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CONVERGENCE OF A RADIAL SOLUTION TO AN INITIAL-BOUNDARY VALUE PROBLEM OF p-GINZBURG-LANDAU TYPE

Yutian Lei

Abstract. This paper is concerned with the asymptotic behavior of the solution u_ε of a p-Ginzburg-Landau system with the radial initial-boundary data. The author proves that the zeros of u_ε in the parabolic domain $B_1(0) \times (0,T]$ locate near the axial line $\{0\} \times (0,T]$. In addition, the author also consider the Holder convergence of the solution when the parameter ε tends to zero. The convergence is derived by establishing a uniform gradient estimate for the regularized solution of the system.

1. INTRODUCTION

Let $n \geq 3$ and $B = \{x \in \mathbf{R}^n; |x| < 1\}$. Write $B_T = B \times (0,T]$ where $T \in (0,\infty)$. We are concerned with the asymptotic behavior of the weak solution u_ε of the following problem

$$u_t = div(|\nabla u|^{p-2}\nabla u) + \frac{1}{\varepsilon^p}u(1-|u|^2), \quad (x,t) \in B_T \tag{1.1}$$

$$u(x,t) = g(x), \quad x \in \partial B, \quad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in B \tag{1.3}$$

when $\varepsilon \to 0$, where $g \in C^\infty(\partial B, S^{n-1})$, $u_0 \in W^{1,p}(B, S^{n-1})$ and $u_0|_{\partial B} = g(x)$. Recall that a weak solution of (1.1) is a measurable function $u : B_T \to \mathbf{R}^n$, such that

$$\text{esssup}_{t\in(0,T)}\|u(\cdot,t)\|^2_{L^2(B)} + \|\nabla u\|^p_{L^p(B_T)} < \infty,$$

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and for any $\phi \in C_0^\infty(B_T)$,

$$(1.4) \qquad \int_{B_T} u\phi_t dxdt = \int_{B_T} |\nabla u|^{p-2}\nabla u \nabla \phi dxdt - \frac{1}{\varepsilon^p}\int_{B_T} u\phi(1-|u|^2)dxdt.$$

Moreover, if a function u is a weak solution of (1.1), and

$$u(x,t) = g(x), \quad x \in \partial B, t \geq 0,$$
$$\lim_{t\to 0^+}\int_B |u(x,t) - u_0(x)|dx = 0,$$

then u solves (1.1)-(1.3) in weak sense.

In the case of $p = n = 2$, the problem is the well-known evolution system of Ginzburg-Landau type. It can be used to described the properties of vortices in the study of the phase transition, such as the theories of superconductor, superfliuds and XY-magnetism. Its background can be seen in [2] and the references therein. The motivation of studying this problem is also from the aspect of mathematics, since it is connected tightly with the study of the heat flow of harmonic maps (cf. [8]). Under the assumption of $u_0(x) \in C^\infty(\overline{B}, S^1)$, some uniform estimates and the Holder convergence of u_ε is derived in [7], which showed that the limit of the solution u_ε is a heat flow of harmonic maps. Clearly, this assumption implies $\deg(u_0, \partial B) = d = 0$. If $d \neq 0$, there exist zeros of the solution u_ε in B, which are so-called Ginzburg-Landau vortices. Under some general assumption on $u(x,0)$, many works studied the properties of vortices (cf. [6],[8]). These works showed that the solution u_ε of (1.1)-(1.3) converges to the heat flow of harmonic map u_* when $\varepsilon \to 0$. At the same time, the Ginzburg-Landau vortices also converge to the singularities of the heat flow of harmonic maps u_*.

In the case of $p \neq 2$, the works in [4] and [12] showed that, the existence of the heat flow of p-harmonic map u_p can be obtained via taking the limit of the solution u_ε of (1.1)-(1.3) as $\varepsilon \to 0$. We expect that the system (1.1) can also be used to research the partial regularity of u_p. When the initial data u_0 satisfies $\deg(u_0, \partial B) = d = 0$, it was proved that ∇u_p has $C^{\alpha,\alpha/2}$ regularity everywhere in any compact subset of B_T (cf. [10]). When $d \neq 0$, u_p only has the partial regularity. To find out the singularities of u_p as doing in the case $p = n = 2$, it is important to investigate the location of zeros of the solution u_ε when ε is sufficiently small. However, it seems to be a very difficult problem when $p \neq 2$, since the main part of (1.1) is degenerate and nonlinear. There may not be any classical solution for the system. Thus, the methods in the case $p = n = 2$ can not be used now. Under the general assumption of u_0 and g, for example, the result as Proposition 2.3 is difficult to obtain. As a try, we shall consider the properties of the solution under the assumption of $\max\{2, n/2\} < p < n$, $u_0(x) = \frac{x}{|x|}$ and $g(x) = x$ in this paper. We expect it can provide an way to handle the more complicate initial problem on a general bounded domain.

To investigate the convergence of the weak solution u_ε to (1.1)-(1.3), we should overcome the singularity of the penalization term $\frac{1}{\varepsilon^p}u(1-|u|^2)$, since it may converge to $+\infty$ near the domain where $|u_\varepsilon| = 0$. Therefore, we have to pick out the zeros of u_ε from B_T. In §2, we will present the location of zeros when $\max\{2, n/2\} < p < n$. The main technique benefits from [1].

Theorem 1.1. *Assume* $\max\{2, n/2\} < p < n$, $u_0(x) = \frac{x}{|x|}$ *and* $g(x) = x$. *Let* u_ε *be the weak solution of* $(1.1) - (1.3)$ *with* $T > 0$. *Then for any* $\sigma > 0$, *there exists a constant* h *independent of* ε, *such that*

$$Z_\varepsilon = \{(x,t) \in B\times[0,T]; |u_\varepsilon(x,t)| < 1/2\} \subset (\overline{B(0,h\varepsilon)}\times[\sigma,T])\cup(\overline{B(0,\sigma)}\times[0,\sigma]).$$

In particular the zeros of u_ε *are contained in* $(\overline{B(0,h\varepsilon)}\times[\sigma,T])\cup(\overline{B(0,\sigma)}\times[0,\sigma])$.

Remark. This theorem implies that all the zeros are located near $\{0\} \times [0,T]$ when ε and σ are sufficiently small. Namely, there does not exist any zero in the domain far away from $\{0\} \times [0,T]$, and

(1.5)
$$|u_\varepsilon(x,t)| \geq 1/2, \quad \forall (x,t) \in ([\overline{B} \setminus \overline{B(0,h\varepsilon)}] \times [\sigma,T]) \cup ([\overline{B} \setminus B(0,\sigma)] \times [0,\sigma]).$$

The Holder convergence of ∇u_ε is given by the following theorem, which is proved in Sections 3 and 4.

Theorem 1.2. *Under the same assumption of Theorem 1.1, we have*

$$\lim_{\varepsilon\to 0} \nabla u_\varepsilon = \nabla \frac{x}{|x|}, \quad in \quad C^{\alpha,\frac{\alpha}{2}}_{loc}([\overline{B} \setminus \{0\}] \times (0,T], \mathbf{R}^n), \quad for\ some \quad \alpha\in(0,1).$$

Remark. Comparing with the results in [10], we can obtain the convergence of u_ε is up to the boundary ∂B, Although the Holder estimate of ∇u_ε near the boundary for the system (1.1) is still open. The reason is once the solution u_ε has the radial structure, the system (1.1) can be simplified as a single equation. Its regularity estimate was already drived (cf. [11]). In addition, the restriction of $p > 4$ in [10] can be removed here, since the embedding result becomes better when the dimension is lower.

2. Proof of Theorem 1.1

Proposition 2.1. *Assume* $p > 1$. *If* $g(x) \in C^\infty(\partial B, S^{n-1})$ *and* $u_0(x) \in W^{1,p}(B, S^{n-1})$, *then the weak solution of* $(1.1) - (1.3)$ *is unique. Furthermore, if* $u_0(x) = \frac{x}{|x|}$ *and* $g(x) = x$, *then there exists a function* $f(r,t) \in V$, *such that the weak solution of* $(1.1) - (1.3)$ *can be written as*

(2.1) $$u_\varepsilon(x,t) = f_\varepsilon(r,t)\frac{x}{r}, \quad r = |x|,$$

where

$$\begin{aligned} V &= \{f(r,t) \in L^\infty_{loc}(0,T;L^2(0,1)) \cap L^p_{loc}(0,T;W^{1,p}_{loc}(0,1]);\\ &\quad r^{\frac{n-p-1}{p}} f, r^{\frac{n-1}{p}} f_r \in L^p_{loc}((0,T;L^p(0,1)),\\ &\quad f(0,t) = 0, f(1,t) = 1,\ for\ t \geq 0, \quad f(r,0) = 1,\ a.e.\ on\ (0,1]\}. \end{aligned}$$

Proof. Set $0 < t_1 < t_2 < T$. By a limit process we see that the test function ϕ in (1.4) can be any member of $W^{1,p}_0(B,\mathbf{R}^n)$. Then the weak solution satisfies

(2.2) $$\begin{aligned} &\int_B (u(x,t_2) - u(x,t_1))\phi(x)dx + \int_{t_1}^{t_2}\int_B |\nabla u|^{p-2}\nabla u \nabla\phi dxdt\\ &= \frac{1}{\varepsilon^p}\int_{t_1}^{t_2}\int_B u\phi(1-|u|^2)dxdt. \end{aligned}$$

Fix $\tau \in (0,T)$. Choose h satisfying $0 < \tau < \tau + h < T$. Taking $t_1 = \tau$, $t_2 = \tau + h$ in (2.2) and multiplying with h^{-1} we obtain

(2.3) $$\begin{aligned} &\int_B (u_h(x,\tau))_\tau \phi(x)dx + \int_B (|\nabla u|^{p-2}\nabla u)_h(x,\tau)\nabla\phi(x)dx\\ &= \frac{1}{\varepsilon^p}\int_B \phi[u(1-|u|^2)]_h dx \end{aligned}$$

for any $\phi \in W^{1,p}_0(B,\mathbf{R}^n)$, where u_h is the Steklov's mean value of u, i.e. $u_h = \frac{1}{h}\int_t^{t+h} u(x,\tau)d\tau$ as $t \in (0, T-h)$; $u_h = 0$ as $t > T-h$. Assume both u_1 and u_2 are weak solution of (1.1)-(1.3), write $w = u_1 - u_2$. In virtue of (2.3), we see that for any $\phi \in W^{1,p}_0(B,\mathbf{R}^n)$ and any given $\tau \in (0,T)$,

$$\begin{aligned} &\int_B w_{h\tau}\phi dx + \int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h \nabla\phi dxdt\\ &= \frac{1}{\varepsilon^p}\int_{B_T} \phi[u_1(1-|u_1|^2) - u_2(1-|u_2|^2)]_h dxdt. \end{aligned}$$

Taking $\phi(x) = w_h(x,\tau)$ and integrating on $(0,t)$ we have

$$\begin{aligned} &\int_B w_h^2 dx - \int_B w_h^2(x,0)dx\\ &= -\int_0^t\int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)_h(\nabla u_1 - \nabla u_2)_h dxd\tau\\ &\quad + \frac{1}{\varepsilon^p}\int_0^t\int_B w_h^2 dxd\tau - \frac{1}{\varepsilon^p}\int_0^t\int_B (u_1|u_1|^2 - u_2|u_2|^2)_h w_h dxd\tau. \end{aligned}$$

Letting $h \to 0$, we get $\int_B w_h^2(x,0)dx \to 0$ and

$$\int_B w^2(x,t)dx = -\int_0^t \int_B (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(\nabla u_1 - \nabla u_2)dxd\tau$$
$$+\frac{1}{\varepsilon^p}\int_0^t \int_B w^2 dxd\tau - \frac{1}{\varepsilon^p}\int_0^t \int_B (u_1|u_1|^2 - u_2|u_2|^2)(u_1 - u_2)dxd\tau.$$

Applying Lemma 1.2 in [4] we know that the first and the third terms of the right hand side of the equality above are no more than zero. Using the Gronwall inequality we obtain

$$\int_B |u_1 - u_2|^2 dx = 0, \quad \forall t \in (0,T)$$

which implies that the weak solution of (1.1)-(1.3) is unique.

The existence of the weak solution can be verified by the regularized idea (cf. [13]). When $u_0(x) = \frac{x}{|x|}$ and $g(x) = x$, we can find a non-negative function f_ε^M solving the problem

$$\begin{aligned} f_t &= r^{1-n}(r^{n-1}A^{(p-2)/2}f_r)_r - r^{-2}A^{(p-2)/2}f \\ &\quad + \frac{1}{\varepsilon^p} f(1-f^2), \ a.e. \ on \ (0,1) \times (0,T] \end{aligned} \tag{2.4}$$

$$f(0,t) = 0, \quad f(1,t) = 1, \quad t \geq 0; \quad \lim_{t \to 0} \|(f(|x|,t) - 1)\frac{x}{|x|}\|_{W^{1,p}(B)} = 0, \tag{2.5}$$

where $A = f_r^2 + (n-1)r^{-2}f^2 + \frac{1}{M}$ with $M > 0$, since (2.4) is a regularized equation (i.e. $A > 0$ for the given M). If letting $M \to \infty$, then there is a function $f_\varepsilon(r,t)$ such that

$$f_\varepsilon^M \to f_\varepsilon, \quad weakly^* \quad in \quad L^\infty_{loc}(0,T;W^{1,p}(K)),$$

$$(f_\varepsilon^M)_t \to (f_\varepsilon)_t, \quad weakly \quad in \quad L^2_{loc}(0,T;L^2(K)),$$

where K is an arbitrary compact subset of $(0,1]$. Set

$$u_\varepsilon(x,t) = f_\varepsilon(r,t)\frac{x}{r}, \quad r = |x| \in [0,1].$$

It is not difficult to prove that u_ε is a weak solution of (1.1)-(1.3). According to the uniqueness of the weak solution, it is just $f_\varepsilon(r,t)\frac{x}{r}$. Proposition 2.1 is complete.

Hereafter, the solution of (1.1)-(1.3) with $g(x) = x$ and $u_0(x) = \frac{x}{|x|}$ will be called the radial solution of (1.1)-(1.3).

Proposition 2.2. *Assume* $p > 1$, *and* f_ε^M *solves* (2.4) *and* (2.5). *Then for any compact subset* K *of* $[0,1] \times (0,T]$, *there holds*

$$f_\varepsilon^M \le 1 \quad on \quad K. \tag{2.6}$$

Moreover, if $p \in (1,n)$, *then there exists a constant* $C > 0$ *which is independent of* ε, M *and* T, *such that the following energy inequality holds,*

$$\begin{aligned} &\sup_{t\in(0.T]} \left[\int_0^t \int_0^1 r^{n-1} \left|\frac{\partial}{\partial \tau} f_\varepsilon^M(r,\tau)\right|^2 dr d\tau + E_\varepsilon^M(f_\varepsilon^M(r,t)) \right] \\ &\le E_\varepsilon^M(f(r,0)) = C. \end{aligned} \tag{2.7}$$

Here

$$E_\varepsilon^M(f) = \frac{1}{p}\int_0^1 r^{n-1} A^{p/2} dr + \frac{1}{4\varepsilon^p}\int_0^1 r^{n-1}(1-|f|^2)^2 dr.$$

Proof. Although the initial data $u_0(x)$ is not continuous at $x = 0$, the solution f_ε^M is still continuous up to the boundary $r = 1$ for $[\sigma, T]$ with $\sigma > 0$, since (2.4) is regularized equation. Multiplying (2.4) with $r^{n-1}f$ we have

$$\begin{aligned} r^{n-1}\frac{1}{2}(f^2)_t &= \frac{1}{2}\left[r^{n-1}A^{(p-2)/2}(f^2)_r\right]_r - r^{n-1}A^{(p-2)/2}(f_r^2 + \frac{f^2}{r^2}) \\ &\quad + \frac{1}{\varepsilon^p} r^{n-1} f^2(1-f^2). \end{aligned} \tag{2.8}$$

If $\psi = f^2$ achieves its maximum on $\{(r,t); r \in \{0,1\}, t \ge 0\}$, then the proposition can be seen easily. Suppose ψ achieves its maximum at (r_0,t_0) in $(0,1) \times [\sigma,T]$ for any $\sigma > 0$. Then $\psi_r(r_0,t_0) = 0$; $\psi_{rr}(r_0,t_0) \le 0$ and $\psi_t(r_0,t_0) \ge 0$. We claim that $\psi \le 1$ over $(0,1) \times [\sigma,T]$. Otherwise, it follows from (2.8) that at (r_0,t_0),

$$\begin{aligned} 0 \le r_0^{n-1}\frac{1}{2}\psi_t &= \frac{1}{2}(r_0^{n-1}A^{(p-2)/2})_r\psi_r + \frac{1}{2}r_0^{n-1}A^{(p-2)/2}\psi_{rr} \\ &\quad - r_0^{n-1}A^{(p-2)/2}(f_r^2 + \frac{\psi}{r^2}) + \frac{1}{\varepsilon^p} r_0^{n-1}\psi(1-\psi) < 0, \end{aligned}$$

which is impossible. Thus (2.6) is proved.

Write $f = f_\varepsilon^M$. Multiplying (2.4) with $r^{n-1}f_t$ and integrating over $(0,1)$ we have

$$\begin{aligned} \int_0^1 r^{n-1} f_t^2 dr &= \int_0^1 f_t[(r^{n-1}A^{(p-2)/2}f_r)_r - r^{n-3}A^{(p-2)/2}f]dr \\ &\quad + \frac{1}{\varepsilon^p}\int_0^1 r^{n-1} f f_t(1-f^2) dr. \end{aligned}$$

From (2.5) we see $f_t(0,t) = f_t(1,t) = 0$. Thus $\int_0^1 r^{n-1} f_t^2 dr = -\frac{\partial}{\partial t} E_\varepsilon^M(f)$. Integrating from σ to t for any $0 < \sigma < t \le T$ and letting $\sigma \to 0$, we can deduce (2.7) from (2.5) since $p < n$. Proposition is complete.

Remark. Noting $p > 2$ which implies $\frac{1}{p} + \frac{1}{2} < 1$, we can use the anisotropic Sobolev embedding theorem. Thus, from (2.6) and (2.7), we see that for any $R \in (0,1)$ and for some $\alpha_0 \in (0,1)$, there exists $C > 0$ (independent of ε and M) such that

$$\begin{aligned} &\|f_\varepsilon^M\|_{C^{\alpha_0}([R,1]\times[0,T])} \\ (2.9) \qquad &\le C(\|(f_\varepsilon^M)_r\|_{L^p([R,1]\times(0,T])} \\ &+\|f_\varepsilon^M\|_{L^p([R,1]\times(0,T])} + \|(f_\varepsilon^M)_t\|_{L^2([R,1]\times(0,T])}) \le C. \end{aligned}$$

Let $M \to \infty$, then there is a constant $C_1 > 0$, such that $\|f_\varepsilon\|_{C^\alpha([R,1]\times[0,T])} \le C_1$, where $\alpha \in (0, \alpha_0)$. From this it is easy to deduce that, if $\sigma \in (0, (2C)^{-1/\alpha})$, then for any $r \in [1-\sigma, 1]$ and $t \in [0,T]$, $|f_\varepsilon(1,t) - f_\varepsilon(r,t)| \le C(1-r)^\alpha \le \frac{1}{2}$. Similarly, for any $r \in [\sigma, 1]$ and $t \in [0,\sigma]$, $|f_\varepsilon(r,t) - f_\varepsilon(r,0)| \le Ct^\alpha \le \frac{1}{2}$. Noting $f(1,t) = 1$ for $t \in [0,T]$ and $f(r,0) = 1$ for $r \in [\sigma, 1]$, we have

$$(2.10) \qquad \begin{aligned} |u_\varepsilon(x,t)| \ge \frac{1}{2}, \qquad &for \quad (x,t) \in [\overline{B} \setminus B(0, 1-\sigma)] \times [0,T] \\ &or \quad (x,t) \in [\overline{B} \setminus B(0,\sigma)] \times [0,\sigma]. \end{aligned}$$

Proposition 2.3. *Let u_ε be the weak radial solution of* (1.1) − (1.3). *If $p \in (n/2, n)$. Then there exists a positive constant C independent of $\varepsilon \in (0,1)$ and T, such that*

$$\sup_{t\in(0,T]} \frac{1}{\varepsilon^n} \int_B (1 - |u_\varepsilon(x,t)|^2)^2 dx \le C.$$

Proof. For any given $t > 0$, write

$$E_\varepsilon^M(u(x,t), B) = \frac{1}{p} \int_B \left(|\nabla u(x,t)|^2 + \frac{1}{M} \right)^{p/2} dx + \frac{1}{\varepsilon^p} \int_B (1 - |u(x,t)|^2)^2 dx.$$

By means of (2.7), we have

$$E_\varepsilon^M \left(u_\varepsilon^M, B\right) \le E_\varepsilon^M \left(\frac{x}{|x|}, B\left(0, \varepsilon\right) \right) + E_\varepsilon^M \left(\frac{x}{|x|}, B \setminus B(0,\varepsilon) \right).$$

Thus, if write $u = u_\varepsilon^M$,

$$E_\varepsilon^M(u, B) \le C\varepsilon^{n-p} + \frac{1}{p} \int_{B\setminus B(0,\varepsilon)} \left(|\nabla \frac{x}{|x|}|^2 + \frac{1}{M} \right)^{p/2}.$$

Combining this with

$$\int_B \left(|\nabla u|^2+\frac{1}{M}\right)^{p/2} \geq \int_B |\nabla |u||^p + \int_{B\setminus B(0,\varepsilon)} \left(|u|^2|\nabla \frac{x}{|x|}|^2+\frac{1}{M}\right)^{p/2},$$

and letting $M \to \infty$, we have

$$\frac{1}{\varepsilon^p}\int_B (1-|u|^2)^2 dx \leq C\varepsilon^{n-p} + C\int_{B\setminus B(0,\varepsilon)} |\nabla \frac{x}{|x|}|^p (1-|u|^2)dx.$$

Here $u = u_\varepsilon$. Then for any $\delta \in (0,1)$,

$$\begin{aligned}
&\frac{1}{\varepsilon^p}\int_B (1-|u|)^2 dx \leq \frac{1}{\varepsilon^p}\int_B (1-|u|^2)^2 dx\\
&\leq C\varepsilon^{n-p} + C\left[\int_{B\setminus B(0,\varepsilon)} |\nabla \frac{x}{|x|}|^{2p}dx\right]^{1/2}\left[\int_B (1-|u|)^2 dx\right]^{1/2}\\
&\leq C\varepsilon^{n-p} + C\left[\varepsilon^{n-p}+\varepsilon^p\right]^{1/2}\left[\frac{1}{\varepsilon^p}\int_B (1-|u|)^2 dx\right]^{1/2}\\
&\leq C\varepsilon^{n-p} + C(\delta)\varepsilon^{n-p} + \delta\left[\frac{1}{\varepsilon^p}\int_B (1-|u|)^2 dx\right]
\end{aligned}$$

since $\frac{n}{2} < p$. Choose δ sufficiently small, then $\frac{1}{\varepsilon^p}\int_B (1-|u_\varepsilon|)^2 dx \leq C\varepsilon^{n-p}$. Multiplying this with ε^{p-n}, and taking the supremum, we obtain the conclusion.

Proposition 2.4. *Assume* $p > 1$, *and* u_ε *is a weak solution of* (1.1) $-$ (1.3). *Then for any* $\sigma > 0$ *and* $\rho > 0$, *there exists* $C_0 > 0$ *independent of* ε, *such that for* $x_0 \in \overline{B}$, *and* $t_0 \in [\sigma, T]$,

$$|u_\varepsilon(x,t) - u_\varepsilon(x_0,t_0)| \leq C_0\varepsilon^{-\alpha}(|x-x_0|^\alpha + |t-t_0|^{\alpha/p}).$$

Here $\alpha \in (0,1)$, *and* $x \in B(x_0,\rho\varepsilon)\cap B$, $t \in (t_0-\rho\varepsilon^p, t_0+\rho\varepsilon^p)\cap[\sigma,T]$.

Proof. Let $(y,s) = (\varepsilon^{-1}x, \varepsilon^{-p}t)$ in (1.1), and define $v(y,s) = u(x,t)$. Then (1.1) becomes

$$v_s = div(|\nabla v|^{p-2}\nabla v) + v(1-|v|^2) \quad \text{in the weak sence}$$

in $B_T^\varepsilon = \{(y,s);(x,t)\in B_T\}$. According to Theorem 1 in [3], for any $\rho > 0$, there exists $C > 0$ such that as $y_0 \in \overline{B_T^\varepsilon}$, $s_0 \in (0, T\varepsilon^{-p}]$,

$$|v(y,s) - v(y_0,s_0)| \leq C(|y-y_0|^\alpha + |s-s_0|^{\alpha/p})$$

for $y \in B(y_0, \rho) \cap B(0, \varepsilon^{-1})$, $s \in (s_0 - \rho, s_0 + \rho) \cap (0, T\varepsilon^{-p}]$, where $\alpha \in (0, 1)$. Letting $x = y\varepsilon$ and $t = s\varepsilon^p$ in the result above, we have the desired consequence. Proposition 2.4 is complete.

In the following argument, we borrow the ideas of [1] to discuss the location of zeros of u_ε.

Proposition 2.5. *Let u_ε be the weak solution of* $(1.1) - (1.3)$*, and write $D = B(0, 1 - \sigma/2)$, where σ is the constant in* (2.10). *There exist positive constants λ, μ independent of $\varepsilon \in (0, 1)$ and t, such that if*

$$\frac{1}{\varepsilon^n} \int_{D \cap B^{2l\varepsilon}} (1 - |u_\varepsilon(x, t)|^2)^2 dx \le \mu, \quad \forall t \in I := [\sigma, T], \tag{2.11}$$

where $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \ge \lambda$, and σ is the constant in (2.10), then

$$|u_\varepsilon(x, t)| \ge 1/2, \quad \forall (x, t) \in [D \cap B^{l\varepsilon}] \times I. \tag{2.12}$$

Proof. First, we observe that there exists a constant $\beta > 0$, such that for any $x \in D$ and $0 < \rho \le 1$, $|D \cap B(x, \rho)| \ge \beta \rho^n$. Next, we choose $\lambda = (\frac{1}{2C_0})^{1/\alpha}$, $\mu = \frac{\beta}{16}\lambda^n$, where C_0 is the constant of Proposition 2.4. Suppose that (2.12) is not true, then there is a point $(x_0, t_0) \in [D \cap B^{l\varepsilon}] \times [\sigma, T]$ such that $|u_\varepsilon(x_0, t_0)| < 1/2$. According to Proposition 2.4 it follows

$$|u_\varepsilon(x, t_0) - u_\varepsilon(x_0, t_0)| \le C_0 \varepsilon^{-\alpha} |x - x_0|^\alpha \le C_0 \lambda^\alpha < \frac{1}{4}, \quad \forall x \in B(x_0, \lambda\varepsilon).$$

Hence $(1 - |u_\varepsilon(x, t_0)|^2)^2 > \frac{1}{16}$ $\forall x \in B(x_0, \lambda\varepsilon)$, and

$$\int_{B(x_0, \lambda\varepsilon) \cap D} (1 - |u_\varepsilon(x, t_0)|^2)^2 dx > \frac{1}{16} |D \cap B(x_0, \lambda\varepsilon)| \ge \beta \frac{1}{16} (\lambda\varepsilon)^n = \mu\varepsilon^n.$$

Since $x_0 \in B^{l\varepsilon} \cap D$, and $(B(x_0, \lambda\varepsilon) \cap D) \subset (B^{2l\varepsilon} \cap D)$, the inequality above implies

$$\int_{B^{2l\varepsilon} \cap D} (1 - |u_\varepsilon(x, t_0)|^2)^2 dx > \mu\varepsilon^n,$$

which contradicts (2.11) and thus Proposition 2.5 is proved.

Hereafter, we always assume that u_ε is the weak radial solution of (1.1)-(1.3). Let λ, μ be the constants in Proposition 2.5. If

$$\frac{1}{\varepsilon^n} \int_{B(x^\varepsilon, 2\lambda\varepsilon) \cap D} (1 - |u_\varepsilon(x, t)|^2)^2 dx \le \mu$$

for all $t \in I$, then $B(x^\varepsilon, \lambda\varepsilon) \times I$ is called a good cylinder. Otherwise $B(x^\varepsilon, \lambda\varepsilon) \times I$ is called a bad cylinder.

Remark. Recalling the results in [6,§4] and [8, Chapter III], we know that the points (x,t) such that $|u_\varepsilon(x,t)| < 1/2$ are located in the distinct *time layers*. The time layer is higher, the number of points such that $|u_\varepsilon| < 1/2$ is larger. When $p \neq 2$, we can hardly describe so elaborately as in [6] and [8]. Indeed, for example, (4.3) in [6] seems to be difficult to obtain now. We have to use the rough argument of the good and the bad cylinders to locate the points where $|u_\varepsilon| < 1/2$. On the other hand, the radial structure of u_ε offsets the rough argument, and we may still prove the zeros of u_ε are located near the axial $\{0\} \times (0,t]$.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in \mathbf{N}\}$ is a family of balls satisfying

$$\begin{aligned} &(i): && x_i^\varepsilon \in D, i \in \mathbf{N}; \\ &(ii): && D \subset \cup_{i\in\mathbf{N}} B(x_i^\varepsilon, \lambda\varepsilon) \\ &(iii): && B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \neq j \end{aligned} \tag{2.13}$$

Write $J_\varepsilon = \{i \in \mathbf{N}; B(x_i^\varepsilon, \lambda\varepsilon) \times I$ is a bad cylinder$\}$.

Proposition 2.6. *There exists a positive integer N_0 independent of $\varepsilon \in (0,1)$, such that the number of bad cylinders $Card\ J_\varepsilon \leq N_0$.*

Proof. By the definition of bad cylinders, it is easy to deduce that, if $B(x_i^\varepsilon, \lambda\varepsilon) \times I$ is a bad cylinder for any given i, then there exists $t_i \in I$ such that

$$\int_{B(x_i^\varepsilon, 2\lambda\varepsilon)\cap D} (1 - |u_\varepsilon(x,t_i)|^2)^2 dx > \mu\varepsilon^n,$$

where μ is independent of ε and t_i. Since (2.13) implies that every point in D can be covered by finite, say m (independent of ε) balls. Thus, it follows from Proposition 2.3 that

$$\begin{aligned} \mu\varepsilon^n Card J_\varepsilon &\leq \sum_{i\in J_\varepsilon} \int_{B(x_i^\varepsilon, 2\lambda\varepsilon)\cap D} (1 - |u_\varepsilon(x,t_i)|^2)^2 dx \\ &\leq m \int_{\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, 2\lambda\varepsilon)\cap D} (1 - |u_\varepsilon(x,t_i)|^2)^2 dx \\ &\leq m \sup_{t\in[\sigma,T]} \int_B (1 - |u_\varepsilon(x,t)|^2)^2 dx \leq mC\varepsilon^n \end{aligned}$$

and hence $Card\ J_\varepsilon \leq \frac{mC}{\mu}$. By taking $N_0 = [\frac{mC}{\mu}] + 1$ we can see the conclusion.

This proposition shows that, no matter how small the parameter ε, the number of the bad cylinders is always finite. At the same time, the number of the good cylinders becomes larger when $\varepsilon \to 0$. Thus, we can collect the bad cylinders closing to each other, and put them into a bigger bad cylinder, such that all the bad cylinders locate near some finite line segments.

Proposition 2.7. *There exist a subset $J \subset J_\varepsilon$ and a constant $h \geq \lambda$ such that*

$$
\begin{aligned}
&[\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \times I] \subset [\cup_{i\in J} B(x_j^\varepsilon, h\varepsilon) \times I],\\
&|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, \quad i, j \in J, \quad i \neq j.
\end{aligned}
\tag{2.14}
$$

Proof. If there are two points x_1, x_2 such that (2.14) is not true with $h = \lambda$, we take $h_1 = 9\lambda$ and $J_1 = J_\varepsilon \setminus \{1\}$. Now, if (2.14) holds, then we are done. Otherwise, we continue to choose a pair points x_3, x_4 which does not satisfy (2.14). Take $h_2 = 9h_1$ and $J_2 = J_\varepsilon \setminus \{1, 3\}$. After at most N_0 steps, we can choose $h \in [\lambda, \lambda 9^{N_0}]$ and complete this proposition.

Proof of Theorem 1.1. Applying proposition 2.7, we may modify the family of bad cylinders, such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon) \times I; i \in J\}$, satisfies

$$
[\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \times I] \subset [\cup_{i\in J} B(x_i^\varepsilon, h\varepsilon) \times I], \quad \lambda \leq h; \quad Card\ J \leq Card\ J_\varepsilon
$$

$$
|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i, j \in J, i \neq j
$$

The last condition implies that every two cylinders in the new family are not intersected.

Suppose there exists a point

$$
(x_0, t_0) \in \{(x, t) \in D \times [\sigma, T]; |u_\varepsilon(x, t)| < 1/2\},
$$

such that (x_0, t_0) is not in $\overline{B(0, h\varepsilon)} \times [\sigma, T]$. Then, all points on the circle $S_0 = \{x \in D;\ |x| = |x_0|\}$ satisfy $|u_\varepsilon(x, t_0)| < 1/2$. Hence, in virtue of Proposition 2.5, if $x \in S_0$, then (x, t_0) must be contained in bad cylinders. However, since $|x_0| > h\varepsilon$, $\{(x, t_0); x \in S_0\}$ can not be included in a single bad cylinder. It must be contained in at least two disjoint bad cylinders. This is impossible. The contradiction means

$$
\{(x, t) \in D \times [\sigma, T]; |u_\varepsilon(x, t)| < 1/2\} \subset B(0, h\varepsilon) \times [\sigma, T].
$$

Combining this with (2.10) yields the consequence of Theorem 1.1.

3. Uniform Estimate

To obtain the global estimate of $\|(f_\varepsilon)_r\|_{L^\infty[R,1]}$ for any $R \in (0, 1)$, we first set up the inner estimate of $\|(f_\varepsilon)_r\|_{L^l}$ for all $l \geq 1$.

Proposition 3.1. *Assume $f_\varepsilon^M \geq 0$ solves (2.4) and (2.5). Then for any compact subset $K \subset (0,1)\times(0,T]$ and $l > 1$, there exists a constant $C > 0$ which is independent of M and ε, such that $\|\frac{\partial}{\partial r} f_\varepsilon^M\|_{L^l(K,\mathbf{R})} \leq C = C(K,l)$.*

Proof. Denote the solution f_ε^M of (2.4) by f. Differentiate (2.4) with respect to r, then

$$\frac{\partial f_r}{\partial t} = (A^{(p-2)/2} f_r)_{rr} + [A^{(p-2)/2}(\frac{n-1}{r} f_r - \frac{f}{r^2})]_r + \frac{1}{\varepsilon^p}[f(1-f^2)]_r. \tag{3.1}$$

Let $B_{2\rho} \subset (0,1)$ be an interval with the length 2ρ. Take $\zeta \in C^\infty(\overline{B_{2\rho}} \times (0,T])$ satisfying

$$\begin{aligned}
&0 \leq \zeta(r,\tau) \leq 1, && (r,\tau) \in B_{2\rho}\times(0,T];\\
&\zeta(r,\tau) = 1, && (r,\tau) \in B_\rho \times (t,T);\\
&\zeta(r,\tau) = 0, && (r,\tau) \in \partial B_{2\rho}\times(0,T] \quad \text{or} \quad \tau \leq t/2;\\
&|\zeta_r| \leq C\rho^{-1}, && |\zeta_t| \leq Ct^{-1}.
\end{aligned}$$

Multiplying (3.1) by $\zeta^2 A^b f_r (b \geq 0)$ and integrating over $B_{2\rho}\times[0,t_0]$ we obtain

$$\begin{aligned}
&\frac{1}{2(b+1)}\int_{B_{2\rho}} \zeta^2(r,t_0) A^{b+1}(r,t_0)dr + \int_0^{t_0}\int_{B_{2\rho}} \zeta^2 (A^b f_r)_r (A^{(p-2)/2} f_r)_r dr d\tau\\
&= \frac{1}{b+1}\int_0^{t_0}\int_{B_{2\rho}} \zeta\zeta_t A^{b+1} dr d\tau - 2\int_0^{t_0}\int_{B_{2\rho}} \zeta\zeta_r A^b f_r (A^{(p-2)/2} f_r)_r dr d\tau\\
&\quad + \int_0^{t_0}\int_{B_{2\rho}} \left[A^{(p-2)/2}\left(\frac{f}{r^2} - \frac{n-1}{r} f_r\right)\right]_r \zeta^2 A^b f_r dr d\tau\\
&\quad + \frac{1}{\varepsilon^p}\int_0^{t_0}\int_{B_{2\rho}} (1-f^2)\zeta^2 A^b f_r^2 dr d\tau - \frac{1}{2\varepsilon^p}\int_0^{t_0}\int_{B_{2\rho}} [(f^2)_r]^2 \zeta^2 A^b dr d\tau\\
&:= \Sigma_{i=1}^5 I_i.
\end{aligned} \tag{3.2}$$

From (2.4) and (1.5) it follows

$$I_4 \leq \int_0^{t_0}\int_{B_{2\rho}} f^{-1}[|f_t| + |(A^{(p-2)/2} f_r)_r| + A^{(p-2)/2}|\frac{f}{r^2} - \frac{n-1}{r} f_r|]\zeta^2 A^{b+1} dr d\tau.$$

By means of Holder inequality and (2.7), we deduce that

$$\int_0^{t_0}\int_{B_{2\rho}} |f_t|\zeta^2 A^{b+1} dr d\tau \leq C(\int_0^{t_0}\int_{B_{2\rho}} \zeta^4 A^{2b+2} dr d\tau)^{1/2}.$$

When $2b+4 \le p$, we have

$$\left(\int_0^{t_0}\int_{B_{2\rho}} A^{2b+2}drd\tau\right)^{1/2} \le C\left(\int_0^{t_0}\int_{B_{2\rho}} A^{(p+2b)/2}drd\tau\right)^{(2b+2)/(p+2b)}.$$

When $2b+4>p$, we use the embedding inequality (cf. [5, Ch 1, Proposition 3.1])

$$\int_0^T\int_0^1 |F(r,t)|^q drdt \le C\left(\int_0^T\int_0^1 |F_r(r,t)|^{h_1}drdt\right)\left(\sup_t\int_0^1 |F(r,t)|^{h_2}dr\right)^{h_1} \tag{3.3}$$

for $F \in L^\infty(0,T;L^q((0,1),\mathbf{R}))\cap L^p(0,T;W_0^{1,p}((0,1),\mathbf{R}))$, where C only depends on $q=h_1(h_2+1)$. By taking $q=2, h_1=h_2=1$ and $F=\zeta^2A^{b+1}$, we derive that, for any $\delta\in(0,1)$,

$$\begin{aligned}
&\left(\int_0^{t_0}\int_{B_{2\rho}} \zeta^4A^{2b+2}drd\tau\right)^{1/2}\\
&\le \delta\left(\sup_t\int_{B_{2\rho}}\zeta^2A^{b+1}dr\right)+\left(\int_0^{t_0}\int_{B_{2\rho}}A^{(p+2b)/2}drd\tau\right)^{(2b+2)/(p+2b)}\\
&\quad +\delta\int_0^{t_0}\int_{B_{2\rho}}\zeta^2A^{(p+2b-2)/2}(f_{rr})^2drd\tau\\
&\quad +C\left(\int_0^{t_0}\int_{B_{2\rho}}A^{(p+2b)/2}drd\tau\right)^{(2b+4-p)/(p+2b)}.
\end{aligned}$$

Combining these estimates, and using Young inequality and (2.7), we obtain that for any $\delta\in(0,1)$,

$$\begin{aligned}
I_4 &\le \delta I_0+\delta\left(\sup_t\int_0^{t_0}\zeta^2A^{b+1}dr\right)\\
&\quad +C\left(\int_0^{t_0}\int_{B_{2\rho}}A^{(p+2b)/2}drd\tau+\int_0^{t_0}\int_{B_{2\rho}}\zeta^2A^{(p+2b+2)/2}drd\tau\right),
\end{aligned}$$

where

$$\begin{aligned}
I_0=\int_0^{t_0}\int_{B_{2\rho}}&\zeta^2[A^{(p+2b-2)/2}(f_{rr})^2+(p+2b-2)A^{(p+2b-4)/2}f_r^2(f_{rr})^2\\
&+2b(p-2)A^{(p+2b-6)/2}f_r^4(f_{rr})^2]drd\tau.
\end{aligned}$$

Similarly, we also have, by applying Young inequality again,

$$|I_2|+|I_3|\le \delta I_0+C\left(\int_0^{t_0}\int_{B_{2\rho}}A^{(p+2b)/2}drd\tau+\int_0^{t_0}\int_{B_{2\rho}}\zeta^2A^{(p+2b+2)/2}drd\tau\right).$$

Substituting these (with δ sufficiently small) and $I_5 \le 0$ into (3.2), and calculating its left hand side we deduce that

(3.4)
$$\begin{aligned}&\frac{1}{4(b+1)}\int_{B_{2\rho}}\zeta^2(r,t_0)A^{b+1}(r,t_0)dr + I_0\\&\le C\left(\int_0^{t_0}\int_{B_{2\rho}}(\zeta_r^2+1)A^{(p+2b)/2}drd\tau + \int_0^{t_0}\int_{B_{2\rho}}\zeta^2A^{(p+2b+2)/2}drd\tau\right)\end{aligned}$$

To estimate the last term of the right hand side of (3.4), we take $F=\zeta^{2/q}|f_r|^{(p+2b+2)/q}$ in (3.3). Set $h_1=1$, $q\in(1,2)$, then $h_2=q-1$. Thus,

$$\begin{aligned}&\int_0^{t_0}\int_{B_{2\rho}}\zeta^2|f_r|^{p+2b+2}drd\tau \le C\sup_{0<t<t_0}\int_{B_{2\rho}}\zeta^{2(q-1)/q}|f_r|^{(1-1/q)(p+2b+2)}dr)\\&\cdot\left[\int_0^{t_0}\int_{B_{2\rho}}\zeta^{2/q-1}|\zeta_r||f_r|^{(p+2b+2)/q}drd\tau\right.\\&+\left(\int_0^{t_0}\int_{B_{2\rho}}\zeta^2|f_r|^{p+2b-2}|f_{rr}|^2drd\tau\right)^{1/2}\\&\left.\left(\int_0^{t_0}\int_{B_{2\rho}}\zeta^{4/q-2}|f_r|^{(2p+4b+4)/q-(p+2b)}drd\tau\right)^{1/2}\right].\end{aligned}$$

Applying Holder inequality we have

$$\begin{aligned}&\sup\nolimits_{0<t<t_0}\int_{B_{2\rho}}\zeta^{2(q-1)/q}|f_r|^{(1-1/q)(p+2b+2)}dr\\&\le(\sup\nolimits_{0<t<t_0}\int_{B_{2\rho}}\zeta^2|f_r|^{2b+2}dr)^{(q-1)/q}(\sup_{0<t<t_0}\int_{B_{2\rho}}|f_r|^{p(q-1)}dr)^{1/q}.\end{aligned}$$

Since $p>2$, we may choose $q\in(1+\frac{2}{p},2)$. Hence $\frac{p+2b+2}{q}\le p+2b$, $\frac{2p+4b+4}{q}-(p+2b)\le p+2b$ and $(q-1)p\le p$. Applying Holder inequality again, and from (2.7) we derive that for any $\delta\in(0,1)$,

$$\begin{aligned}&\int_0^{t_0}\int_{B_{2\rho}}\zeta^2|f_r|^{p+2b+2}drd\tau\\&\le\delta\int_0^{t_0}\int_{B_{2\rho}}\zeta^2|f_r|^{p+2b-2}|f_{rr}|^2drd\tau+\delta\sup_t\int_{B_{2\rho}}A^{(p+2b)/2}dr\\&+C\left(\delta)(\int_0^{t_0}\int_{B_{2\rho}}|f_r|^{p+2b}drd\tau\right)^C.\end{aligned}\tag{3.5}$$

Substituting this into (3.4), choosing δ sufficiently small, we get

$$\int_{B_{2\rho}}\zeta^2A^{b+1}(r,t_0)dr+I_0\le C+C\left(\int_0^{t_0}\int_{B_{2\rho}}|f_r|^{p+2b}drd\tau\right)^C.$$

Since it always true for all $t_0 \in (0,T)$, we obtain

$$\sup_{0<\tau<T}\int_{B_{2\rho}} \zeta^2 A^{b+1}(r,\tau)dr + I_0 \le C + C\left(\int_0^T\int_{B_{2\rho}} |f_r|^{p+2b} dr d\tau\right)^C . \tag{3.6}$$

First, take $b=0$. Using (2.7) and choosing δ sufficiently small, we have

$$\sup_{0<\tau<T}\int_{B_{2\rho}} (\zeta^{2/\gamma}w)^\gamma(r,\tau)dr + \int_0^T\int_{B_{2\rho}} (\zeta^{2/\gamma}|w_r|)^2 dr d\tau \le C,$$

where $w = A^{(p+2b)/4}$ and $\gamma = \frac{4(b+1)}{p+2b}$. Applying the embedding inequality (3.3) to $\zeta^{2/\gamma}w$ with $h_1 = 2$ and $h_2 = \gamma$, we have $\int_0^T\int_{B_{2\rho}} (\zeta^{2/\gamma}w)^s dx d\tau \le C$, which implies $\int_t^T\int_{B_\rho} w^s dx d\tau \le C$, where $s = 2+\frac{8}{p}$. This means $f_r \in L^{p+4}(B_\rho\times(t,T))$ and

$$\int_t^T\int_{B_\rho} |f_r|^{p+4} dr d\tau \le C.$$

Next, take $b=1$. Applying the result above and proceeding in the same way just now, we can see that $f_r \in L^{p+10}(B_\rho\times(t,T))$ and $\int_t^T\int_{B_\rho}|f_r|^{p+10}drd\tau \le C$. For any given $l>0$, arguing inductively we may find some s_k such that $p+s_k>l$, and deduce that $f_r \in L^{p+s_k}(B_\rho\times(t,T))$, $\int_t^T\int_{B_\rho}|f_r|^{p+s_k}drd\tau \le C$. Substituting this into (3.6) with some sufficiently large b, we see that $\sup_{t<\tau<T}\int_{B_{2\rho}}|f_r|^l dr \le C$. Thus Proposition 3.1 is proved.

By means of the Moser iteration, from Proposition 3.1 we can further prove the following Proposition.

Proposition 3.2. *Assume $f_\varepsilon^M \ge 0$ solves* (2.4) *and* (2.5). *For any compact subset $K \subset (0,1)\times(0,T]$, there exists a constant $C>0$ which is independent of M and ε, such that*

$$\|\frac{\partial}{\partial r} f_\varepsilon^M\|_{L^\infty(K,\mathbf{R})} \le C = C(K).$$

Proof. Assume $R\in(0,1)$, $t_1\in(0,T)$. Let $t_0\in(t_1,T)$, $t_0>4t_1$ and $r_0\in(0,1)$. Define

$$\begin{aligned}
Q(\rho,t_0) &= (r_0-\rho, r_0+\rho)\times(t_0,T). \quad T_m = \frac{t_0}{2}-\frac{t_0}{2^{m+2}}, \quad \rho_m = \rho+\frac{\rho}{2^m},\\
\overline{\rho}_m &= \frac{1}{2}(\rho_m+\rho_{m+1}) = \rho + \frac{3\rho}{2^{m+2}},\\
B_m &= (r_0-\rho_m, r_0+\rho_m), \quad B'_m = (r_0-\overline{\rho}_m, r_0+\overline{\rho}_m),\\
Q_m &= B_m\times(T_m,T), \quad Q'_m = B'_m\times(T_{m+1},T),
\end{aligned}$$

and take $\zeta_m(r,t) \in C_0^1(Q_m)$ satisfying

$$\zeta_m = 0, \forall t \le T_m; \quad \zeta_m = 1, \forall (r,t) \in Q'_m$$

$$|\zeta_{mr}| \le \rho^{-1} 2^{m+2}, \quad 0 \le \zeta_{mt} \le t_0^{-1} 2^{m+3}.$$

Here $\rho > 0$ is sufficiently small such that $(r_0 - 2\rho, r_0 + 2\rho) \subset (0,1)$ and $|Q_m| \le 1$. Multiplying (3.1) with $\zeta_m^2 A^b f_r$ $(b \ge 1)$ and integrating over Q_m we obtain

$$\begin{aligned}
&\frac{1}{2(b+1)} \int_{B_m} \zeta_m^2 A^{b+1}(r,t)dr + \int_{Q_m} \zeta_m^2 (A^b f_r)_r (A^{(p-2)/2} f_r)_r dr d\tau \\
&= \frac{1}{b+1} \int_{Q_m} \zeta_m \zeta_{m\tau} A^{b+1} dr d\tau - 2 \int_{Q_m} \zeta_m \zeta_{mr} A^b f_r (A^{(p-2)/2} f_r)_r dr d\tau \\
&+ \frac{1}{\varepsilon^p} \int_{Q_m} (1-f^2) \zeta_m^2 A^b (f_r)^2 dr d\tau - \frac{1}{2\varepsilon^p} \int_{Q_m} \zeta_m^2 A^b ((f^2)_r)^2 dr d\tau.
\end{aligned}$$

Similar to the derivation of (3.4), we may also get

$$\begin{aligned}
&\frac{1}{4(b+1)} \int_{B_m} \zeta^2 A^{b+1}(r,t)dr + J_1 \\
&\le C\Bigg(\int_{Q_m} |\zeta_{m\tau}| A^{b+1} dr d\tau + \int_{Q_m} A^{(p+2b)/2} |\zeta_{mr}|^2 dr d\tau \\
&\quad + \frac{p+2b-2}{2} \int_{Q_m} \zeta^2 A^{(p+2b+2)/2} dr d\tau \Bigg)
\end{aligned} \tag{3.7}$$

with the constant C independent of M, ε, b and m. Here

$$\begin{aligned}
J_1 = \int_{Q_m} & \zeta_m^2 [A^{(p+2b-2)/2} f_{rr}^2 + (p+2b-2) A^{(p+2b-4)/2} f_r^2 f_{rr}^2 \\
& + 2b(p-2) A^{(p+2b-6)/2} f_r^4 f_{rr}^2] dr d\tau.
\end{aligned}$$

To estimate $\int_{Q_m} \zeta^2 A^{(p+2b+2)/2} dr d\tau$, we take $F = \zeta_m^{2/q} |f_r|^{(p+2b+2)/q}$ in (3.3) with $h_1 = 1$, $h_2 = q-1$ and $q \in (1 + \frac{2}{p}, 2)$. Then, using Holder inequality, and noticing $|Q_m| \le 1$ we obtain

$$\int_{Q_m} \zeta_m^2 A^{(p+2b+2)/2} dr d\tau \le C J_3 \left[\frac{2^m}{\rho} J_4^{\frac{p+2b+2}{q(p+2b)}} + \frac{p+2b+2}{q} J_1^{\frac{1}{2}} J_4^{\frac{p+2b+2}{q(p+2b)} - \frac{1}{2}} \right] \tag{3.8}$$

(in which we replace t_0 by $t \in (T_m, T)$ since (3.8) still holds for all $t_0 \in (T_m, T)$), where

$$J_3 = sup_{T_m < t < T} \int_{B_m} \zeta_m^{2(q-1)/q} |f_r|^{(1-1/q)(p+2b+2)} dr, \quad J_4 = \int_{Q_m} |f_r|^{p+2b} dr d\tau.$$

Set $J_2 = \sup_{T_m<t<T} \int_{B_m} \zeta_m^2 A^{b+1} dr$. By means of Holder inequality and (2.7) we have

$$J_3 \le J_2^{(q-1)/q} (\sup_t \int_{Q_m} |f_r|^{p(q-1)} dr)^{1/q} \le C J_2^{(q-1)/q}.$$

Substituting this and (3.8) into (3.7), and similar to the derivation of (3.6) we also obtain

$$\begin{aligned} J_1 + J_2 \le C \Bigg\{ & \frac{2^m}{t_0} J_4^{\frac{2(b+1)}{p+2b}} + \left(\frac{2^m}{\rho}\right)^2 J_4 + \frac{p+2b-2}{2} J_2^{\frac{q-1}{q}} \\ & \left[\frac{2^m}{\rho} J_4^{\frac{p+2b+2}{q(p+2b)}} + \frac{p+2b+2}{q} J_1^{\frac{1}{2}} J_4^{\frac{p+2b+2}{q(p+2b)} - \frac{1}{2}} \right] \Bigg\}. \end{aligned}$$

Using Young inequality, yields

$$\begin{aligned} J_1 + J_2 \quad \le C & \left[\frac{2^m}{t_0} J_4^{\frac{b+1}{p+2b}} + \left(\frac{2^m}{\rho}\right)^2 J_4 + (p+2b-2)^q \left(\frac{2^m}{\rho}\right)^q J_4^{\frac{p+2b+2}{p+2b}} \right. \\ & \left. + \left(\frac{p+2b+2}{q}\right) J_4^{(\frac{p+2b+2}{q(p+2b)} - \frac{1}{2})\left(\frac{2q}{2-q}\right)} \right]. \end{aligned}$$

Since $\frac{b+1}{p+2b} \le 1 \le \frac{p+2b+2}{p+2b} \le (\frac{p+2b+2}{q(p+2b)} - \frac{1}{2})(\frac{2q}{2-q})$, by Young inequality it follows that

$$J_1 + J_2 \le C_1 C_2^m \left(1 + J_4^{1 + \frac{C_3}{(p-2)/2 + 3^m/2}}\right), \tag{3.9}$$

where $3^m = 2(b+1)$, $C_1 > 0$, $C_3 = \frac{2}{2-q}$, and C_2 depends on t_0, ρ, q. Set $w = A^{(p+2b)/4}$, $\gamma = \frac{4(b+1)}{p+2b}$. Then

$$\sup_{T_m<\tau<T} \int_{B_m} (\zeta_m^{2/\gamma} w)^\gamma (x,\tau) dx + \int_{Q_m} (\zeta_m^{2/\gamma} |w_r|)^2 dx d\tau \le J_1 + J_2.$$

Combining this with (3.9), and applying the embedding inequality (3.3) with $h_1 = 2$ and $h_2 = \gamma$, we have

$$\begin{aligned} & \int_{Q_{m+1}} A^{(p-2)/2 + 3^{m+1}/2} dr d\tau \\ & \le \left(C_1 C_2^m)^2 (1 + \int_{Q_m} A^{(p-2)/2 + 3^m/2} dr d\tau \right)^{2(1 + C_3/((p-2)/2 + 3^m/2))}. \end{aligned} \tag{3.10}$$

If there exists a subsequence of positive integers $\{m_i\}$ with $m_i \to \infty$ such that $J_4 = \int_{Q_{m_i}} A^{(p-2)/2 + 3^{m_i+1}/2} dr d\tau < 1$, then letting $m_i \to \infty$ yields immediately

$$\|A\|_{L^\infty(Q_\infty, \mathbf{R})} \le C. \tag{3.11}$$

Otherwise, there must be a positive integer m_0 such that $J_4 = \int_{Q_m} A^{\frac{p-2}{2}+3^{m+1}/2} drd\tau \geq 1$ for $m \geq m_0$. This, together with (3.10), implies

$$\int_{Q_{m+1}} A^{(p-2)/2+3^{m+1}/2} drd\tau$$
$$\leq (C_1 C_2^m)^2 \left(\int_{Q_m} A^{(p-2)/2+3^m/2} drd\tau \right)^{2(1+C_3/((p-2)/2+3^m/2))}.$$

Using Proposition 3.1 and an iteration lemma (Proposition 2.3 in [9]), we can also deduce the estimate (3.11). Proposition 3.2 is complete.

Now, the global estimation is given by the following Proposition.

Proposition 3.3. *Assume* $f_\varepsilon^M \geq 0$ *solves* (2.4) *and* (2.5). *For any compact subset* $K \subset [R,1] \times [R,T]$, *there exists a constant* $C > 0$ *which is independent of* M *and* ε, *such that*

$$\| \frac{\partial}{\partial r} f_\varepsilon^M \|_{L^\infty(K,\mathbf{R})} \leq C = C(K).$$

Proof. In virtue of Proposition 3.2, it only need to consider the estimation near the boundary point $r = 1$. Write $H(r,t) = f(r+1,t) - 1$. Set $\tilde{H}(r,t) = H(r,t)$ as $-1 \leq r \leq 0$, $\tilde{H}(r,t) = -H(-r,t)$ as $0 < r \leq 1$. If denoting still $\tilde{H}(r-1,t)+1$ in $[0,2]$ by $f(r,t)$, then we see that $f(r,t)$ solves (2.4) on $[0,2]$. Take $R < \frac{1}{4}$. Assume that $\zeta \in C^\infty((0,1] \times (0,T])$ satisfies $\zeta = 1$ when $r \geq 1-R$; $\zeta = 0$ when $r \leq 2R$. Multiplying (3.1) by $f_r A^b \zeta^2 (b \geq 0)$, and integrating the left hand side by parts on t, and integrating the first and the second terms of the right hand side by parts on r, we obtain

$$\frac{1}{2(b+1)} \int_R^1 \zeta^2(r,t_0) A^{b+1}(r,t_0) dr + \int_0^{t_0} \int_R^1 \zeta^2 (A^b f_r)_r (A^{(p-2)/2} f_r)_r drd\tau$$
$$= \frac{1}{b+1} \int_0^{t_0} \int_R^1 \zeta \zeta_t A^{b+1} drd\tau - 2 \int_0^{t_0} \int_R^1 \zeta \zeta_r A^b f_r (A^{(p-2)/2} f_r)_r drd\tau$$
$$+ \int_0^{t_0} \int_R^1 \left[A^{(p-2)/2} \left(\frac{f}{r^2} - \frac{n-1}{r} f_r \right) \right]_r \zeta^2 A^b f_r drd\tau$$
$$+ \frac{1}{\varepsilon^p} \int_0^{t_0} \int_R^1 (1-f^2) \zeta^2 A^b f_r^2 drd\tau - \frac{1}{2\varepsilon^p} \int_0^{t_0} \int_R^1 \left[(f^2)_r \right]^2 \zeta^2 A^b drd\tau$$
$$+ \int_0^{t_0} \left[\left(\left(A^{(p-2)/2} f_r \right)_r + [A^{(p-2)/2} \left(\frac{f}{r^2} - \frac{n-1}{r} f_r \right) \right]_r) \zeta^2 A^b f_r \right]_{r=1} \zeta^2 A^b f_r^2 d\tau,$$

since $\zeta(R,t)=0$. Noting $f(1,t)=1$, from (2.4) we deduce that

$$\begin{aligned}&\left[\left((A^{(p-2)/2}f_r\right)_r+[A^{(p-2)/2}\left(\frac{f}{r^2}-\frac{n-1}{r}f_r\right)\right]_r)\zeta^2A^bf_r]_{r=1}\\ &=\frac{\partial f(1,t)}{\partial t}-\frac{1}{\varepsilon^p}f(1,t)(1-f^2(1,t))=0.\end{aligned}$$

This means the last term of the right hand side of the inequality above is zero. Combining this with the inequality above, and by the same argument as the inner estimation, we can also derive (3.4). Now, take $F(r,t)=\zeta^{2/q}f_r^{(p+2)/q}$ in the embedding inequality (cf. [5, Ch1, Proposition 3.2])

$$\|F(r,t)\|_{L^2(B_T)}\le C(\|F(r,t)_r\|_{L^1(B_T)}+\sup_t\|F(r,t)\|_{L^1(B)})$$

instead of (3.3) (in fact, (3.3) is not valid since $F\neq 0$ near $r=1$). Thus, (3.6) can be still derived. The rest proof is same as the proof of the inner estimation.

4. Proof of Theorem 1.2

Proposition 4.1. *Assume u_ε is a radial solution of $(1.1)-(1.3)$, $f_\varepsilon(r)=|u_\varepsilon(x)|$. Then for any $R\in(0,1)$ and $T>0$, there exists a positive constant C (independent of ε), such that*

$$\sup_{(R,1]\times(R,T]}(1-f_\varepsilon^2)\le C\varepsilon^p. \tag{4.1}$$

Proof. Suppose $f=f_\varepsilon^M$ solves (2.4) and (2.5). Set $\psi=\frac{1-f^2}{\varepsilon^p}$. Multiplying (2.4) by f we obtain

$$r^{n-1}ff_t-(r^{n-1}A^{(p-2)/2}ff_r)_r+r^{n-3}f^2A^{(p-2)/2}=r^{n-1}f^2\psi. \tag{4.2}$$

Substituting $\psi_r=\frac{-2}{\varepsilon^p}ff_r$ into (4.2) we have

$$\begin{aligned}&\frac{\varepsilon^p}{2}\left[r^{n-1}\left(A^{(p-2)/2}\psi_{rr}-\psi_t\right)+\left(r^{n-1}A^{(p-2)/2}\right)_r\psi_r\right]+r^{n-1}A^{(p-2)/2}\left(f_r^2+\frac{f^2}{r^2}\right)\\ &=r^{n-1}f^2\psi.\end{aligned}$$

Suppose $\psi(r,t)$ achieves its maximum at the point (r_0,t_0). If $r_0=1$, (4.1) is true obviously. If $(r_0,t_0)\in(R,1)\times(R,T]$. Then $\psi_r(r_0,t_0)=0$, $\psi_{rr}(r_0,t_0)\le 0$ and $\psi_t(r_0,t_0)\ge 0$. Thus, it is deduced that, from Proposition 3.3,

$$\psi(r,t)\le\psi(r_0,t_0)\le r_0^{n-1}A^{(p-2)/2}\left(f_r^2+\frac{f^2}{r^2}\right)|_{(r_0,t_0)}\le C,$$

which implies

$$\sup_{(R,1]\times(R,T]}(1-f^2)\leq C\varepsilon^p \tag{4.3}$$

with the constant $C>0$ independent of ε and M. Combining this with (2.6) we can see that the second term of the right hand side of

$$u_t = div\left[\left((|\nabla u|^2+\frac{1}{M})^{(p-2)/2}\nabla u\right]+\frac{1}{\varepsilon^p}u(1-|u|^2\right)$$

is bounded on $\Omega := [\overline{B}\setminus B(0,R)]\times(R,T]$. According to [3, Theorem 1], we can find $C>0$ independent of ε and M, such that for the solution u_ε^M of the system above, $\|u_\varepsilon^M\|_{C^{\beta,\beta/2}(\Omega)}\leq C$ with some $\beta\in(0,1)$. From (2.4) it is not difficult to deduce that $u_\varepsilon^M = f_\varepsilon^M\frac{x}{|x|}$ on Ω. Hence,

$$\lim_{M\to\infty} f_\varepsilon^M = f_\varepsilon, \quad in \quad C^{\alpha,\frac{\alpha}{2}}((R,1]\times(R,T]),$$

where $\alpha\in(0,\beta)$. Letting $M\to\infty$ in (4.3) we can obtain (4.1). Proposition is proved.

Proof of Theorem 1.2.. By (4.1) and Proposition 3.3, the second and the third terms of

$$f_t = (A^{(p-2)/2}f_r)_r + A^{(p-2)/2}\left(\frac{n-1}{r}f_r-\frac{f}{r^2}\right)+\frac{1}{\varepsilon^p}f(1-f^2) \tag{4.4}$$

are bounded on $(R,1]\times(R,T]$ uniformly in ε. Here $A=(f_r)^2+(n-1)r^{-2}f^2$. Thus, according to Theorem 1 in [11], we see that for some $\beta\in(0,1)$,

$$\|(f_\varepsilon)_r\|_{C^{\beta,\frac{\beta}{2}}((R,1]\times(R,T])}\leq C,$$

where f_ε is a solution of (4.4), and the constant $C>0$ does not depend on ε. It is easy to verify that $u_\varepsilon(x)=f_\varepsilon(r)\frac{x}{|x|}$ is a weak solution of (1.1). By Proposition 2.1, u_ε is also the unique solution. This means that

$$\|\nabla u_\varepsilon\|_{C^{\beta,\frac{\beta}{2}}([\overline{B}\setminus B(0,R)]\times[R,T],\mathbf{R}^n)}\leq C.$$

Thus, we may find a function u_* and a subsequence u_{ε_k}, such that

$$\lim_{k\to\infty}\nabla u_{\varepsilon_k} = u_*,\ in\ C^{\alpha,\frac{\alpha}{2}}([\overline{B}\setminus B(0,R)]\times[R,T],\mathbf{R}^n), \quad \alpha\in(0,\beta).$$

This, together with (4.3), implies that $u_* = \nabla \frac{x}{|x|}$. Since any subsequence of u_ε contains a convergent subsequence in $C^{\alpha,\frac{\alpha}{2}}([\overline{B} \setminus B(0,R)] \times (0,T], \mathbf{R}^n)$ and the limit is the same function $\frac{x}{|x|}$, we may assert

$$\lim_{\varepsilon \to 0} \nabla u_\varepsilon = \nabla \frac{x}{|x|}, \ in \ C^{\alpha,\frac{\alpha}{2}}([\overline{B} \setminus B(0,R)] \times [R,T], \mathbf{R}^n),$$

and the theorem is proved.

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Yutian Lei
Department of Mathematics,
Nanjing Normal University,
Nanjing, Jiangsu 210097,
P. R. China
E-mail: leiyutian@njnu.edu.cn