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THE NON-RUIN PROBABILITY FOR THE RISK RESERVE PROCESS WITH EXPONENTIAL TYPE CLAIMS

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Abstract. In this paper, we consider the risk reserve process with claims. In order to have the non-ruin probability in finite time, we use the skeleton process studied by T.Mikosch. Furthermore, we find a general formula that derives the non-ruin probability for the model in which the claim inter-arrival time has an exponential distribution and the claim size has an exponential distribution with distinct rate.

1. Introduction

Recently, many banks or insurance companies are combined to avoid the ruin. It is a matter of common knowledge that a bank has been ruined in Japan. Thus it is a pressing need to have the non-ruin probability and the non-ruin policy for them. Usually they reserve the fund for the ruin risk. We call it as the risk reserve process.

For the risk reserve process, in the steady state, the expected ruin time and the ruin probability have been studied by Doi [1] and [2], respectively.

In this paper, we consider the risk reserve process with exponential type claims and we find the non-ruin probability depending on the initial state in finite time.

Let us denote by $U(t)$ the reserve level at time t, where $\{U(t)\}_{t\geq 0}$ is called the risk reserve process. If the reserve level is zero, the process ruins. The fluctuation of $U(t)$ is controlled by three elements : the claim inter-arrival time, the claim size and the premium rate. Let us assume that the claim inter-arrival time and the claim size are independent and identically distributed random variables, respectively. We

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also assume the premium rate is a constant. In the same way of Mikosch [3] we introduce the notation :

$$\begin{cases} U(t) & : \text{ the reserve level at time } t, \\ u & : \text{ the initial reserve level } (u = U(0) > 0), \\ X_n & : \text{ the } n\text{-th claim size }, \\ T_n & : \text{ the } n\text{-th claim arrival time } (T_0 = 0), \\ W_n & : \text{ the claim inter-arrival time} \\ & \qquad \text{between } (n-1)\text{-th and } n\text{-th claim arrival times}, \\ c & : \text{ the premium rate}, \end{cases}$$

where X_n and W_n are independent.

The total claim amount process $\{S(t)\}_{t\geq 0}$ and the premium income $I(t)$ are defined as follows :

The total claim amount process $\{S(t)\}_{t\geq 0}$ is define as

$$S(t) = \sum_{n=1}^{N(t)} X_n \quad (t \geq 0),$$

where $\{N(t)\}_{t\geq 0}$ is the claim number process defined by

$$N(t) = \max\{n \geq 1 : T_n \leq t\} \quad (t \geq 0).$$

We define the premium income by I(t) = ct, which is the accumulated income by time t.

Therefore, we obtain the expression of risk reserve process $\{U(t)\}_{t\geq 0}$ as follows:

$$U(t) = u + I(t) - S(t), \quad (t \geq 0).$$

In the next section, we make a mathematical model to get the non-ruin probability in finite time.

2. Mathematical Model and Analysis

In the risk reserve process $\{U(t)\}_{t\geq 0}$, the ruin can occur only at the time $t = T_n$ for some $n \geq 1$, since $\{U(t)\}_{t\geq 0}$ linearly increases in the intervals $[T_n,\ T_{n+1})$. We call the sequence $\{U(T_n)\}_{n\geq 0}$ the skeleton process of the risk reserve process $\{U(t)\}_{t\geq 0}$ (see Mikosch [3]). By use of the skeleton process, we can express the event $\{\text{ruin}\}$ in terms of the inter-arrival times W_n, the claim sizes X_n, the initial reserve level and the premium rate c, as follows :

$$\begin{aligned}\{\text{ruin}\} &= \left\{\inf_{t>0} U(t) < 0\right\}\\ &= \left\{\inf_{n\geq 1} [u + I(T_n) - S(T_n)] < 0\right\}\\ &= \left\{\inf_{n\geq 1} \left[u - \sum_{i=1}^{n} (X_i - cW_i)\right] < 0\right\}.\end{aligned}$$

Now, we define

$$Z_n = X_n - cW_n, \quad (n \geq 1),$$

$$S_n = Z_1 + \cdots + Z_n, \quad (n \geq 1,\ S_0 = 0).$$

In this section, we propose R|Ex|Ex model where R means the risk reserve process, the first Ex means that the claim inter-arrival time W_n has an exponential distribution with rate λ, and the next Ex means that the claim size X_n has an exponential distribution with rate μ. In what follows, we omit the subscript n.

2.1. Probability Density Function of Z for R|Ex|Ex Model

We find the probability distribution of random variable Z for this model. First, let $Y = cW$, which has the probability density function as follows :

$$f_Y(y) = \begin{cases} \dfrac{\lambda}{c} e^{-\frac{\lambda}{c} y} & (y \geq 0) \\ 0 & (y < 0). \end{cases}$$

Next, let us denote

$$\begin{cases} Z = X - Y \\ V = Y\,. \end{cases}$$

Since X and Y are independent random variables, we obtain the joint probability density function with respect to Z and V

$$f_{ZV}(z, v) = \mu e^{-\mu(z+v)} \cdot \frac{\lambda}{c} e^{-\frac{\lambda}{c} v},$$

where the domain of v is

$$\begin{cases} 0 \leq v < \infty & (z \geq 0) \\ -z \leq v < \infty & (z < 0). \end{cases}$$

We obtain the probability density function $g(z)$ of Z as follows :

$$g(z) = \begin{cases} \dfrac{\lambda\mu}{\lambda + c\mu} e^{-\mu z} & (z \geq 0) \\ \dfrac{\lambda\mu}{\lambda + c\mu} e^{\frac{\lambda}{c} z} & (z < 0). \end{cases} \tag{1}$$

2.2. Non-Ruin Probability in Finite Time

We denote by $r_n(u, c)$ the non-ruin probability that the risk reserve process does not ruin till n-th claim arrival time given the initial reserve level u and the premium rate c, that is,

$$\begin{aligned} r_n(u, c) = P(&Z_1 < u, Z_2 < u - S_1, \cdots, Z_n < u - S_{n-1} \\ &|U(0) = u, T_1 < T_2 < \cdots < T_n < \infty). \end{aligned} \tag{2}$$

For the general formula of $r_n(u, c)$, we have the following theorem.

Theorem 1. *For R|Ex|Ex model, we obtain the probability* $r_n(u, c)$ *as follows:*

$$\begin{aligned} r_0(u, c) &= 1, \\ r_n(u, c) &= r_{n-1}(u, c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda + c\mu}\right)^n \\ &\qquad e^{-u\mu} \sum_{i=0}^{n-1} K_{n,n-i} \frac{u^i}{i!} \left(\frac{c}{\lambda + c\mu}\right)^{n-i-1} \quad (n \geq 1), \end{aligned} \tag{3}$$

where

$$\begin{cases} K_{n,1} = 1, & (n \geq 1) \\ K_{n,n} = K_{n,n-1}, & (n \geq 2) \\ K_{n,l} = K_{n-1,l} + K_{n,l-1}, & (n \geq 3,\ 2 \leq l \leq n-1) \\ K_{n,l} = 0, & (\textit{others}). \end{cases} \tag{4}$$

Proof. Preliminarily, we describe the following two lemmas.

Lemma 1. *For any non-negative integer* i*, the following relation holds.*

$$\int_{-\infty}^{u} g(z)(u - z)^i e^{\mu z} dz = \frac{\lambda\mu}{\lambda + c\mu} \sum_{j=0}^{i+1} \frac{i!}{(i - j + 1)!} \left(\frac{c}{\lambda + c\mu}\right)^j u^{i-j+1}. \tag{5}$$

Proof of Lemma 1. In the case of $i = 0$, we find the left-hand side is equal to the right-hand side as follows :

$$\begin{aligned}\int_{-\infty}^{u} g(z)(u-z)^0 e^{\mu z} dz &= \int_{-\infty}^{u} g(z) e^{\mu z} dz \\ &= \int_0^u \frac{\lambda\mu}{\lambda+c\mu} e^{-\mu z} e^{\mu z} dz + \int_{-\infty}^{0} \frac{\lambda\mu}{\lambda+c\mu} e^{\frac{\lambda}{c}z} e^{\mu z} dz \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left(u+\frac{c}{\lambda+c\mu}\right) \\ &= \frac{\lambda\mu}{\lambda+c\mu}\sum_{j=0}^{1}\frac{0!}{(-j+1)!}\left(\frac{c}{\lambda+c\mu}\right)^j u^{-j+1}.\end{aligned}$$

In the case of $i = k$, we assume that (5) holds.

We prove that (5) is true if $i = k+1$ as follows:

$$\begin{aligned}&\int_{-\infty}^{u} g(z)(u-z)^{k+1} e^{\mu z} dz \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left\{\int_0^u (u-z)^{k+1} dz + \int_{-\infty}^{0} (u-z)^{k+1} e^{(\frac{\lambda}{c}+\mu)z} dz\right\} \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left(\frac{u^{k+2}}{k+2}+\frac{c}{\lambda+c\mu}u^{k+1}\right) + \frac{c}{\lambda+c\mu}(k+1)\int_{-\infty}^{0} g(z)(u-z)^k e^{\mu z} dz \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left(\frac{u^{k+2}}{k+2}+\frac{c}{\lambda+c\mu}u^{k+1}\right) \\ &\quad + \frac{c}{\lambda+c\mu}(k+1)\left\{\int_{-\infty}^{u} g(z)(u-z)^k e^{\mu z} dz - \int_0^u g(z)(u-z)^k e^{\mu z} dz\right\} \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left(\frac{u^{k+2}}{k+2}+\frac{c}{\lambda+c\mu}u^{k+1}\right) + \frac{c}{\lambda+c\mu}(k+1) \\ &\quad \cdot\left\{\frac{\lambda\mu}{\lambda+c\mu}\sum_{j=0}^{k+1}\frac{k!}{(k-j+1)!}\left(\frac{c}{\lambda+c\mu}\right)^j u^{k-j+1} - \frac{\lambda\mu}{\lambda+c\mu}\int_0^u (u-z)^k dz\right\} \\ &= \frac{\lambda\mu}{\lambda+c\mu}\left\{\frac{(k+1)!}{(k+2)!}\left(\frac{c}{\lambda+c\mu}\right)^0 u^{k+2} + \frac{c}{\lambda+c\mu}u^{k+1}\right. \\ &\quad \left. + \sum_{j=0}^{k+1}\frac{(k+1)!}{(k-j+1)!}\left(\frac{c}{\lambda+c\mu}\right)^{j+1} u^{k-j+1} - \frac{c}{\lambda+c\mu}u^{k+1}\right\} \\ &= \frac{\lambda\mu}{\lambda+c\mu}\sum_{j=0}^{(k+1)+1}\frac{(k+1)!}{\{(k+1)-j+1\}!}\left(\frac{c}{\lambda+c\mu}\right)^j u^{(k+1)-j+1}.\end{aligned}$$

Therefore, (5) holds for any non-negative integer i. ■

Lemma 2. *For any natural numbers m and n $(m \leq n)$, the following relation holds.*

$$\sum_{i=1}^{m} K_{n,i} = K_{n+1,m}. \tag{6}$$

Proof of Lemma 2. In the case of $m = 1$, by use of (4), we find the left hand side is equal to the right hand side as follows :

$$\sum_{i=1}^{1} K_{n,i} = K_{n,1} = 1 = K_{n+1,1}.$$

In the case of $m = l$, we assume that (6) holds.

Next, using (4), we prove that (6) is true if $m = l + 1$.

$$\sum_{i=1}^{l+1} K_{n,i} = \sum_{i=1}^{l} K_{n,i} + K_{n,l+1} = K_{n+1,l} + K_{n,l+1} = K_{n+1,l+1}.$$

Therefore, (6) holds for any natural numbers m and n $(m \leq n)$. ■

We proceed the proof of Theorem 1.

For $n = 1$, the left hand side of (3) is obtained by the definition.

$$r_1(u, c) = P(Z_1 < u | U(0) = u,\ T_1 < \infty) = \int_{-\infty}^{u} g(z_1) dz_1 = 1 - \frac{\lambda}{\lambda + c\mu} e^{-u\mu}.$$

On the other hand, the right hand side of (3) is reduced to

$$\begin{aligned} r_0(u, c) - \frac{1}{\mu} \left(\frac{\lambda\mu}{\lambda + c\mu} \right)^1 e^{-u\mu} \sum_{i=0}^{0} K_{1,1-i} \frac{u^i}{i!} \left(\frac{c}{\lambda + c\mu} \right)^{1-i-1} \\ = 1 - \frac{\lambda}{\lambda + c\mu} e^{-u\mu} K_{1,1} \frac{u^0}{0!} \left(\frac{c}{\lambda + c\mu} \right)^0 \\ = 1 - \frac{\lambda}{\lambda + c\mu} e^{-u\mu}. \end{aligned}$$

Therefore, (3) holds for $n = 1$.

We assume that (3) holds for $n = m$. For $n = m+1$, we have

$$
\begin{aligned}
r_{m+1}(u,c) &= \int_{-\infty}^{u} g(z_1) \int_{-\infty}^{u-s_1} g(z_2) \cdots \int_{-\infty}^{u-s_m} g(z_{m+1}) dz_{m+1} \cdots dz_2 dz_1 \\
&= \int_{-\infty}^{u} g(z_1) r_m(u-z_1,c) dz_1 \\
&= \int_{-\infty}^{u} g(z_1) r_{m-1}(u-z_1,c) dz_1 \\
&- \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^m e^{-u\mu} \sum_{i=0}^{m-1} K_{m,m-i} \frac{1}{i!}\left(\frac{c}{\lambda+c\mu}\right)^{m-i-1} \int_{-\infty}^{u} g(z_1)(u-z_1)^i e^{\mu z_1} dz_1 \,.
\end{aligned}
$$

Thus we have the following.

$$
\begin{aligned}
&r_{m+1}(u,c) \\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^m e^{-u\mu} \sum_{i=0}^{m-1} K_{m,m-i} \frac{1}{i!}\left(\frac{c}{\lambda+c\mu}\right)^{m-i-1} \\
&\cdot \int_{-\infty}^{u} g(z_1)(u-z_1)^i e^{\mu z_1} dz_1 \,.
\end{aligned}
\tag{7}
$$

By use of Lemma 1, (7) is reduced to

$$
\begin{aligned}
&r_{m+1}(u,c) \\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^m \cdot e^{-u\mu} \sum_{i=0}^{m-1} K_{m,m-i} \frac{1}{i!}\left(\frac{c}{\lambda+c\mu}\right)^{m-i-1} \\
&\quad \cdot \frac{\lambda\mu}{\lambda+c\mu} \sum_{j=0}^{i+1} \frac{i!}{(i-j+1)!}\left(\frac{c}{\lambda+c\mu}\right)^j u^{i-j+1} \\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1} \\
&\quad \cdot e^{-u\mu} \sum_{i=0}^{m-1} \sum_{j=0}^{i+1} K_{m,m-i} \frac{u^{i-j+1}}{(i-j+1)!}\left(\frac{c}{\lambda+c\mu}\right)^{m-(i-j+1)} \\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1} \cdot e^{-u\mu} \sum_{i=0}^{m-1} \sum_{j=0}^{i+1} K_{m,m-i} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{m-j} \\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1} \cdot e^{-u\mu} \\
&\quad \cdot \left\{ \sum_{j=1}^{m} \sum_{i=j-1}^{m-1} K_{m,m-i} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{m-j} + \sum_{i=0}^{m-1} K_{m,m-i}\left(\frac{c}{\lambda+c\mu}\right)^m \right\}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&r_{m+1}(u,c)\\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1} \cdot e^{-u\mu}\\
&\cdot \left\{ \sum_{j=1}^{m} \sum_{i=1}^{m-j+1} K_{m,i} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{m-j} + \sum_{i=1}^{m} K_{m,i}\left(\frac{c}{\lambda+c\mu}\right)^{m} \right\}.
\end{aligned} \tag{8}
$$

Furthermore, by use of Lemma 2, (8) is reduced to

$$
\begin{aligned}
&r_{m+1}(u,c)\\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1}\\
&\quad \cdot e^{-u\mu} \cdot \left\{ \sum_{j=1}^{m} K_{m+1,m-j+1} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{m-j} + K_{m+1,m}\left(\frac{c}{\lambda+c\mu}\right)^{m} \right\}\\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1}\\
&\quad \cdot e^{-u\mu} \cdot \left\{ \sum_{j=1}^{m} K_{m+1,m-j+1} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{m-j} + K_{m+1,m+1}\left(\frac{c}{\lambda+c\mu}\right)^{m} \right\}\\
&= r_m(u,c) - \frac{1}{\mu}\left(\frac{\lambda\mu}{\lambda+c\mu}\right)^{m+1}\\
&\quad \cdot e^{-u\mu} \sum_{j=0}^{(m+1)-1} K_{m+1,(m+1)-j} \frac{u^j}{j!}\left(\frac{c}{\lambda+c\mu}\right)^{(m+1)-j-1}.
\end{aligned}
$$

Thus, in the case of $n = m+1$, (3) holds.
Hence, Theorem 1 is proved for any natural number n. ■

3. Numerical Examples

In this section we consider the non-ruin policy. Assume the case where the mean claim inter-arrival time $1/\lambda = 1$ unit of time and the mean claim size $1/\mu = 1$ unit of amount. Then we need to decide the premium rate c under the condition that the probability $r_n(u,c)$ is greater than a certain value, where the number of claims n and the initial reserve level u are given. Now, we suppose n and u are equal to 100 and 10, respectively. From Theorem 1, we plot the graph of $r_{100}(10,c)$ against the premium rate c (Figure 1). In order to hold $r_{100}(10, c)$, the non-ruin probability that the process does not ruin till 100-th claim arrival time given the initial level 10 and the premium rate c, to be greater than 0.8, we must give the premium rate c the value of greater than 1.137.

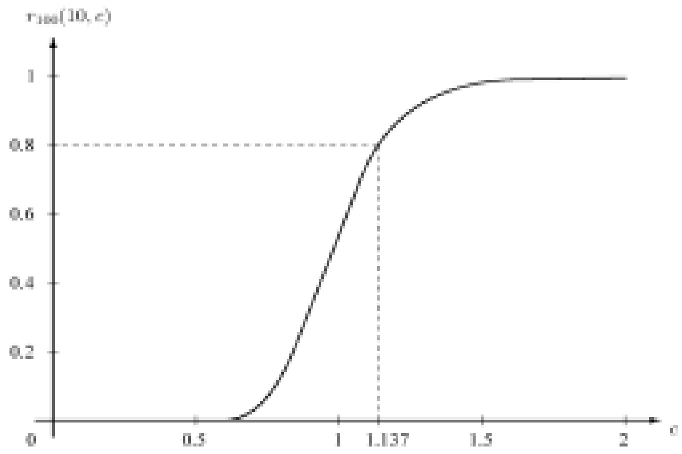

Figure 1.

Next we need to decide the initial reserve level u under the condition that the probability $r_n(u, c)$ is greater than a certain value, where the number of claims n and the premium rate c are given similarly above. Now, we suppose n and c are equal to 100 and 1.1, respectively. We plot the graph of $r_{100}(u, 1.1)$ against the initial reserve level u (Figure 2). In order to hold $r_{100}(u, 1.1)$ to be greater than 0.8, we must give the initial reserve level u the value of greater than 11.57.

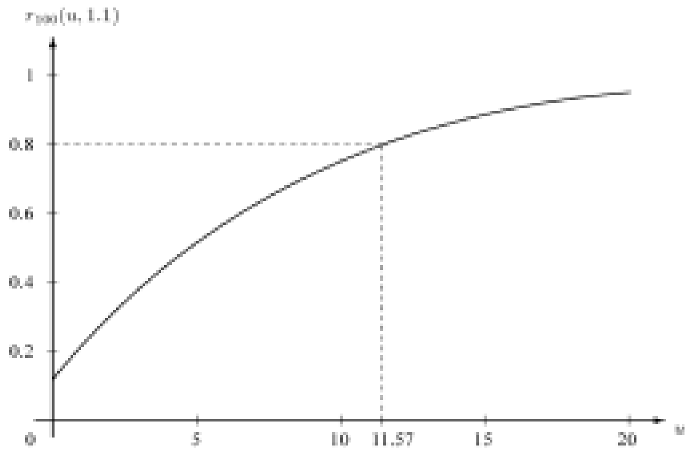

Figure 2.

4. Concluding Remarks

Assuming that the inter-arrival time and the claim size have the exponential distributions with constant rates λ and μ, respectively, we have analytically got the non-ruin probability for the R|Ex|Ex model in finite time. On the non-homogeneous case where both rates are time dependent functions $\lambda(t)$ and $\mu(t)$, we could construct the mathematical model, however, it may be difficult to have the analytical solution so that we have to use the Monte Carlo simulation or numerical analysis.

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