)^*(x_1 \otimes y_1) = 0, \qquad && \forall y_1, y_2 \in H \\ \text{implies} \quad & \theta(x_2 \otimes y_2)^*\theta(x_1 \otimes y_1) = 0, && \forall y_1, y_2 \in H \\ \text{implies} \quad & (Sx_2 \otimes Ry_2)^*(Sx_1 \otimes Ry_1) = 0, && \forall y_1, y_2 \in H \\ \text{implies} \quad & \langle Sx_1, Sx_2 \rangle = 0. \end{aligned}$$

For any two orthogonal norm one elements x, y in H, we have $\langle x, y \rangle = \langle x + y, x - y \rangle = 0$. This gives $\langle Sx, Sy \rangle = \langle Sx + Sy, Sx - Sy \rangle = 0$, and therefore $\|Sx\| = \|Sy\|$. It follows that $S = \lambda U$ for a nonzero scalar λ and a unitary operator U from H onto K. Renaming λR by R, we will have

$$\theta(x \otimes y) = Ux \otimes Ry, \quad \forall x, y \in H.$$

To get the boundedness of R we need to utilize the extra assumptions on A now. Suppose first that A is unital. For any norm one element e in H, as $(e \otimes e)(1 - e \otimes e) = 0$, we have $\theta(e \otimes e)^*(\theta(1) - \theta(e \otimes e)) = 0$. It follows $Re \otimes \theta(1)^*Ue = \langle Ue, Ue \rangle Re \otimes Re = Re \otimes Re$, and consequently, $Re = \theta(1)^*e$. So $R = \theta(1)^*U$ is bounded.

Suppose then that A contains all trace class operators on H and H is of infinite dimension. Suppose on contrary that there were an orthonormal sequence $\{x_n\}$ in

H such that $\|Rx_n\| \geq n^3$ for $n = 1, 2, 3, \ldots$. Define a trace class operator W on H by $W = \sum_n x_n \otimes x_n / n^2$. Since $(x_n \otimes x_n)(n^2 W - x_n \otimes x_n) = 0$, we have $\theta(x_n \otimes x_n)^*(n^2\theta(W) - \theta(x_n \otimes x_n)) = 0$. It follows $n^2 Rx_n \otimes \theta(W)^* Ux_n = \langle Ux_n, Ux_n \rangle Rx_n \otimes Rx_n = Rx_n \otimes Rx_n$. As a result, $\|\theta(W)^*\| \geq \|\theta(W)^* Ux_n\| = \|Rx_n\|/n^2 \geq n$ for all $n = 1, 2, 3, \ldots$. This contradiction ensures again that R is bounded.

Let $a \in A$. For any $x \neq 0$ in H, let $y \in H$ such that $\langle x, y \rangle = 1$. Set $b = a - (y \otimes a^* x)$. Observe $b^*(x \otimes y) = 0$. Thus,

$$\begin{aligned} 0 = \theta(b)^*\theta(x \otimes y) &= (\theta(b)^* Ux) \otimes Ry \\ &= ([\theta(a)^* - \theta(y \otimes a^* x)^*]Ux) \otimes Ry \\ &= (\theta(a)^* Ux - (Ra^* x \otimes Uy)Ux) \otimes Ry. \end{aligned}$$

This implies

$$\theta(a)^* Ux = (Ra^* x \otimes Uy)Ux = Ra^* x, \quad \forall x \in H.$$

Hence,

$$\theta(a) = UaR^*, \quad \forall a \in A.$$

Setting $T = R^*$, we are done, as the boundedness of θ is now clear. ∎

Next, we consider the other cases θ transforming the domain/range orthogonality to the domain/range orthogonality.

Theorem 2. *Let A, B be standard operator algebras on Hilbert spaces H, K, respectively. Suppose A is unital, or A contains all trace class operators on H. Let $\theta : A \to B$ be a surjective linear map.*

(a) *Assume that $ab^* = 0$ if and only if $\theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exists a bounded invertible linear operator $S : H \to K$ and a unitary operator $V : K \to H$ such that*
$$\theta(a) = SaV, \quad \forall a \in A.$$

(b) *Assume that $a^* b = 0$ if and only if $\theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exist a bounded invertible linear operator $S : H \to K$ and a unitary operator $V : K \to H$ such that*
$$\theta(a) = Sa^t V, \quad \forall a \in A.$$

(c) *Assume that $ab^* = 0$ if and only if $\theta(a)^*\theta(b) = 0$. Then there exist a unitary operator $U : H \to K$ and a bounded invertible linear operator operator $T : K \to H$ such that*
$$\theta(a) = Ua^t T, \quad \forall a \in A.$$

Proof. For a fixed orthonormal basis, we can define three range orthogonality preserving surjective linear maps respectively by setting

$$a \mapsto \theta(a^t)^t, \quad a \mapsto \theta(a)^t, \quad \text{and } a \mapsto \theta(a^t).$$

Then Theorem 1 applies. ■

Finally, we will investigate the doubly orthogonality preservers. A map θ is called a *doubly orthogonality preserver* if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$ whenever $a^*b = ab^* = 0$. Bounded doubly orthogonality preservers between C*-algebras and JB*-algebras are studied in [5, 2]. Note also that like the case of the zero product preservers, we do not need to assume A is unital or A contains any trace class operator on H in this case.

Theorem 3. *Let $\theta : A \to B$ be a surjective linear map between standard operator algebras on Hilbert space H, K, respectively, such that $a^*b = ab^* = 0$ if and only if $\theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0$. Then θ is bounded, and there exist a nonzero scalar λ and unitary operators $U : H \to K$ and $V : K \to H$ such that either*

$$\theta(a) = \lambda UaV, \quad \forall a \in A,$$

or

$$\theta(a) = \lambda Ua^tV, \quad \forall a \in A.$$

Proof. Put for all nonzero a in A that

$$a^{\dashv} = \{c \in A : c^*a = 0\} \quad \text{and} \quad a^{\vdash} = \{c \in A : ac^* = 0\}.$$

Set $a^+ = a^{\dashv} \cap a^{\vdash}$. For any a and b in A, it is not difficult to see that that $a^+ \subseteq b^+$ if and only if the closure of the range space of a contains that of b, and the initial space of a contains that of b. Define a partial order on A by saying $a \leq b$ if and only if $a^+ \subseteq b^+$. In this partial order, a is a maximum if and only if a is of rank one. By the doubly orthogonality preserving property of θ and θ^{-1}, we see that both of them preserves this partial order, and thus sends the maxima onto the maxima. In other words, both θ and θ^{-1} send rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist invertible linear or conjugate linear maps $S : H \to K$ and $R : K \to H$ such that either

$$\theta(x \otimes y) = Sx \otimes Ry, \quad \forall x, y \in H,$$

or

$$\theta(x \otimes y) = Ry \otimes Sx, \quad \forall x, y \in H.$$

By replacing θ with the map $a \mapsto \theta(a)^t$ if necessary, we can assume that the first case happens.

Arguing as in the proof of Theorem 1, we will see that there exist nonzero scalars λ_1, λ_2 such that $U = \lambda_1^{-1}S$ is a unitary operator from H onto K, and $W = \lambda_2^{-1}R$ is a unitary operator from K onto H. Put $\lambda = \lambda_1\lambda_2$ and $V = W^*$, we will have

$$\theta(a) = \lambda UaV, \quad \forall a \in \mathcal{F}(H).$$

In general, let $a \in A$. For any x in H with $a^*x \neq 0$, let $y \in H$ such that $\langle x, ay\rangle = 1$. Set $b = a - (ay \otimes a^*x)$. Observe $b^*(x \otimes y) = b(x \otimes y)^* = 0$. Thus,

$$\begin{aligned} 0 = \theta(b)^*\theta(x \otimes y) &= \lambda(\theta(b)^*Ux) \otimes V^*y \\ &= \lambda([\theta(a)^* - \theta(ay \otimes a^*x)^*]Ux) \otimes V^*y \\ &= \lambda(\theta(a)^*Ux - \bar{\lambda}(V^*a^*x \otimes Uay)Ux) \otimes V^*y. \end{aligned}$$

This implies

$$\theta(a)^*Ux = \bar{\lambda}(V^*a^*x \otimes Uay)Ux = \lambda V^*a^*x, \quad \forall x \in H.$$

Hence,

$$\theta(a) = \lambda UaV, \quad \forall a \in A.$$

The map θ is clearly bounded. ∎

References

1. J. Araujo and K. Jarosz, Biseparating maps between operator algebras, *J. Math. Anal. Appl.*, **282** (2003), 48-55.
2. M. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno and A. M. Peralta, Orthogonality preservers in C*-algebras, JB*-algebras and JB*-triples, *J. Math. Anal. Appl.*, **348** (2008), 220-233.
3. C.-W. Leung and N.-C. Wong, Zero Product Preserving Linear Maps of CCR C*-algebras with Hausdorff Spectrum, *J. Math. Anal. Appl.*, **361** (2010), 187-194.
4. M. Omladic and P. Semrl, Additive mappings preserving operators of rank one, *Linear Algebra Appl.*, **182** (1993), 239-256.
5. N.-C. Wong, Triple homomorphisms of operators algebras, *Southeast Asian Bulletin of Math.*, **29** (2005), 401-407.
6. Liankuo Zhao and Jinchuan Hou, Additive Maps Preserving Indefinite Semi-Orthogonality, *Journal of Systems Science and Mathematical Sciences* (*in Chinese*), **27(5)** (2007), 697-702.

Chung-Wen Tsai and Ngai-Ching Wong
Department of Applied Mathematics,
National Sun Yat-sen University,
Kaohsiung 80424, Taiwan
E-mail: tsaicw@math.nsysu.edu.tw
wong@math.nsysu.edu.tw