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CONTINUITY OF RESTRICTIONS OF (a, k)-REGULARIZED RESOLVENT FAMILIES TO INVARIANT SUBSPACES

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Abstract. Let X be a Banach space which is continuously embedded in another Banach space Y and is an invariant subspace for an (a, k)-regularized resolvent family $R(\cdot)$ of operators on Y. It is shown that the restriction of $R(\cdot)$ to X is strongly continuous with respect to the norm of X if and only if all its partial orbits are relatively weakly compact in X. This property is shared by many particular cases of (a, k)-regularized resolvent families, such as integrated solution families, integrated semigroups, and integrated cosine functions.

1. Introduction

Let Y be a Banach space with norm $\|\cdot\|_Y$ and let $X \subset Y$ be a linear subspace. Suppose X is equipped with a norm $\|\cdot\|_X$ such that $(X, \|\cdot\|_X)$ becomes a Banach space and such that $(X, \|\cdot\|_X)$ is continuously embedded in Y, i.e., the identity map from $(X, \|\cdot\|_X)$ onto $(X, \|\cdot\|_Y)$ is continuous, or equivalently, $\|x\|_Y \le M\|x\|_X$ for some $M > 0$ and all $x \in X$. Let $B(Y)$ and $B(X)$ denote the Banach algebras of all bounded linear operators on Y and on X, respectively.

For a C_0-semigroup $\{T(\cdot); t \ge 0\} \subset B(Y)$ of linear operators on Y which leaves X invariant, S.C. Hille [3] gives a characterization of strong continuity of the restricted semigroup $\{T(t)_X := T(t)|_X; t \ge 0\} \subset B(X)$ in terms of norm and weak compactness of the partial orbits

$$\mathcal{O}_x(\tau) := \{T(t)_X x; 0 \le t \le \tau\} \subset X$$

for $\tau > 0$ and all $x \in X$.

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The purpose of this paper is to prove this same property for cosine operator functions and more generally for an (a, k)-regularized resolvent family.

Let $a, k \in L^1_{loc}([0,\infty))$ be positive functions, and let A be a densely defined closed linear operator in Y. Consider the Volterra equation of convolution type

$$VE(a, A) \qquad u(t) = \int_0^t a(t-s)Au(s)ds + f(t),\ t \geq 0.$$

A strongly continuous function $\{R(t); t \geq 0\} \subset B(Y)$ is called an (a, k)*-regularized resolvent family* on Y for VE(a, A) if it satisfies the conditions:

$(R1)$ $R(0) = k(0)I$;

$(R2)$ $R(t)y \in D(A)$ and $AR(t)y = R(t)Ay$ for all $y \in D(A)$ and $t \geq 0$;

$(R3)$ $(a * R)(t)y \in D(A)$ and $R(t)y = k(t)y + (a * R)(t)Ay$ for all $y \in D(A)$ and $t \geq 0$.

It is easy to see that $(a * R)(t)y \in D(A)$ and

$$R(t)y = k(t)y + A(a * R)(t)y \text{ for all } y \in Y \text{ and } t \geq 0. \tag{1.1}$$

The notion of a (a, k)-regularized resolvent family was introduced and studied in [6, 7, 8]. See also [5, 14]. It contains α-times integrated solution families ($k(t) = t^\alpha/\Gamma(\alpha + 1)$) [9], resolvent families ($k(t) \equiv 1$) [10], α-times integrated semigroups ($a \equiv 1, k(t) = t^\alpha/\Gamma(\alpha + 1)$) [4], C_0-semigroups ($a = k \equiv 1$) [2], and α-times integrated cosine functions ($a(t) = t, k(t) = t^\alpha/\Gamma(\alpha + 1)$) [13] as special cases. In each of these particular cases, the operator A is just the generator of the respective family.

In particular, a $(t, 1)$-regularized resolvent family for VE(t, A) is just a *cosine operator function* $\{C(t); t \geq 0\}$ (cf. [12, 15]), which is defined as a strongly continuous function on $[0, \infty)$ satisfying

$$C(0) = I \text{ and } C(s + t) + C(s - t) = 2C(s)C(t) \text{ for all } s \geq t \geq 0.$$

By extending $C(\cdot)$ to the whole real line $\mathbb{R}$ as an even function, we see that the above equality holds for all $s, t \in \mathbb{R}$.

The main theorem (Theorem 2.4), to be proved in Section 2, asserts that when an (a, k)-regularized resolvent family $R(\cdot)$ on Y for VE(a, A) leaves the subspace X invariant, the restricted family $R(\cdot)_X$ forms an (a, k)-regularized resolvent family on X for VE(a, A_X) if and only if for all $x \in X$ and $\tau > 0$ the partial orbits

$$\mathcal{O}_x(\tau) := \{R(t)_X x; 0 \leq t \leq \tau\} \subset X$$

are relatively weakly compact in X. Here A_X denote the part A_X of A in X (i.e., $D(A_X) = \{x \in X; x \in D(A) \text{ and } Ax \in X\}$ and $A_X x = Ax$ for $x \in D(A_X)$) and X_w denote the weak topology of the Banach space X.

Clearly the necessity part of the above theorem is obvious. The sufficiency part comprises two implications: the first one from the X_w-compactness of partial orbits to the X_w-continuity of the orbits, and the second one from the X_w-continuity to the norm continuity. It will be seen in Proposition 2.3 that the first implication holds for any strongly continuous operator functions which leave X invariant. But the second implication seems not to hold in general. The proof of the second implication in Theorem 2.4 involves the operator A and condition (R3). However, a proof of the second implication without involving the generator A is possible for C_0-semigroups and cosine operator functions. One can refer to [2, Theorem 5.8] and [16, p. 233] for such proofs for C_0-semigroups. For cosine operator functions we will give in Theorem 3.1 an alternative proof without using the generator A.

2. Main Result

For a $\tau > 0$ let $u : [0, \tau] \to Y$ be a strongly continuous function such that $u[0, \tau] := \{u(t); t \in [0, \tau]\} \subset X$.

Lemma 2.1.

(i) *If $\{x_\alpha\} \subset X$ is a net X_w-convergent (X_w being the weak topology of X) to some $x \in X$, then $\{x_\alpha\}$ is also Y_w-convergent to x.*

(ii) *Every Y_w-closed subset of X is also X_w-closed, and every $[0, \tau] \times Y_w$-closed subset of $[0, \tau] \times X$ is also $[0, \tau] \times X_w$-closed.*

(iii) *Let $u : [0, \tau] \to X$ be X_w-continuous as well as Y_w-continuous. If $u(\cdot)$ is X_w-Riemann integrable (i.e., there is a unique $x \in X$ such that $\langle x, x^* \rangle = \int_0^\tau \langle u(t), x^* \rangle$, existing as a Riemann integral), then it is also Y_w-Riemann integrable, and X_w-$\int_0^\tau u(t)dt = Y_w$-$\int_0^\tau u(t)dt$.*

Proof. (i) For any $y^* \in Y^*$, the functional $x^* := y^*|_X$ is continuous on $(X, \|\cdot\|_X)$ because the topology of X is stronger than the topology of Y restricted to X. Hence $x^* \in X^*$ and so we have $\langle x_\alpha, y^* \rangle = \langle x_\alpha, x^* \rangle \to \langle x, x^* \rangle = \langle x, y^* \rangle$. This means that x_α is Y_w-convergent to x.

(ii) and (iii) follow from (i). ■

Lemma 2.2. *Let (S, σ) be a Hausdorff topological space. A function $u : [0, \tau] \to S$ is continuous if and only if $u[0, \tau]$ is relatively compact in S and the graph $G(u, [0, \tau]) := \{(t, u(t)); 0 \le t \le \tau\}$ is closed in $[0, \tau] \times S$.*

Proof. Necessity. The mappings $t \to u(t)$ and $t \to (t, u(t))$ are continuous functions from $[0, \tau]$ to S and to $[0, \tau] \times S$, respectively. Hence $u[0, \tau]$ is compact in S and $G(u, [0, \tau])$ is compact and hence closed in $[0, \tau] \times S$.

Sufficiency. For any $t_0 \in [0, \tau]$ and for any sequence $\{t_n\} \subset [0, \tau]$ such that $t_n \to t_0$, the relative compactness of $u[0, \tau]$ implies that $\{t_n\}$ contains a subsequence $\{t_{n_k}\}$ such that $u(t_{n_k})$ converges to some $x \in S$. By the closedness of $G(u, [0, \tau])$ in $[0, \tau] \times S$, we must have that $x = u(t_0)$. Then $u(t_n)$ must converge to $u(t_0)$, otherwise we can choose a subsequence of $\{u(t_n)\}$ which contains no subsequence with limit $u(t_0)$. This is a contradiction. Since $\{t_n\}$ is arbitrary, this shows that $u(\cdot)$ is continuous at t_0.

Let $S(\cdot) = \{S(t); t \geq 0\}$ be a strongly continuous function of linear operators on Y, and suppose X is invariant under $S(\cdot)$. Then $S(\cdot)_X = \{S(t)|_X; t \geq 0\}$ is a function of operators on $(X, \|\cdot\|_X)$. As shown by Lemma 2.1, for each $x \in X$ the orbit $\mathcal{O}_x(\tau) := \{S(t)x; 0 \leq t \leq \tau\}$ of $S(\cdot)_X x$ is weakly closed in X, and the graph of $S(\cdot)_X x$ is weakly closed in $[0, \infty) \times X$. However, $S(\cdot)_X$ is not necessarily continuous. The following theorem gives characterizations for $S(\cdot)_X$ to be strongly continuous.

Proposition 2.3. *The following conditions satisfy the relations:* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d).

(a) $S(\cdot)_X$ *is strongly continuous on* X.

(b) *For each* $x \in X$ *and for all* $\tau > 0$, $\mathcal{O}_x(\tau) := \{S(t)x; 0 \leq t \leq \tau\}$ *is compact in* X.

(c) *For each* $x \in X$ *and for all* $\tau > 0$, $\mathcal{O}_x(\tau)$ *is relatively* X_w*-compact (resp. bounded, when* X *is reflexive).*

(d) $S(\cdot)_X$ *is weakly continuous on* X.

Proof. "(a) $\Rightarrow$ (b) $\Rightarrow$ (c)" and "(d) $\Rightarrow$ (c)" are obvious.

(c) $\Rightarrow$ (d). Since $S(\cdot)x$ is strongly continuous in Y, $G(S(\cdot)x, [0, \tau])$ is strongly compact, and hence it is a $[0, \tau] \times Y_w$-compact subset of $[0, \tau] \times X$. By Lemma 2.1, $G(S(\cdot)x, [0, \tau])$ is $[0, \tau] \times X_w$-closed. This fact together with (c) implies (d), by Lemma 2.2. ∎

Let $a, k \in L^1_{loc}[0, \infty)$ be positive functions, and let A be a densely defined closed operator in Y. Let $R(\cdot) = \{R(t); t \geq 0\}$ be a (a, k)-resolvent family on Y for VE(a, A). Suppose X is invariant under $R(\cdot)$. Then $R(\cdot)_X = \{R(t)|_X; t \geq 0\}$ is a function of operators on $(X, \|\cdot\|_X)$. The following theorem gives characterizations for $R(\cdot)_X$ to be a (a, k)-resolvent family of operators on X.

Theorem 2.4. *For a* (a, k)*-resolvent family* $R(\cdot)$ *of operators on* Y *for VE*(a, A) *such that* X *is invariant under* $R(\cdot)$*, the following conditions are equivalent:*

(a) $R(\cdot)_X$ *is strongly continuous on* X.

(b) *For each $x \in X$ and for all $\tau > 0$, $\mathcal{O}_x(\tau) := \{R(t)x; 0 \leq t \leq \tau\}$ is compact in X.*

(c) *For each $x \in X$ and for all $\tau > 0$, $\mathcal{O}_x(\tau)$ is relatively X_w-compact (resp. bounded, when X is reflexive).*

(d) *$R(\cdot)_X$ is weakly continuous on X.*

Moreover, in this case A_X is a densely defined operator in X and $R(\cdot)_X$ is a (a, k)-resolvent family of operators on X for VE(a, A_X).

Proof. Because of Proposition 2.3, it remains to prove "(d) $\Rightarrow$ (a)".

(d) $\Rightarrow$ (a). First note that the X_w-continuity of $R(\cdot)_X x$ implies that $\mathcal{O}_x(\tau)$ is X_w-compact, and hence so is its X_w-closed convex hull $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$, by Krein's theorem.

For every $x \in X$, we consider the vectors $x_r := \frac{1}{(a*1)(r)} \int_0^r a(r-s)R(s)xds$, $r > 0$, defined as Riemann integrals in $\|\cdot\|_Y$. Then $x_r \in D(A)$, by (1.1). x_r is also equal to the Pettis integral

$$X_w\text{-}\int_0^r \frac{1}{(a*1)(r)} a(r-s)R(s)_X x ds \ (\in X)$$

of the X_w-continuous function $R(\cdot)_X x$ on $[0, r]$, which exists and lies in $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$ $(\subset X)$ by the X_w-continuity of $R(\cdot)_X x$, the X_w-compactness of $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$, and the fact that $\frac{1}{(a*1)(r)} \int_0^r a(r-s)ds = 1$ (cf. [11, Theorem 3.27]). Thus $x_r \in D(A) \cap X$. Since $(a*1)(r)Ax_r = A\int_0^r a(r-s)R(s)xds = R(r)x - k(r)x \in X$, x_r belongs to $D(A_X)$ and $(a*1)(r)A_X x_r = R(r)_X x - k(r)x$.

Hence $D := \{x'_r := \frac{(a*1)(r)}{(a*k)(r)} x_r; x \in X, r > 0\}$ and span(D) are subsets of $D(A_X)$. Clearly, the X_w-continuity of $R(\cdot)_X x$ at 0 imply that

$$\begin{aligned} |\langle x'_r - x, x^* \rangle| &\leq \frac{1}{(a*k)(r)} \int_0^r a(r-s)|\langle R(s)_X x - k(s)x, x^* \rangle| ds \\ &\leq \sup_{0 \leq s \leq r} \| \langle R(s)_X x - k(s)x, x^* \rangle| \to 0 \end{aligned}$$

as $r \to 0^+$ for all $x^* \in X^*$, i.e., $x'_r \to x$ weakly as $r \to 0^+$. Hence D is X_w-dense in X and the same are span(D) and $D(A_X)$. As linear subspaces of X, both span(D) and $D(A_X)$ are also strongly dense in X, by the Hahn-Banach theorem.

Since the weak continuity of $R(\cdot)_X$ implies it is locally bounded, to show that $R(\cdot)_X$ is strongly continuous, it remains to show that $\|R(t+h)_X x_r - R(t)_X x_r\|_X \to 0$ as $h \to 0$ (with $t + h \geq 0$) for all $x \in X$, $t \geq 0$, and $r > 0$.

Since $R(\cdot)_X x_r$ is assumed to be X_w-continuous, by the above argument and (R3), we see that the Pettis integral $X_w\text{-}\int_0^t a(t-s)R(s)_X A_X x_r ds$ exists and

$$R(t)_X x_r - k(t)x_r = R(t)x_r - k(t)x_r = \int_0^t a(t-s)R(s)Ax_r ds$$
$$= X_w\text{-}\int_0^t a(t-s)R(s)_X A_X x_r ds.$$

It follows that for any fixed $t \geq 0$ and all $|h| < 1$ such that $t + h \geq 0$

$$\begin{aligned}
&|\langle R(t+h)_X x_r - R(t)_X x_r, x^*\rangle| \\
= &\Big| \int_0^{t+h} a(t+h-s)\langle R(s)_X A_X x_r, x^*\rangle ds \\
&- \int_0^t a(t-s)\langle R(s)_X A_X x_r, x^*\rangle ds \Big| \\
\leq &\left| \int_t^{t+h} a(t+h-s)\langle R(s)_X A_X x_r, x^*\rangle ds \right| \\
&+ \left| \int_0^t (a(t+h-s) - a(t-s))\langle R(s)_X A_X x_r, x^*\rangle ds \right| \\
\leq &\left(\int_0^h a(s)ds + \int_0^t |a(t+h-s) - a(t-s)|\, ds \right) \\
&\cdot \sup_{0\leq s\leq t+1} \|R(s)_X\| \|A_X x_r\|_X \|x^*\|
\end{aligned}$$

for all $x^* \in X^*$, so that

$$\begin{aligned}
&\|R(t+h)_X x_r - R(t)_X x_r\|_X \\
\leq &\left(\int_0^h a(s)ds + \int_0^t |a(t+h-s) - a(t-s)|\, ds \right) \\
&\cdot \| \sup_{0\leq s\leq t+1} \|R(s)_X\| \|A_X x_r\|_X,
\end{aligned}$$

which converges to 0 as $h \to 0$, by Lebesgue's Dominated Convergence Theorem. Hence $R(\cdot)_X x$ is strongly continuous at t.

Finally, to show that $R(\cdot)_X$ is a (a, k)-resolvent family for VE(a, A_X), let $x \in D(A_X)$. Then $x \in D(A) \cap X$ and $Ax \in X$ so that $R(s)_X x = R(s)x \in D(A) \cap X$ and $AR(s)_X x = AR(s)x = R(s)Ax = R(s)A_X x = R(s)_X A_X x \in X$, which means that $R(s)_X x \in D(A_X)$ and $A_X R(s)_X x = R(s)_X A_X x$ for all $x \in D(A_X)$. Moreover, by (R3) we have

$$
\begin{aligned}
X\text{-}\int_0^t a(t-s)A_X R(s)_X x ds &= X\text{-}\int_0^t a(t-s)R(s)_X A_X x ds \\
&= Y\text{-}\int_0^t a(t-s)R(s)Ax ds = R(t)x - k(t)x \\
&= R(t)_X x - k(t)x
\end{aligned}
$$

for $x \in D(A_x)$. Hence $R(\cdot)_X$ is a (a, k)-resolvent family of operators on X for VE(a, A_X). The proof is complete.

Corollary 2.5. *The assertion of Theorem 2.4 still holds if $R(\cdot)$ is replaced with an α-times integrated semigroup $T(\cdot)$ or an α-times integrated cosine function $C(\cdot)$.*

3. Another Proof for the Case of Cosine Operator Functions

Let $C(\cdot) = \{C(t); t \in \mathbb{R}\}$ be a strongly continuous cosine operator function on Y with infinitesimal generator A, and suppose X is invariant under $C(\cdot)$. Then $C(\cdot)_X = \{C(t)|_X; t \in \mathbb{R}\}$ is a cosine function of operators on $(X, \|\cdot\|_X)$. The following theorem is a special case of Corollary 2.5 (except the inclusion of condition (b')). Moreover, the part "(d) $\Rightarrow$ (a)" is to be proved without using the generator A.

Theorem 3.1. *For a strongly continuous cosine operator function $C(\cdot)$ on Y such that X is invariant under $C(\cdot)$, the following conditions are equivalent:*

- (a) *$C(\cdot)_X$ is strongly continuous cosine operator function on X.*
- (b) *For each $x \in X$ and for all $\tau > 0$, $\mathcal{O}_x(\tau) := \{C(t)x; 0 \le t \le \tau\}$ is compact in X.*
- (b') *For each $x \in X$ there exists a $\tau_0 > 0$ such that $\mathcal{O}_x(\tau_0)$ is compact in X.*
- (c) *For each $x \in X$ and for all $\tau > 0$, $\mathcal{O}_x(\tau)$ is relatively X_w-compact (resp. bounded, when X is reflexive).*
- (d) *$C(\cdot)_X$ is weakly continuous on X.*

In this case, the infinitesimal generator of $C(\cdot)_X$ is A_X, which is a densely defined closed operator in X.

Proof. In view of Proposition 2.3, we need to prove "(b') $\Rightarrow$ (b)" and "(d) $\Rightarrow$ (a)".

(b') $\Rightarrow$ (b). First, we note that the continuity of $C(\cdot)x$ implies that $\mathcal{O}_x(\tau)$ is closed in $(Y, \|\cdot\|_Y)$, and hence is closed in $(X, \|\cdot\|_X)$ because $(X, \|\cdot\|_X)$ is continuously embedded in Y.

For $n \geq 2$, we have for all $0 \leq r \leq \tau_0$

$$\begin{aligned} C((n-1)\tau_0 + r)_X x &= 2C((n-1)\tau_0)_X C(r)_X x - C((n-1)\tau_0 - r)_X x \\ &\in 2C((n-1)\tau_0)_X \mathcal{O}_x(\tau_0) - \mathcal{O}_x((n-1)\tau_0). \end{aligned}$$

It follows that

$$\mathcal{O}_x(\tau) \subset \mathcal{O}_x((n-1)\tau_0) \cup [2C((n-1)\tau_0)_X \mathcal{O}_x(\tau_0) - \mathcal{O}_x((n-1)\tau_0)]$$

for all $\tau \in [0, n\tau_0]$. Since $C((n-1)\tau_0)_X$ is a continuous operator on X, if $\mathcal{O}_x(\tau_0)$ and $\mathcal{O}_x((n-1)\tau_0)$ are compact in X, then so is the set on the right hand side of the above inclusion. Thus, as a closed subset of a compact set, $\mathcal{O}_x(\tau)$ is compact in X for all $\tau \in [0, n\tau_0]$. Hence, by induction, one can infer (b) from (b').

(d) $\Rightarrow$ (a). Note that the X_w-continuity of $C(\cdot)_X x$ implies that $\mathcal{O}_x(\tau)$ is X_w-compact, and hence so is its X_w-closed convex hull $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$, by Krein's theorem.

To show that $C(\cdot)_X x$ is continuous in norm $\|\cdot\|_X$ on $[0, \infty)$ for every $x \in X$, we consider the vectors $x_r := \frac{1}{r}\int_0^r C(s)_X x ds$, $r > 0$, where the integrals are defined as Pettis integrals, which exist and lie in $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$ $(\subset X)$ by the X_w-continuity of $C(\cdot)_X x$ and the X_w-compactness of $\overline{\text{co}}^w(\mathcal{O}_x(\tau))$ (cf. [11, Theorem 3.27]). Hence $D := \{x_r; x \in X, r > 0\}$ is a subset of X. The X_w-continuity of $C(\cdot)_X x$ at 0 also shows that $x_r \to x$ weakly as $r \to 0^+$. Hence D is X_w-dense in X and its linear span span(D) is weakly (and strongly) dense in X. For $t \in \mathbb{R}$ and all $x^* \in X^*$, we have

$$\begin{aligned} \langle C(t)_X x_r, x^* \rangle &= \langle x_r, (C(t)_X)^* x^* \rangle = \frac{1}{r}\int_0^r \langle C(s)_X x, (C(t)_X)^* x^* \rangle ds \\ &= \frac{1}{r}\int_0^r \langle C(t)_X C(s)_X x, x^* \rangle ds \\ &= \frac{1}{2r}\int_0^r \langle (C(t+s)_X + C(s-t)_X) x, x^* \rangle ds \\ &= \frac{1}{2r}\left(\int_t^{t+r} + \int_{-t}^{r-t}\right) \langle C(s)_X x, x^* \rangle ds \end{aligned}$$

and hence

$$\begin{aligned} &\langle C(t+h)_X x_r - C(t)_X x_r, x^* \rangle \\ &= \frac{1}{2r}\left(\int_{t+h}^{t+h+r} + \int_{-t-h}^{r-t-h} - \int_t^{t+r} - \int_{-t}^{r-t}\right) \langle C(s)_X x, x^* \rangle ds \\ &= \frac{1}{2r}\left(\int_{t+r}^{t+h+r} - \int_t^{t+h} + \int_{r-t}^{r-t-h} - \int_{-t}^{-t-h}\right) \langle C(s)_X x, x^* \rangle ds \end{aligned}$$

for all $|h| < 1$.

Since the weak continuity of $C(\cdot)_X$ implies it is locally bounded, we have

$$\begin{aligned}&\|C(t+h)_X x_r - C(t)_X x_r\|_X \\ \le\ & \frac{1}{2r} 4h \sup\{\|C(s)_X\|; |s| \le |t| + 1 + r\}\|x\|_X \to 0\end{aligned}$$

as $h \to 0$. Thus $\|C(t+h)_X x - C(t)_X x\|_X \to 0$ as $h \to 0$ for all $x \in \text{span}(D)$. Since $\text{span}(D)$ is strongly dense in X and $C(\cdot)_X$ is locally bounded, $\|C(t+h)_X x - C(t)_X x\|_X \to 0$ holds for all $x \in X$, i.e., $C(t+h)_X \to C(t)_X$ in the strong operator topology as $h \to 0$.

Finally, we show that $C(\cdot)_X$ is generated by A_X. Let B be the infinitesimal generator of $C(\cdot)_X$. Since the $\|\cdot\|_X-$topology of X is stronger than the $\|\cdot\|_Y$-topology of X, clearly $B \subset A_X$. To show the converse, we need only to show $D(A_X) \subset D(B)$. Note that $C(\cdot)$ and $C(\cdot)_X$ are exponentially bounded, so that for sufficiently large $\lambda > 0$ we have

$$\lambda(\lambda^2 - B)^{-1}x = \int_0^\infty e^{-\lambda t} C(t)_X x dt = \int_0^\infty e^{-\lambda t} C(t) x dt = \lambda(\lambda^2 - A)^{-1}x$$

for all $x \in X$, where the first Riemann integral is in the sense of $\|\cdot\|_X$ and the second one is in the sense of $\|\cdot\|_Y$. If $x \in D(A_X)$, then $(\lambda^2 - A)x \in X$, and so

$$x = (\lambda^2 - A)^{-1}(\lambda^2 - A)x = (\lambda^2 - B)^{-1}(\lambda^2 - A)x \in D(B).$$

Hence $B = A_X$ and the proof is complete.

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