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NONEXPANSIVE RETRACTIONS ONTO CLOSED CONVEX CONES IN BANACH SPACES

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a closed convex subset of the dual space E^* of E and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R_{C^*} defined by $R_{C^*} = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$, where J is the normalized duality mapping on E. In this paper, we first prove that if K is a closed convex cone in E and P is the nonexpansive retaction of E onto K, then P a sunny generalized nonexpansive retraction of E onto K. Using this result, we obtain an equivalent condition for a closed half-space of E to be a nonexpansive retract of E.

1. INTRODUCTION

Let E be a smooth, Banach space and let E^* be the dual space of E. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for each $x, y \in E$, where J is the normalized duality mapping from E into E^*. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then, T is called generalized nonexpansive if the set $F(T)$ of fixed points of T is nonempty and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$; see Ibaraki and Takahashi [22]. Such nonlinear operators are connected with the resolvents of maximal monotone operators in Banach

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spaces. When E is a smooth, strictly convex and reflexive Banach space and C is a nonempty closed convex subset of E, Alber [1] also defined a nonlinear projection Π_C of E onto C called the generalized projection. Motivated by Alber [1] and Ibaraki and Takahashi [22], Kohsaka and Takahashi [29] proved the following result: Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.

When E is a Hilbert space and C is a closed convex subset of E, the metric projection (the nearest point projection) of E onto C, a sunny nonexpansive retraction of E onto C, the generalized projection of E onto C and a sunny generalized nonexpansive retraction of E onto C are all same; see [36]. However, it is known [32] that if the metric projections are nonexpansive whenever they exist for closed convex subsets C of a Banach space E with $\dim(E) \geq 3$, then E must be a Hilbert space. Moreover, it is also known [34] that if every closed convex subset of a Banach space E with $\dim(E) \geq 3$ is a nonexpansive retract of E, then E is necessarily a Hilbert space; see also [30].

Motivated by Ibaraki and Takahashi [22], Honda and Takahashi [18, 19] obtained the relation between nonexpansive retractions and sunny generalized nonexpansive retractions in a Banach space when their retracts of E are closed linear subspaces.

In this paper, we study the relation between nonexpansive retractions and sunny generalized nonexpansive retractions in a Banach space when their retarcts of E are closed convex cones. Furthermore, we obtain an equivalent condition for a closed half space of a Banach space E to be a nonexpansive retract of E.

2. Preliminaries

Throughout this paper, E is a real Banach space with the dual E^*. For any subset A of E, $\overline{A}$ denotes the closure of A with respect to the norm topology, Int$A$ denotes the set of interior points of A with respect to the norm topology and ∂A denotes the set of boundary points of A with respect to the norm topology. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x \neq y$. A Banach space E is said to be smooth provided

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$; see [36] for more details. Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. We also define the function $\phi_* : E^* \times E^* \to \mathbb{R}$ by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle x^*, J^{-1}y^* \rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$. It is easy to see that $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \geq 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.1}$$

for all $x, y, z \in E$. Further, we have

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.2}$$

for all $x, y, z, w \in E$. It is easy to see that

$$\phi(x, y) = \phi_*(Jy, Jx) \tag{2.3}$$

for all $x, y \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.4}$$

The following lemma is well-known.

Lemma 2.1. ([28]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z,x) = \min_{y \in C} \phi(y,x)\}$$

is always nonempty and a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y \in C} \phi(y,x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [28].

Lemma 2.2. ([1, 28]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x,z) \in E \times C$. Then, the following hold:*

(a) $z = \Pi_C x$ *if and only if* $\langle y - z, Jx - Jz \rangle \le 0$ *for all* $y \in C$*;*

(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(z,x)$.

Let D be a nonempty closed convex subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [22] if $F(T)$ is nonempty and $\phi(Tx,u) \le \phi(x,u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \to E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = r(T)$, where $r(T)$ is the range of T. When a mapping T is a retraction, the subset $r(T)$ is said to be a retract. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \ge 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [22].

Lemma 2.3. ([22]). *Let C be a nonempty closed subset of a smooth, strictly convex and reflexisve Banach space E and let R be a retraction from E onto C. Then, the following are equivalent:*

(a) R *is sunny and generalized nonexpansive;*

(b) $\langle x - Rx, Jy - JRx \rangle \le 0$ *for all* $(x,y) \in E \times C$.

Lemma 2.4. ([22]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5. ([22]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then, the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz\rangle \leq 0$ for all $y \in C$;*

(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C; see [36]. The following lemma is in [36].

Lemma 2.6. ([36]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z)\rangle \leq 0$ for all $y \in C$.*

An operator $A : E \to 2^{E^*}$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $r(A) = \cup\{Ax : x \in D(A)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for any $(x, x^*), (y, y^*) \in A$. The operator A is said to be strictly monotone if $\langle x - y, x^* - y^*\rangle > 0$ for any $x, y \in E$, $x^* \in Ax$, $y^* \in Ay$. A monotone operator A is said to be maximal if its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is not properly contained in the graph of any other monotone operator. If A is maximal monotone, then the set $A^{-1}0 = \{u \in E : 0 \in Au\}$ is closed and convex (see [37] for more details). Let J be the normalized duality mapping from E into E^*. Then, J is monotone. If E is strictly convex, then J is one to one and strictly monotone. The following theorem is well-known; for instance, see [36].

Theorem 2.1. *Let E be a reflexive, strictly convex and smooth Banach space and let $A\colon E \to 2^{E^*}$ be a monotone operator. Then A is maximal if and only if $r(J + rA) = E^*$ for all $r > 0$. Further, if $r(J + A) = E^*$, then $r(J + rA) = E^*$ for all $r > 0$.*

3. Nonexpansive Retractions onto Closed Convex Cones

In this section, we discuss some relations between a nonexpansive retraction onto a closed convex cone and sunny generalized nonexpansive retraction. We start with two theorems proved by Kohsaka and Takahashi [29].

Theorem 3.1. ([29]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 3.2. ([29]). *Let E be a smooth, reflexive and strictly convex Banach space and let D be a nonempty subset of E. Then, the following conditions are equivalent.*

(1) *D is a sunny generalized nonexpansive retract of E;*

(2) *D is a generalized nonexpansive retract of E;*

(3) *JD is closed and convex.*

In this case, D is closed.

From these theorems, we can represent sunny generalized nonexpansive retraction by using generalized projections. Let E be a reflexive, strictly convex and smooth Banach space and let J be the normalized duality mapping from E onto E^*. Let C^* be a closed convex subset of the dual space E^* of E. Then, the sunny generalized nonexpansive retraction R_{C^*} with respect to C^* is defined as follows:

$$R_{C^*} := J^{-1}\Pi_{C^*}J,$$

where Π_{C^*} is the generalized projection from E^* onto C^*.

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

In a reflexive Banach space, both concepts coincide with each other.

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then, a mapping T of C into itself is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. A mapping T of C into itself with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|Tx - m\| \le \|x - m\|$ for all $m \in F(T)$ and $x \in C$. It is clear that any nonexpansive mapping with fixed points is quasi-nonexpansive.

Motivated by previous theorems, the authors obtained following theorems.

Theorem 3.3. ([3, 18]). *Let E be a reflexive, strictly convex and smooth Banach space and let I be the identity operator of E into itself. Let Y^* be a closed linear subspace of the dual space E^* and let R_{Y^*} be the sunny generalized nonexpansive retraction with respect to Y^*. Then, the mapping $I - R_{Y^*}$ is the metric projection of E onto $Y^*_\perp$. Conversely, let Y be a closed linear subspace of E and let P_Y be the metric projection of E onto Y. Then, the mapping $I - P_Y$ is the generalized conditional expectation $R_{Y^\perp}$ with respect to $Y^\perp$, i.e., $I - P_Y = R_{Y^\perp}$.*

Theorem 3.4. ([19]). *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E. If the sunny generalized nonexpansive retraction R_{Y^*} is a quasi-nonexpansive projection of E onto $J^{-1}Y^*$, then it is a norm one linear projection and $J^{-1}Y^*$ is a closed linear subspace in E. Conversely, any norm one linear projection is a quasi-nonexpansive sunny generalized nonexpansive retraction with respect some closed linear subspace in E^*.*

We shall generalize these theorems and obtain a nonlinear retraction which is both "nonexpansive" and "sunny generalized nonexpansive".

A subset K of a Banach space is called a cone if it satisfies that $\lambda x \in K$ when $x \in K$ and $\lambda \geq 0$. Any cone contains the origin. When a cone contains a non-zero element, we call it nontrivial.

Theorem 3.5. *Let E be a reflexive and smooth Banach space and let K be a closed convex cone in E If $T : K \to K$ is a quasi-nonexpansive mapping such that $F(T)$ is a cone, then T is generalized nonexpansive.*

Proof. We first show that for any $x \in K$ and $m \in F(T)$,

$$\langle x - Tx, Jm \rangle \leq 0, \tag{3.1}$$

where J is the normalized duality mapping of E.

For the case of $m = 0$, it is obvious that $\langle x - Tx, Jm \rangle = 0$.

Fix $x \in K \setminus F(T)$ and $m \in F(T)$ such that $m \neq 0$. We have that for all $\alpha \in \mathbb{R}$ with $\alpha > 0$,

$$x \in F(T) \Leftrightarrow \alpha x \in F(T).$$

So, we have that $\frac{x}{k} - m \neq 0$ for any $k > 0$. We have from the Hahn-Banach theorem that there exists $\xi_k \in E^*$ such that $\left\langle \frac{x}{k} - m, \xi_k \right\rangle = \left\| \frac{x}{k} - m \right\|$ and $\|\xi_k\| = 1$. Then, we have that

$$\left\langle \frac{Tx}{k} - m, \xi_k \right\rangle \leq \left\| \frac{Tx}{k} - m \right\| = \frac{1}{k} \|Tx - km\|$$

$$\leq \frac{1}{k}\|x - km\| = \left\|\frac{x}{k} - m\right\|$$
$$= \left\langle \frac{x}{k} - m, \xi_k \right\rangle .$$

So, we have $\left\langle \frac{x}{k} - \frac{Tx}{k}, \xi_k \right\rangle \geq 0$ and hence

$$\langle x - Tx, \xi_k \rangle \geq 0.$$

Take a positive sequence $\{k_n\}$ with $k_n \to \infty$. Put $x_n = \frac{x}{k_n} - m$ and $\xi_n = \xi_{k_n}$. Then, we have $\dfrac{x}{k_n} - m \to -m$. Since E is a reflexive Banach space and $\{\xi_n\}$ is bounded, there exists a subsequence $\{\xi_{n_i}\}$ of $\{\xi_n\}$ converging to some $\xi \in E^*$ in weak topology.

We may show that ξ satisfies $\langle m, -\xi \rangle = \|m\|$ and $\|\xi\| = 1$. Since the norm of E^* is lower semicontinuous in the weak topology, we have

$$\|\xi\| \leq \liminf_{i\to\infty} \|\xi_{n_i}\| = 1.$$

On the other hand, we have that

$$|\langle -m, \xi\rangle - \|x_{n_i}\|| = |\langle -m, \xi\rangle - \langle x_{n_i}, \xi_{n_i}\rangle|$$
$$\leq |\langle -m, \xi - \xi_{n_i}\rangle| + |\langle -m - x_{n_i}, \xi_{n_i}\rangle|.$$

Since $\langle -m, \xi - \xi_{n_i}\rangle \to 0$ and $\langle -m - x_{n_i}, \xi_{n_i}\rangle \to 0$, we have

$$\|x_{n_i}\| \to -\langle m, \xi\rangle = \langle m, -\xi\rangle.$$

Since $\|x_{n_i}\| \to \|m\|$, we have $\langle m, -\xi\rangle = \|m\|$. So we have

$$\|m\| = \langle m, -\xi\rangle \leq \|m\|\|\xi\|$$

and hence $\|\xi\| \geq 1$. Therefore, we have $\|\xi\| = 1$ and $\langle m, -\xi\rangle = \|m\|$. Then, without loss of generality, there exists a positive sequence $\{k_n\}$ such that

$$k_n \to \infty,$$
$$\frac{x}{k_n} - m \to -m$$

and

$$\xi_{k_n} \rightharpoonup \xi$$

in weak topology, where ξ is an element of E^* such that $\langle m, -\xi\rangle = \|m\|$ and $\|\xi\| = 1$.

Putting $\xi_0 = -\xi$, we have $\langle m, \xi_0\rangle = \|m\|$, $\|\xi_0\| = 1$ and

$$\langle x - Tx, \xi_0\rangle \leq 0.$$

Since E^* is smooth and

$$\big\| \|m\|\xi_0 \big\|^2 = \|m\|^2 = \|m\|\langle m, \xi_0\rangle = \langle m, \|m\|\xi_0\rangle,$$

we know that $\|m\|\xi_0 = Jm$, where J is the normalized duality mapping on E. Then for any $x \in K \setminus F(T)$ and $m \in F(T) \setminus \{0\}$, we have $\|m\|\langle x - Tx, \xi_0\rangle \le 0$ and hence

$$\langle x - Tx, Jm\rangle \le 0.$$

We also have for $x \in F(T)$ and $m \in F(T)$ with $m \neq 0$, $\langle x - Tx, Jm\rangle = 0$.

So, the inequality (3.1) holds for any $x \in K$ and $m \in F(T)$. This implies that for any $x \in K$ and $m \in F(T)$,

$$\langle x, Jm\rangle \le \langle Tx, Jm\rangle.$$

Since T is quasi-nonexpansive and $0 \in F(T)$, we have $\|Tx\| \le \|x\|$. Then for any $x \in E$ and $m \in K$, we have $\|Tx\|^2 - 2\langle Tx, Jm\rangle + \|m\|^2 \le \|x\|^2 - 2\langle x, Jm\rangle + \|m\|^2$ and hence

$$\phi(Tx, m) \le \phi(x, m).$$

This means that T is a generalized nonexpansive mapping. ∎

From this theorem, we obtain following corollaries.

Corollary 3.1. *Let E be a smooth and reflexive Banach space and let $T : E \to E$ be a norm one linear operator. Then, T is generalized nonexpansive.*

Corollary 3.2. *Let E be a strictly convex, smooth and reflexive Banach space and let K be a cone in E. If K is a nonexpansive retract of E, then K is a closed convex cone in E, K is a sunny generalized nonexpansive retract and JK is a closed convex cone in E^*.*

Proof. Since K is a nonexpansive retract of E, there exists a nonexpansive retraction T with $T(E) = F(T) = K$. So, from [24], $F(T) = K$ must be closed and convex. From Theorem 3.5, we also know that T is a generalized nonexpansive retraction of E onto K. From Theorem 3.2, K is a sunny generalized nonexpansive retract and JK is a closed convex subset in E^*. Since for any $x \in E$ and $\alpha \in \mathbb{R}$ we have $J(\alpha x) = \alpha Jx$ from [36], JK is a cone. ∎

We shall extend Theorem 3.3; see also Alber [2], Hudzik, Wang and Sha [21]. First we shall introduce two new nonlinear operators. We call a mapping $T : E \to E$ a *firmly generalized nonexpansive type* [23], if it satisfies

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(x, Tx) + \phi(y, Ty) \le \phi(x, Ty) + \phi(y, Tx)$$

for all $x, y \in E$. We call a mapping $S : E \to E$ a *firmly metric operator* [38], if it satisfies

$$\begin{aligned}&\phi(x - Sx, y - Sy) + \phi(y - Sy, x - Sx)\\ \le\ &\phi(x, y - Sy) + \phi(y, x - Sx) - \phi(x, x - Sx) - \phi(y, y - Sy)\end{aligned}$$

for all $x, y \in E$.

Let C be a nonempty subset of a Banach space E and let C^* be a nonempty subset of the dual space E^*. Then, we define the dual cone (or the polar cone) $C^*_\circ$ of C^* and the dual cone (or the polar cone) C° of C as follows:

$$C^*_\circ = \{x \in E : f(x) \le 0 \text{ for all } f \in C^*\}$$

and

$$C^\circ = \{f \in E^* : f(x) \le 0 \text{ for all } x \in C\}.$$

Both of them are closed convex cones. In a reflexive Banach space, both concepts coincide with each other.

Lemma 3.1. *Let E be a strictly convex, smooth and reflexive Banach space, let C be a nonempty closed convex subset of E and let P_C be the metric projection of E onto C. Then the mapping $T = I - P_C$ is a firmly generalized nonexpansive type of E into E. In particular, if $0 \in C$, then $F(T) = P_C^{-1}0 = J^{-1}C^\circ$ and $JF(T)$ is a closed convex cone in E^*.*

Proof. From Lemma 2.6, we have that for any $x, y \in E$,

$$\langle J(x - P_Cx), P_Cx - P_Cy\rangle \ge 0$$

and

$$\langle J(y - P_Cy), P_Cy - P_Cx\rangle \ge 0.$$

Then we have

$$\langle J(x - P_Cx) - J(y - P_Cy), P_Cx - P_Cy\rangle \ge 0.$$

Since $Tx = x - P_Cx$ and $Ty = y - P_Cy$, we obtain

$$\langle JTx - JTy, x - Tx - (y - Ty)\rangle \ge 0.$$

From (2.2), we have

$$\begin{aligned}0 &\le 2\langle JTx - JTy, x - Tx - (y - Ty)\rangle\\ &= 2\langle JTx - JTy, x - y\rangle - 2\langle JTx - JTy, Tx - Ty\rangle\\ &= \phi(x, Ty) + \phi(y, Tx) - \phi(x, Tx) - \phi(y, Ty) - \phi(Tx, Ty) - \phi(Ty, Tx).\end{aligned} \tag{3.2}$$

So, T is a firmly generalized nonexpansive type on E. If $0 \in C$, we have that

$$\begin{aligned} &P_C x = 0 \\ \Leftrightarrow &x - P_C x = x \\ \Leftrightarrow &Tx = x. \end{aligned}$$

Then $F(T) = P_C^{-1}0$. From Lemma 2.6, we have

$$\begin{aligned} x \in F(T) &\Leftrightarrow x \in P_C^{-1}0 \\ &\Leftrightarrow \langle J(x-0), 0-y\rangle \geq 0 \text{ for any } y \in C \\ &\Leftrightarrow \langle J(x), y\rangle \leq 0 \text{ for any } y \in C \\ &\Leftrightarrow Jx \in C^{\circ}. \end{aligned}$$

Then we obtain

$$JF(T) = C^{\circ} = \cap_{y\in C}\{x^* \in E^* : \langle x^*, y\rangle \leq 0\}.$$

This is the intersection of closed convex cones of E^*. So, $JF(T)$ is a closed convex cone in E^*. ∎

Lemma 3.2. *Let E be a strictly convex, smooth and reflexive Banach space and let $T : E \to E$ be a firmly generalized nonexpansive type such that $JF(T)$ is a nonempty closed convex subset in E^* and $T(E) = F(T)$. Then, T is a sunny generalized nonexpansive retraction of E onto $F(T)$.*

Proof. From (3.2), we know that a mapping $T : E \to E$ satisfies that

$$\langle JTx - JTy, x - Tx - (y - Ty)\rangle \geq 0.$$

From assumptions of T, $F(T) \neq \emptyset$. For any $x \in E$ and $m \in F(T)$, we have

$$\langle JTx - Jm, x - Tx\rangle \geq 0.$$

Since $Tx \in F(T)$ and $JF(T)$ is closed and convex in E^*, we have, from Lemma 2.3, that T is a sunny generalized nonexpansive retraction of E onto $F(T)$. ∎

Lemma 3.3. *Let E be a strictly convex, smooth and reflexive Banach space and let $T : E \to E$ be a firmly metric operator such that $F(T)$ is a nonempty closed convex subset in E and $T(E) = F(T)$. Then T is the metric projection of E onto $F(T)$.*

Proof. From (3.2), for any $x, y \in E$, we have

$$\langle J(x - Tx) - J(y - Ty), Tx - Ty\rangle \geq 0.$$

Then for any $x \in E$ and $m \in F(T)$, we have

$$\langle J(x - Tx), Tx - m\rangle \geq 0.$$

Since $F(T)$ is closed and convex and $Tx \in F(T)$, the mapping T is the metric projection of E onto $F(T)$. ∎

Theorem 3.6. *Let E be a strictly convex, smooth and reflexive Banach space. Let K be a closed convex cone of E and let P_K be the metric projection of E onto K. Then the mapping $T = I - P_K$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}K^\circ$, where K° is the dual cone of K.*

Proof. From Lemma 2.6, we have

$$\langle J(x - P_Kx), P_Kx - m\rangle \geq 0$$

for any $x \in E$ and $m \in K$. From $0 \in K$, we have

$$\langle J(x - P_Kx), P_Kx\rangle \geq 0.$$

From $2P_Kx \in K$, we also have

$$\langle J(x - P_Kx), P_Kx\rangle \leq 0.$$

From these inequalities, we have

$$\langle J(x - P_Kx), P_Kx\rangle = 0.$$

So, we have, for any $x \in E$ and $m \in K$,

$$\begin{aligned} &\langle J(x - P_Kx), P_Kx - m\rangle \geq 0 \\ \Rightarrow &\langle J(x - P_Kx), P_Kx\rangle - \langle J(x - P_Kx), m\rangle \geq 0 \\ \Rightarrow &\langle J(x - P_Kx), m\rangle \leq 0 \\ \Rightarrow &\langle JTx, m\rangle \leq 0. \end{aligned}$$

Then for any $x \in E$, we have $JTx \in K^\circ$. We have $T(E) \subset J^{-1}K^\circ$ and hence

$$F(T) \subset T(E) \subset J^{-1}K^\circ.$$

From Lemma 3.1, we have that T is a firmly generalized nonexpansive type, $JF(T)$ is a closed convex cone in E^* and $F(T) = J^{-1}K^\circ$. Since $T(E) = F(T) = J^{-1}K^\circ$, from Lemma 3.2, T is a sunny generalized nonexpansive retraction of E onto $F(T) = J^{-1}K^\circ$. ∎

Theorem 3.7. *Let E be a strictly convex, smooth and reflexive Banach space. Let K^* be a closed convex cone of E^* and let $R_{K^*} = J^{-1}\Pi_{K^*}J$ be the sunny generalized nonexpansive retraction of E onto $J^{-1}K^*$, where Π_{K^*} is the generalized projection of E^* onto K^*. Then, the mapping $T = I - R_{K^*}$ is the metric projection of E onto the dual cone $K^*_\circ$ of K^*.*

Proof. Since $0 \in J^{-1}K^*$, from Lemma 2.3, we have

$$\begin{aligned} x \in R_{K^*}^{-1}0 &\Leftrightarrow R_{K^*}x = 0 \\ &\Leftrightarrow \langle x - 0, J0 - JJ^{-1}m^* \rangle \geq 0 \text{ for any } m^* \in K^* \\ &\Leftrightarrow \langle x, m^* \rangle \leq 0 \text{ for any } m^* \in K^* \\ &\Leftrightarrow x \in K^*_\circ. \end{aligned}$$

Then we have that

$$R_{K^*}^{-1}0 = K^*_\circ.$$

From assumptions, we have

$$\begin{aligned} &R_{K^*}x = 0 \\ \Leftrightarrow &x - R_{K^*}x = x \\ \Leftrightarrow &Tx = x. \end{aligned}$$

Then we have that

$$F(T) = R_{K^*}^{-1}0.$$

So, we obtain that

$$F(T) = K^*_\circ.$$

Since a sunny generalized nonexpansive retraction is a firmly generalized nonexpansive type, T is a firmly metric operator such that $F(T) = K^*_\circ$. To obtain the desired result, from Lemma 3.3, it is sufficient to show that $T(E) \subset F(T) = K^*_\circ$. From $0, 2R_{K^*}x \in J^{-1}K^*$ and Lemma 2.3, we have

$$\langle x - R_{K^*}x, JR_{K^*}x \rangle = 0.$$

So, we have for any $x \in E$ and $m^* \in K^*$, $\langle x - R_{K^*}x, JR_{K^*}x - JJ^{-1}m^* \rangle \geq 0$ and hence

$$\langle x - R_{K^*}x, m^* \rangle \leq 0.$$

Then we have that for any $x \in E$ and $m^* \in K^*$,

$$\langle Tx, m^* \rangle \leq 0.$$

Then we obtain that $Tx \in K^*_\circ$ for any $x \in E$. This implies $T(E) \subset K^*_\circ$. Therefore, $T = P_{K^*_\circ}$. This completes the proof. ■

Remark 3.1. In a Hilbert space, Theorem 3.3 is called the Riesz decomposition and Theorems 3.6 and 3.7 are called the Moreau decomposition; see Hudzik, Wang and Sha [21].

From Corollary 3.2and Theorem 3.7, we have the following corollary.

Corollary 3.3. *Let E be a strictly convex, reflexive and smooth Banach space and let K be a closed convex cone of E. If there exists a sunny nonexpansive retraction R of E onto K, then $I-R$ is the metric projection of E onto $\{JK\}_\circ$, where I is the identity mapping on E.*

4. Nonexpansive Retractions onto Closed Half-spaces

Let E be a strictly convex, reflexive and smooth Banach space. Calvert [10] showed that a closed linear subspace Y in E is a 1-complemented subspace (i.e. the range of a norm one linear projection) if and only if JY is a closed linear subspace in E^*; see also [18]. Using our theorems in the preivious section, we can extend this result.

Let E be a Banach space. A subset $V \subset E$ is called a linear manifold if it is of the form $V = \{x_0 + g : g \in G\}$, where x_0 is some element of E and G is a linear subspace of E. We call a closed linear manifold M a closed hyperplane if there exists no closed linear manifold $M_1 \subset E$ such that $M \subset M_1$ and $M \neq M_1 \neq E$. We know that M is a closed hyperplane if and only if there exist a nonzero bounded linear functional $f \in E^*$ and $\alpha \in \mathbb{R}$ such that $M = \{x \in E : f(x) = \alpha\}$; see Singer [35]. A subset $H \subset E$ is called a closed half-space if it is of the form $H = \{x \in E : f(x) \leq \alpha\}$, where f is a nonzero bounded linear functional $f \in E^*$ and $\alpha \in \mathbb{R}$. In particular, in this paper, a closed half-space means only the case $\alpha = 0$.

Theorem 4.1. *Let E be a strictly convex, smooth and reflexive Banach space and let H be a closed half-space of E such that for some $z^* \in E^* \setminus \{0\}$*

$$H = \{x \in E : \langle x, z^* \rangle \leq 0\}.$$

Then, H is a nonexpansive retract of E if and only if JH is a closed half-space in E^*.

To prove this theorem, we need some definitions and lemmas. Let E be a real Banach space. The definition of orthogonality that we use is that of Birkhoff [7] and James [25, 26, 27]; for $x, y \in E$, x is said to be *orthogonal* to y, denoted by $x \perp y$, if

$$\|x + \lambda y\| \geq \|x\| \tag{4.1}$$

for all $\lambda \in \mathbb{R}$. x is said to be *acute* to y if (4.1) holds for all $\lambda \geq 0$. When E is smooth, we know that

$$x \text{ is orthogonal to } y \Leftrightarrow \langle Jx, y \rangle = 0$$

and

$$x \text{ is acute to } y \Leftrightarrow \langle Jx, y \rangle \geq 0;$$

see [36]. Let F be a closed subset of E. A retraction R of E onto F is *orthogonal*; see Bruck [9], if for each $x \in E$ and $m \in F$, $Rx - m$ is acute to $x - Rx$;

$$\|(1-\lambda)Rx + \lambda x - m\| \geq \|Rx - m\|$$

for all $\lambda \geq 0$.

Using this orthogonal retraction, we show a following lemma.

Lemma 4.1. *Let E be a strictly convex, smooth and reflexive Banach space and let H be a closed half-space of E such that for some $z^* \in E^* \setminus \{0\}$*

$$H = \{x \in E : \langle x, z^* \rangle \leq 0\}.$$

Then, H is a nonexpansive retract of E if and only if JH is a closed convex cone in E^*.

Proof. A closed half-space H is a closed convex cone. If H is a nonexpansive retract of E, from Corollary 3.2, JH is a closed convex cone in E^*.

Conversely, if JH is a closed convex cone in E^*, from Theorem 3.2, there exists the sunny generalized nonexpansive retraction $R_{JH} = J^{-1}\Pi_{JH}J$ of E onto H, where Π_{JH} is the generalized projection of E^* onto JH. We shall show that R_{JH} is nonexpansive. Since R_{JH} is sunny, we have for any $x \in E$,

$$R_{JH}\left(R_{JH}x + \lambda\left(x - R_{JH}x\right)\right) = R_{JH}x,$$

for $\lambda \geq 0$. When $z \in E \setminus H = \{x \in E : \langle x, z^* \rangle > 0\}$, we have that $R_{JH}z \in \{x \in E : \langle x, z^* \rangle = 0\}$. In fact, if $R_{JH}z \in \{x \in E : \langle x, z^* \rangle < 0\}$, then $z - R_{JH}z \in \{x \in E : \langle x, z^* \rangle > 0\}$. For a sufficiently small $\lambda > 0$, we have

$$R_{JH}z + \lambda\left(z - R_{JH}z\right) \in \{x \in E : \langle x, z^* \rangle < 0\} \subset H.$$

Then we have that

$$R_{JH}z = R_{JH}\left(R_{JH}z + \lambda\left(z - R_{JH}z\right)\right) = R_{JH}z + \lambda\left(z - R_{JH}z\right)$$

and hence $\lambda\left(z - R_{JH}z\right) = 0$. From $\lambda > 0$, we have $z - R_{JH}z = 0$ and hence $z \in H = \{x \in E : \langle x, z^* \rangle \leq 0\}$. This contradicts to $z \in \{x \in E : \langle x, z^* \rangle > 0\}$.

So, for any $m \in H$ and $z \notin H$, we have

$$m - R_{JH}z \in \{x \in E : \langle x, z^* \rangle \leq 0\} = H.$$

Then $J(m - R_{JH}z) \in JH$. From Theorem 3.7, the mapping $P = I - R_{JH}$ is the metric projection of E onto $(JH)_\circ$. Then we have, for any $m \in H$ and $z \notin H$,

$$\begin{aligned} &\langle J(m - R_{JH}z), Pz \rangle \leq 0 \\ \Rightarrow &\langle J(m - R_{JH}z), z - R_{JH}z \rangle \leq 0 \\ \Rightarrow &\langle J(R_{JH}z - m), z - R_{JH}z \rangle \geq 0. \end{aligned}$$

From this, we obtain that $R_{JH}z - m$ is acute to $z - R_{JH}z$. When $z \in H$, $z - R_{JH}z = 0$ and $R_{JH}z - m$ is acute to $z - R_{JH}z$ obviously. This means that R_{JH} is an orthogonal retraction of E onto H. Since R_{JH} is an orthogonal retraction of E onto H, for any $x, y \in E$, we have

$$\langle J(R_{JH}x - R_{JH}y), x - R_{JH}x \rangle \geq 0$$

and

$$\langle J(R_{JH}y - R_{JH}x), y - R_{JH}y \rangle \geq 0.$$

Then for any $x, y \in E$, we have

$$\begin{aligned} &\langle J(R_{JH}x - R_{JH}y), x - R_{JH}x \rangle - \langle J(R_{JH}x - R_{JH}y), y - R_{JH}y \rangle \geq 0 \\ \Rightarrow &\langle J(R_{JH}x - R_{JH}y), x - y - (R_{JH}x - R_{JH}y) \rangle \geq 0 \\ \Rightarrow &\langle J(R_{JH}x - R_{JH}y), x - y \rangle \geq \|R_{JH}x - R_{JH}y\|^2 \\ \Rightarrow &\|R_{JH}x - R_{JH}y\| \cdot \|x - y\| \geq \|R_{JH}x - R_{JH}y\|^2 \\ \Rightarrow &\|x - y\| \geq \|R_{JH}x - R_{JH}y\|. \end{aligned}$$

Then R_{JH} is nonexpansive. So, H is a nonexpansive retract of E. ■

Using an idea of Beauzamy [5] and Davis and Enflo [12], we obtain the following lemma.

Lemma 4.2. *Let E be a strictly convex, smooth and reflexive Banach space and let H be a closed half-space of E such that for some $z^* \in E^* \setminus \{0\}$*

$$H = \{x \in E : \langle x, z^* \rangle \leq 0\}.$$

Let $M = \{x \in E : \langle x, z^* \rangle = 0\}$. Then, H is a nonexpansive retract of E if and only if JM is a closed linear subspace of E^*.

Proof. Assume that H is a nonexpansive retract of E. Then, from Corollary 3.2, JH is a closed convex cone in E^*. As in the proof of Lemma 4.1, we may assume that there exists a sunny nonexpasnsive retraction R of E onto H. In this case, we have $R(E) = F(R) = H$. Define a mapping $\hat{R} : E \to E$ by $\hat{R}(x) = -R(-x)$ for all $x \in E$. For any $x \in E$, we have $R(-x) \in H$ and $\hat{R}x \in -H$. When $x \in -H$, we have $-x \in F(R)$ and $\hat{R}x = -R(-x) = -(-x) = x$. Then we have that $\hat{R}(E) = F(\hat{R}) = -H$. For any $x, y \in E$,

$$\begin{aligned}\|\hat{R}x - \hat{R}y\| &= \| - R(-x) + R(-y)\| \\ &\leq \|x - y\|.\end{aligned}$$

Then $\hat{R}$ is a nonexpansive retraction of E onto $-H$. As in the proof of Lemma 4.1, R (resp. $\hat{R}$) maps any point $x \notin H$ (resp. $x \notin -H$) to the boundary $(-H) \cap H = M$. Then $\hat{R} \circ R$ is a nonexpansive retraction onto $(-H) \cap H = M$. Indeed, $\hat{R} \circ R$ is a nonexpansive mapping. So we shall show that it is a retraction of E onto M. If $x \in M$, then $\hat{R} \circ Rx = x \in M$. If $x \in H \setminus M$, then $Rx = x \in H \setminus M$ and $\hat{R} \circ Rx \in M$. If $x \in (-H) \setminus M$, then $Rx \in M$ and $\hat{R} \circ Rx \in M$. Then, we have that $F(\hat{R} \circ R) = \hat{R} \circ R(E) = M$.

From Theorem 3.5, JM is a closed convex cone in E^*. Since M is a closed linear subspace of E, for any $x^* \in J$ and $\alpha \in \mathbb{R}$, we have $\alpha x^* \in JM$. Then JM is a closed linear subspace in E^*.

When JM is a closed linear subspace of E^*, there exists a norm one linear projection P of E onto M; see [10, 18]. We define the new operator $Q : E \to E$ such that

$$Qx = \begin{cases} Px & \text{if } x \notin H, \\ x & \text{if } x \in H. \end{cases} \tag{4.2}$$

Q is a nonlinear retraction of E onto H. We shall show that Q is nonexpansive. When $x, y \in H$ or $x, y \in E \setminus H$, we have $\|Qx - Qy\| \leq \|x - y\|$, obviously. When $x \in H$ and $y \in E \setminus H$, let z be an element of the segument $[x, y]$ such that $z \in M$. We have that

$$\begin{aligned}\|Qx - Qy\| &= \|x - Py\| \leq \|x - z\| + \|z - Py\| \\ &= \|x - z\| + \|Pz - Py\| \leq \|x - z\| + \|z - y\| \\ &= \|x - y\|.\end{aligned}$$

Then, Q is a nonexpansive retraction of E onto H. So, H is a nonexpansive retract of E. ∎

To prove Theorem 4.1, we need more lemmas;

Lemma 4.3. *Let E be a Banach space and let K be a closed convex cone in E such that for some $z^* \in E^* \setminus \{0\}$*

$$K \supset M := \{x \in E : \langle x, z^* \rangle = 0\}.$$

Then K is one of the following four;

(1) *the closed hyperplane M;*

(2) *the closed half-space $H_+ = \{x \in E : \langle x, z^* \rangle \geq 0\}$;*

(3) *the closed half-sapce $H_- = \{x \in E : \langle x, z^* \rangle \leq 0\}$;*

(4) *the whole space E.*

Proof. Suppose that K contains an element $\xi \in E$ such that $\langle \xi, z^* \rangle = a > 0$. For any $y \in E$ such that $0 < \langle y, z^* \rangle < a$, we define y_α as follows:

$$y_\alpha = \alpha(y - \xi) + \xi, \quad \alpha \geq 0.$$

When $\alpha = 0$, we have $\langle y_\alpha, z^* \rangle = a > 0$. As $\alpha \to \infty$, $\langle y_\alpha, z^* \rangle$ decreases strictly and continuously. Furthermore, it tends to $-\infty$. Then there exists $\alpha_0 > 0$ such that $\langle y_{\alpha_0}, z^* \rangle = 0$. This means that there exist $x \in M$ and $\alpha > 0$ such that

$$x = \alpha(y - \xi) + \xi.$$

So, we have

$$y = \frac{1}{\alpha} x + \left(1 - \frac{1}{\alpha}\right) \xi.$$

We can show $1 < \alpha$. In fact, if $\alpha = 1$, then $\langle y, z^* \rangle = \langle x, z^* \rangle = 0$. This is a contradiction. If $0 < \alpha < 1$, then $\langle y, z^* \rangle = \frac{1}{\alpha} \langle x, z^* \rangle + \left(1 - \frac{1}{\alpha}\right) \langle \xi, z^* \rangle = \left(1 - \frac{1}{\alpha}\right) a < 0$. This is a contradiction. So, we have $1 < \alpha$.

Then y is an element of the convex hull of $M \cup \{\xi\}$. So, we have

$$K \supset \{x \in E : 0 \leq \langle x, z^* \rangle < a\}.$$

Since K is a closed convex cone, we have $K \supset H_+$.

Similarly, when K contains an element ζ such that $\langle \zeta, z^* \rangle < 0$, we have $K \supset H_-$. Then if $K \neq M$, then $K \supset H_+$ or $K \supset H_-$. The proof is completed. ∎

Lemma 4.4. *Let E be a Banach space and let M be a hyperplane in E such that for some $z^* \in E^* \setminus \{0\}$,*

$$M = \{x \in E : \langle x, z^* \rangle = 0\}.$$

Then $M^\perp = \overline{\text{span}}\{z^*\}$, where $\overline{\text{span}}\{z^*\} = \{x^* \in E^* : x^* = \alpha z^*, \alpha \in \mathbb{R}\}$.

Proof. It is clear that $M^\perp \supset \overline{\text{span}}\{z^*\}$. It is sufficient to show that there exists a unique non-zero element in E^* up to a scalar multiple, such that it vanishes in M.

Since M is a hyperplane, for $x_0 \in E \setminus M$, we have

$$E = \overline{\text{span}}\{M \cup \{x_0\}\},$$

where $\overline{\text{span}}A$ is a closed linear span generated by A. For any $x \in \text{span}\{M \cup \{x_0\}\}$, we can say $x = \alpha x_0 + m$, where α and m are some real value and some element of M, respectively. Then, we have taht for any $x \in \text{span}\{M \cup \{x_0\}\}$, $\langle x, z^*\rangle = \alpha\langle x_0, z^*\rangle$ and $\langle x_0, z^*\rangle \neq 0$. If $w^* \in M^\perp$, then for any $x \in \text{span}\{M \cup \{x_0\}\}$, $\langle x, w^*\rangle = \alpha\langle x_0, w^*\rangle$. This means that $\langle x, w^*\rangle = \frac{\langle x_0, w^*\rangle}{\langle x_0, z^*\rangle}\langle x, z^*\rangle$. Since w^* and z^* are continuous, we have $\langle x, w^*\rangle = \frac{\langle x_0, w^*\rangle}{\langle x_0, z^*\rangle}\langle x, z^*\rangle$ for any $x \in E$. So, we have $w^* = \frac{\langle x_0, w^*\rangle}{\langle x_0, z^*\rangle} z^*$ and hence $w^* \in \{x^* \in E^* : x^* = \alpha z^*, \alpha \in \mathbb{R}\}$. ■

Let E be a Banach space and let $Y_1, Y_2 \subset E$ be closed linear subspaces. If $Y_1 \cap Y_2 = \{0\}$ and for any $x \in E$ there exists a unique pair $y_1 \in Y_1, y_2 \in Y_2$ such that

$$x = y_1 + y_2,$$

then, we represent the space E as

$$E = Y_1 \oplus Y_2.$$

Lemma 4.5. *Let E be a strictly convex, reflexive and smooth Banach space and let Y^* be a closed linear subspace of the dual space E^* of E such that for any $y_1, y_2 \in J^{-1}Y^*$, $y_1 + y_2 \in J^{-1}Y^*$. Then, $J^{-1}Y^*$ is a closed linear subspace of E and the sunny generalized nonexpansive retraction $R_{Y^*} = J^{-1}\Pi_{Y^*}J$ of E onto $J^{-1}Y^*$, where Π_{Y^*} is the generalized projection of E^* onto Y^*, is a norm one linear projection of E onto $J^{-1}Y^*$. Further, the following holds:*

$$E = J^{-1}Y^* \oplus Y^*_\perp.$$

Proof. By the assumption, for any $y_1, y_2 \in J^{-1}Y^*$, we have $y_1 + y_2 \in J^{-1}Y^*$. Further, for $y \in J^{-1}Y^*$ and $\alpha \in \mathbb{R}$, we have from $J(\alpha y) = \alpha Jy \in Y^*$ that $\alpha y \in J^{-1}Y^*$. So, $J^{-1}Y^*$ is a linear subspace of E. Since J is norm to weak continuous and Y^* is weakly closed subset in E^*, $J^{-1}Y^*$ is closed. Therefore, $J^{-1}Y^*$ is a closed linear subspace of E. For any $x, y \in E$, from Theorem 3.1, we have $R_{Y^*}x, R_{Y^*}y \in J^{-1}Y^*$. Since $J^{-1}Y^*$ is a linear subspace of E, we have $R_{Y^*}x + R_{Y^*}y \in J^{-1}Y^*$. Since Y^* is a closed linear subspace of E^*, from Lemma 2.3, for any $x \in E$, an element $y \in J^{-1}Y^*$ satisfies $y = R_{Y^*}x$ if and only if

$$\langle x - y, m^*\rangle = 0, \quad \forall m^* \in Y^*. \tag{4.3}$$

For $x \in E$ and $\alpha \in \mathbb{R}$, let $y = R_{Y^*}x$. We have that

$$\langle \alpha x - \alpha y, m^* \rangle = 0, \quad \forall m^* \in Y^*.$$

Since $\alpha y \in J^{-1}Y^*$, we have that

$$\alpha y = R_{Y^*}(\alpha x).$$

For $x_1, x_2 \in E$, let $y_1 = R_{Y^*}x_1$ and $y_2 = R_{Y^*}x_2$. Then, we have that

$$\langle x_1 + x_2 - (y_1 + y_2), m^* \rangle = \langle x_1 - y_1, m^* \rangle + \langle x_2 - y_2, m^* \rangle = 0, \quad \forall m^* \in Y^*.$$

Since $y_1 + y_2 \in J^{-1}Y^*$, we obtain that

$$R_{Y^*}(x_1 + x_2) = y_1 + y_2 = R_{Y^*}x_1 + R_{Y^*}x_2.$$

So, the retraction R_{Y^*} is linear. Since $\phi(R_{Y^*}x, m) \leq \phi(x, m)$ for any $x \in E$ and $m \in J^{-1}Y^*$, putting $m = 0 \in J^{-1}Y^*$, we have that

$$\|R_{Y^*}x\| \leq \|x\|.$$

Then, R_{Y^*} is a norm one linear projection of E onto $J^{-1}Y^*$.

From this, we have that

$$E = J^{-1}Y^* \oplus R_{Y^*}^{-1}0,$$

where $R_{Y^*}^{-1}0 = \{x \in E : R_{Y^*}x = 0\}$. It is sufficient to show that $R_{Y^*}^{-1}0 = Y_\perp^*$. From (4.3), we have that

$$x \in R_{Y^*}^{-1}0 \Leftrightarrow \langle x, m^* \rangle = 0, \quad \forall m^* \in Y^*.$$

This means that

$$R_{Y^*}^{-1}0 = Y_\perp^*.$$

■

Proof of Theorem 4.1. Let H be a closed half-space of E such that for some $z^* \in E^* \setminus \{0\}$,

$$H = \{x \in E : \langle x, z^* \rangle \leq 0\}.$$

When JH is a closed half-space in E^*, JH is a closed convex cone in E^*. So, from Lemma 4.1, H is a nonexpansive retract of E. It is sufficient to show that if H is a nonexpansive retract of E, then JH is a closed half-space in E^*.

Assume H is a nonexpansive retract of E. From Lemma 4.1, JH is closed convex cone in E^*. From Lemma 4.2, JM is a closed linear subspace in E^*, where $M = \{x \in E : \langle x, z^* \rangle = 0\}$. Since $M \subset E = E^{**}$ and $J_*^{-1}M = JM$ is a closed linear subspace in E^*, from Lemma 4.5, we have that

$$E^* = J_*^{-1}M \oplus M^\perp,$$

where $M^{\perp} = \{x^* \in E^* : \langle x^*, m\rangle = 0 \quad \forall m \in M\}$. Then, from Lemma 4.4, we have that

$$E^* = JM \oplus \overline{\text{span}}\{z^*\}.$$

This means that the co-dimension of the closed linear subspace JM in E^* is one. Then, JM is a closed hyperplane in E^*.

Since the closed conve cone JH contains the hyperplane JM, the duality mapping J is bijective and both sets $H \setminus M$ and $E \setminus H$ are nonempty, from Lemma 4.3, we obtain that JH is a closed half-space in E^*. This completes the proof. ∎

From this theorem, we obtain the following corollary.

Corollary 4.1. *Let E be a strictly convex, smooth and reflexive Banach space and let H be a closed half-space of E such that for some $z^* \in E^*$*

$$H = \{x \in E : \langle x, z^*\rangle \leq 0\}.$$

Then, JH is a closed convex cone in E^* if and only if JH is a closed half-space in E^*.

Remark 4.1. In a Hilbert space, the normalized duality mapping J is the identity mapping. The image of a closed convex cone by J is always a closed convex cone and the image of a closed half-space by J is always a closed half-space. In this case, any closed convex cone is a nonexpansive retract; see [36].

Remark 4.2. Let E be a strictly convex, smooth and reflexive Banach space, let $z \in E$ and let $M^* = \{\overline{\text{span}}\{z\}\}^{\perp}$. When $P_{\overline{\text{span}}\{z\}}$ is linear, R_{M^*} is a norm one linear projection onto $J^{-1}M^*$; see [10, 18]. Then M^* is a closed hyperplane such that $J^{-1}M^* = J_*M^*$ is a closed linear subspace of E.

In [13, 14], Deutsch showed an equivalent condition for the metric projection $P_{\overline{\text{span}}\{z\}}$ to be linear in L^p spaces; see also [6, 16].

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