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ON A CERTAIN ITERATED MIXED INTEGRAL EQUATION

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Abstract. In this paper we study the stability and asymptotic behavior of solutions of a nonlinear iterated Volterra-Fredholm integral equation by using the well known Krasnoselskii's fixed point theorem.

1. Introduction

Consider the following iterated Volterra-Fredholm integral equation

$$x(t) = h(t) + \sum_{i=1}^{n} A_i x(t) + \sum_{i=1}^{n} B_i x(t) \ , \tag{1.1}$$

where

$$A_i x(t) = \int_0^t \left(\int_0^{t_1} \ldots \left(\int_0^{t_{i-1}} a_i(t, t_1, \ldots, t_i) f_i(t_1, \ldots, t_i, x(t_i)) dt_i \right) \ldots \right) dt_1, \tag{1.2}$$

$$B_i x(t) = \int_0^\infty \left(\int_0^{t_1} \ldots \left(\int_0^{t_{i-1}} b_i(t, t_1, \ldots, t_i) g_i(t_1, \ldots, t_i, x(t_i)) dt_i \right) \ldots \right) dt_1, \tag{1.3}$$

for $i = 1, \ldots, n$, $t \in J = [0, \infty)$ and $h : J \to R^n$, $a_i, b_i : J_{i+1} \to R,$, $f_i, g_i : J_i \times R^n \to R^n$ are continuous functions, in which $J_i = \{(t_1, \ldots, t_i) \in R^i : 0 \le t_i \le \ldots \le t_1 < \infty\}$, $R = (-\infty, \infty)$ and R^n the n-dimensional Euclidean space. Let $|.|$ denote any appropriate norm in R^n and let $BC[0, \infty)$ be the collection of all bounded continuous functions from $[0, \infty)$ into R^n with the sup-norm defined by $\|\phi\| =$

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$\sup_{t \geq 0} |\phi(t)|$ for $\phi \in BC[0,\infty)$. For $\varepsilon > 0$, let $S(\varepsilon) = \{\phi : \phi \in BC[0,\infty),$ $\|\phi\| \leq \varepsilon\}$.

The special versions of the equations of the form (1.1) arise naturally in the study of boundary value problems on the infinite half line. In fact in [3], Gamidov observed that the study of certain boundary value problems of differential equations can be reduced to the study of special version of the equation (1.1). The problems of existence and other properties of solutions of the special versions of equation (1.1) have been studied by many authors by using different techniques, see [1, 2, 5, 6, 7, 9-11] and the references cited therein. In [7] the authors have studied the stability and asymptotic behavior of solutions of equation (1.1) when $n = 1$ and $a_1(t,t_1), b_1(t,t_1)$ are n by n matrices. Motivated by many physical problems arising in wide variety of applications, governed by both ordinary differential and integral equations (see [9, 12], in this paper we study the stability and asymptotic behavior of solutions of more general equation (1.1) by using the fixed point theorem due to Krasnoselskii [5].

2. Statement of Results

In the proofs of our results we make use of the following fixed point theorem of Krasnoselskii [5] which combines both the Contraction mapping principle and the Schauder fixed point theorem (see also [4,8]).

Theorem K. *Let S be a bounded, closed, convex subset of a Banach space and let A and B be operators satisfying:*

(i) *$Ax + By \in S$ whenever $x, y \in S$;*

(ii) *A is a contraction on S;*

(iii) *B is completely continuous on S.*

Then the equation $Ax + Bx = x$ has a solution on S.

For convenience, we list the following hypotheses used in our discussion.

(H_1) For $i = 1, ..., n$

$$\sup_{t \geq 0} \int_0^t \left(\int_0^{t_1} \cdots \left(\int_0^{t_{i-1}} |a_i(t,t_1,...,t_i)|\, dt_i \right) \cdots \right) dt_1 \leq M_i < \infty.$$

(H_2) For $i = 1, ..., n$

$$\sup_{t \geq 0} \int_0^\infty \left(\int_0^{t_1} \cdots \left(\int_0^{t_{i-1}} |b_i(t,t_1,...,t_i)|\, dt_i \right) \cdots \right) dt_1 \leq N_i < \infty.$$

(H_3) For $i = 1, ..., n$

$$f_i(t_1, ..., t_i, 0) \equiv g_i(t_1, ..., t_i, 0) \equiv 0.$$

(H_4) For $i = 1, ..., n$ and each $\gamma_i > 0$, there exists $\delta > 0$ such that

$$|f_i(t_1, ..., t_i, x(t_i)) - f_i(t_1, ..., t_i, y(t_i))| \leq \gamma_i |x(t_i) - y(t_i)|,$$

for all $|x|, |y| \leq \delta$ and uniformly in $t_1, ..., t_i$.

(H_5) For $i = 1, ..., n$ and each $\xi_i > 0$, there exists $\eta > 0$ such that

$$|g_i(t_1, ..., t_i, x(t_i)) - g_i(t_1, ..., t_i, y(t_i))| \leq \xi_i |x(t_i) - y(t_i)|,$$

for all $|x|, |y| \leq \eta$ and uniformly in $t_1, ..., t_i$.

(H_6) For $i = 1, ..., n$ and all $t \in J$,

$$\int_0^\infty \left(\int_0^{t_1} ... \left(\int_0^{t_{i-1}} |b_i(t+k, t_1, ..., t_i) - b_i(t, t_1, ..., t_i)| \, dt_i \right) ... \right) dt_1 \to 0,$$

as $|k| \to 0$.

Our main results are given in the following theorems.

Theorem 1. *Suppose that the hypotheses* $(H_1) - (H_6)$ *are satisfied. Then there exists a number* $\varepsilon_0 > 0$ *such that to any* $\varepsilon \in (0, \varepsilon_0]$ *, there corresponds a* $\delta > 0$ *such that* $\|h\| < \delta$, *there exists a unique solution* $x(t)$ *of equation* (1.1) *on* J *satisfying* $\|x\| \leq \varepsilon$.

Theorem 2. *In addition to the assumptions of Theorem 1, if* $h(t) \to 0$ *and for each* $T > 0$ *and* $i = 1, ..., n$

(H_7)

$$\lim_{t \to \infty} \int_0^T \left(\int_0^{t_1} ... \left(\int_0^{t_{i-1}} |a_i(t, t_1, ..., t_i)| \, dt_i \right) ... \right) dt_1 = 0,$$

and

(H_8)

$$\lim_{t \to \infty} \int_0^T \left(\int_0^{t_1} ... \left(\int_0^{t_{i-1}} |b_i(t, t_1, ..., t_i)| \, dt_i \right) ... \right) dt_1 = 0,$$

then the solution $x(t)$ *of equation* (1.1) *satisfies*

$$\lim_{t\to\infty} x(t) = 0.$$

3. Proofs of Theorems 1 and 2

To prove Theorem 1, fix $\xi_i > 0$ such that $\sum\limits_{i=1}^{n} \xi_i N_i < 1$. By (H_3) and (H_5) we may pick $\eta > 0$ such that for $|x| \le \eta$ and $i = 1, ..., n$ we have $|g_i(t_1, ..., t_i, x(t_i))| \le \xi_i |x(t_i)|$ uniformly in $t_1, ..., t_i$. Let $\gamma_i = \frac{\left(1 - \sum\limits_{j=1}^{n} \xi_j N_j\right)}{2n M_i}$, $i = 1, ..., n$ and choose $\delta > 0$ such that (H_4) holds for all $|x|, |y| \le \delta$ uniformly in $t_1, ..., t_i$. Let $\varepsilon_0 = \min(\eta, \delta)$. Define the operators A and B on $S(\varepsilon)$ as follows :

$$Ax(t) = h(t) + \sum_{i=1}^{n} A_i x(t),$$
$$Bx(t) = \sum_{i=1}^{n} B_i x(t).$$

For $\varepsilon \in (0, \varepsilon_0]$, in the first step we show that there exists $\delta > 0$ such that $Ax + By \in S(\varepsilon)$ for all $x, y \in S(\varepsilon)$ provided that $\|h\| \le \delta$. Note that for all $t \ge 0$ we have

$$\begin{aligned}
&|Ax(t) + By(t)| \\
&\le |h(t)| + \gamma_1 \|x\| \int_0^t |a_1(t, t_1)| dt_1 + \gamma_2 \|x\| \int_0^t \left(\int_0^{t_1} |a_2(t, t_1, t_2)| \, dt_2 \right) dt_1 \\
&\quad + ... + \gamma_n \|x\| \int_0^t \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} |a_n(t, t_1, ..., t_n)| \, dt_n \right) ... \right) dt_1 \\
&\quad + \xi_1 \|y\| \int_0^\infty |b_1(t, t_1)| dt_1 + \xi_2 \|y\| \int_0^\infty \left(\int_0^{t_1} |b_2(t, t_1, t_2)| \, dt_2 \right) dt_1 \\
&\quad + ... + \xi_n \|y\| \int_0^\infty \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} |b_n(t, t_1, ..., t_n)| \, dt_n \right) ... \right) dt_1 \\
&\le \delta + \varepsilon \{\gamma_1 M_1 + \gamma_2 M_2 + ... + \gamma_n M_n\} + \varepsilon \{\xi_1 N_1 + \xi_2 N_2 + ... + \xi_n N_n\} \\
&= \delta + \frac{\varepsilon}{2n} \left\{ 1 - \sum_{j=1}^{n} \xi_j N_j + 1 - \sum_{j=1}^{n} \xi_j N_j + ... + 1 - \sum_{j=1}^{n} \xi_j N_j \right\} + \varepsilon \sum_{j=1}^{n} \xi_j N_j \\
&= \delta + \frac{\varepsilon}{2} \left\{ 1 - \sum_{j=1}^{n} \xi_j N_j \right\} + \varepsilon \sum_{j=1}^{n} \xi_j N_j \le \varepsilon,
\end{aligned}$$

provided $\delta \leq \frac{\varepsilon}{2}\left\{1-\sum\limits_{j=1}^{n}\xi_j N_j\right\}$, which proves that $Ax+By\in S(\varepsilon)$.

In the next step we show that the operator A is a contraction. Let $x,y\in S(\varepsilon)$, then

$$\begin{aligned}
&|Ax(t)-Ay(t)|\\
&\leq \int_0^t |a_1(t,t_1)|\;|f_1(t_1,x(t_1))-f_1(t_1,y(t_1))|dt_1\\
&\quad+\int_0^t\left(\int_0^{t_1}|a_2(t,t_1,t_2)|\;|f_2(t_1,t_2,x(t_2))-f_2(t_1,t_2,y(t_2))|dt_2\right)dt_1\\
&\quad+...+\int_0^t\left(\int_0^{t_1}...\left(\int_0^{t_{n-1}}|a_n(t,t_1,...,t_n)|\,|f_n(t_1,...,t_n,x(t_n))))\right.\right.\\
&\quad\left.\left.-f_n(t_1,...,t_n,y(t_n))|dt_n)...\right)dt_1\\
&\leq \int_0^t |a_1(t,t_1)|\;\gamma_1\;|x(t_1)-y(t_1)|\,dt_1\\
&\quad+\int_0^t\left(\int_0^{t_1}|a_2(t,t_1,t_2)|\;\gamma_2\;|x(t_2)-y(t_2)|dt_2\right)dt_1\\
&\quad+...+\int_0^t\left(\int_0^{t_1}...\left(\int_0^{t_{n-1}}|a_n(t,t_1,...,t_n)|\;\gamma_n\;|x(t_n)-y(t_n)|dt_n\right)...\right)dt_1\\
&\leq(\gamma_1M_1+\gamma_2M_2+...+\gamma_nM_n)\;\|x-y\|\\
&=\frac{1}{2}\left(1-\sum_{j=1}^{n}\xi_jN_j\right)\;\|x-y\|,
\end{aligned}$$

which implies $\|Ax-Ay\|\leq\frac{1}{2}\alpha\|x-y\|$, where $\alpha=1-\sum\limits_{j=1}^{n}\xi_jN_j<1$, i.e. A is a contraction on $S(\varepsilon)$.

The operator B is clearly continuous on $S(\varepsilon)$. Now we shall show that B is completely continuous. Let $\{w_m\}$ be a sequence in $S(\varepsilon)$. From the definition of the operator B and the hypotheses $(H_2),(H_3),(H_5)$ we have

$$|Bw_m(t)|\leq\int_0^\infty|b_1(t,t_1)|\;|g_1(t_1,w_m(t_1))|\,dt_1$$

$$+\int_0^{\infty}\left(\int_0^{t_1}|b_2(t,t_1,t_2)|\,|g_2(t_1,t_2,w_m(t_2))|\,dt_2\right)dt_1$$
$$+\ldots+\int_0^{\infty}\left(\int_0^{t_1}\ldots\left(\int_0^{t_{n-1}}|b_n(t,t_1,\ldots,t_n)|\,|g_n(t_1,\ldots,t_n,w_m(t_n))|\,dt_n\right)\ldots\right)dt_1$$
$$\leq\int_0^{\infty}|b_1(t,t_1)|\,\xi_1\,|w_m(t_1)|\,dt_1+\int_0^{\infty}\left(\int_0^{t_1}|b_2(t,t_1,t_2)|\,\xi_2\,|w_m(t_2)|\,dt_2\right)dt_1$$
$$+\ldots+\int_0^{\infty}\left(\int_0^{t_1}\ldots\left(\int_0^{t_{n-1}}|b_n(t,t_1,\ldots,t_n)|\,\xi_n\,|w_m(t_n)|\,dt_n\right)\ldots\right)dt_1$$
$$\leq\varepsilon\xi_1N_1+\varepsilon\xi_2N_2+\ldots+\varepsilon\xi_nN_n.$$

Hence we have

$$\|Bw_m\|\leq\varepsilon\sum_{j=0}^{n}\xi_jN_j.$$

This means that the sequence $\{Bw_m\}$ is uniformly bounded.

Now we shall show that the sequence $\{Bw_m\}$is equicontinuous. From the definition of operator B and hypotheses (H_3), (H_5), (H_6) we have

$$|Bw_m(t+k)-Bw_m(t)|$$
$$\leq\int_0^{\infty}|b_1(t+k,t_1)-b_1(t,t_1)|\,|g_1(t_1,w_m(t_1))|\,dt_1$$
$$+\int_0^{\infty}\left(\int_0^{t_1}|b_2(t+k,t_1,t_2)-b_2(t,t_1,t_2)|\,|g_2(t_1,t_2,w_m(t_2))|\,dt_2\right)dt_1$$
$$+\ldots+\int_0^{\infty}\left(\int_0^{t_1}\ldots\left(\int_0^{t_{n-1}}|b_n(t+k,t_1,\ldots,t_n)\right.\right.$$
$$\left.\left.-b_n(t,t_1,\ldots,t_n)|\,|g_n(t_1,\ldots,t_n,w_m(t_n))|dt_n\right)\ldots\right)dt_1,$$

which tends to zero as $|k|\to 0$ uniformly, i.e. $\{Bw_m\}$ is equicontinuous (see also [5, p. 19]). Now an application of Theorem K, we conclude that equation (1.1) has a solution in $S(\varepsilon)$.

To prove the uniqueness of solutions of equation (1.1) in $S(\varepsilon)$, let $x(t)$ and $y(t)$ be two solutions of equation (1.1) in $S(\varepsilon)$. and let $z(t)$ be their difference.

Using (H_4) and (H_5) we can estimate as

$$\begin{aligned}|z(t)| \leq & \int_0^t |a_1(t,t_1)|\ \gamma_1\ |z(t_1)|\,dt_1 + \int_0^t \left(\int_0^{t_1} |a_2(t,t_1,t_2)|\ \gamma_2\ |z(t_2)|\,dt_2 \right) dt_1 \\ & + ... + \int_0^t \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} |a_n(t,t_1,...,t_n)|\ \gamma_n\ |z(t_n)|\,dt_n \right) ... \right) dt_1 \\ & + \int_0^\infty |b_1(t,t_1)|\ \xi_1\ |z(t_1)|\,dt_1 + \int_0^\infty \left(\int_0^{t_1} |b_2(t,t_1,t_2)|\ \xi_2\ |z(t_2)|\,dt_2 \right) dt_1 \\ & + ... + \int_0^\infty \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} |b_n(t,t_1,...,t_n)|\ \xi_n\ |z(t_n)|\,dt_n \right) ... \right) dt_1 \\ \leq & \{\gamma_1 M_1 + \gamma_2 M_2 + ... + \gamma_n M_n\} \|z\| + \{\xi_1 N_1 + \xi_2 N_2 + ... + \xi_n N_n\} \|z\| \\ = & \frac{1}{2} \left(1 + \sum_{j=1}^n \xi_j N_j \right) \|z\|.\end{aligned}$$

Taking the supremum over all $t \in J$ in the above estimate we obtain a contradiction

$$\|z\| \leq \frac{1}{2} \left(1 + \sum_{j=1}^n \xi_j N_j \right) \|z\| < \|z\|.$$

Hence $\|z\| = 0$, proving uniqueness of solutions of equation (1.1).

In order to prove Theorem 2, let γ_i be as in the proof of Theorem 1 and assume the contrary. Then

$$\mu = \limsup_{t \to \infty} |x(t)| > 0.$$

Choose T so large that for $t \geq T$ we have $|x(t)| \leq \frac{\mu}{\lambda}$ for fixed λ ,$\frac{1}{2}\left(1 + \sum_{j=1}^n \xi_j N_j\right)$ $< \lambda < 1$. Using these facts we can now estimate the solution $x(t)$ of equation (1.1) as

$$\begin{aligned}|x(t)| \leq & |h(t)| + \varepsilon\gamma_1 \int_0^T |a_1(t,t_1)|\,dt_1 + \varepsilon\gamma_2 \int_0^T \left(\int_0^{t_1} |a_2(t,t_1,t_2)|\,dt_2 \right) dt_1 \\ & + ... + \varepsilon\gamma_n \int_0^T \left(\int_0^{t_1} ... \left(\int_0^{t_{n-1}} |a_n(t,t_1,...,t_n)|\,dt_n \right) ... \right) dt_1\end{aligned}$$

$$+\varepsilon\xi_1\int_0^T |b_1(t,t_1)|\,dt_1+\varepsilon\xi_2\int_0^T\left(\int_0^{t_1}|b_2(t,t_1,t_2)|\,dt_2\right)dt_1$$
$$+\ldots+\varepsilon\xi_n\int_0^T\left(\int_0^{t_1}\ldots\left(\int_0^{t_{n-1}}|b_n(t,t_1,\ldots,t_n)|\,dt_n\right)\ldots\right)dt_1$$
$$+\frac{\mu}{\lambda}\{\gamma_1M_1+\gamma_2M_2+\ldots+\gamma_nM_n\}+\frac{\mu}{\lambda}\{\xi_1N_1+\xi_2N_2+\ldots+\xi_nN_n\}.$$

Taking the lim sup in the above estimate we obtain

$$\mu\le\frac{\mu}{\lambda}\,\frac{1}{2}\left(1+\sum_{j=1}^n\xi_jN_j\right)<\mu,$$

which is the desired contradiction. Hence $\lim_{t\to\infty} x(t)=0$ and the proof is complete.

It is interesting to note that, Theorem 1 may be regarded as a stability result for the solutions of equation (1.1) in the sence : For every sufficiently small $\varepsilon>0$, there exists a $\delta>0$ such that for every $h\in S(\varepsilon)$ with $\|h\|<\delta$, the solution $x(t)$ of equation (1.1) is in $S(\varepsilon)$ i.e. $\|x\|\le\varepsilon$. We also note that Theorem 2 is a type of asymptotic stability theorem for the solutions of equation (1.1).

In many situations Volterra integral equations occur as integrodifferential equations of the form (see [2, 5, 6])

$$x'(t)=f_1(t,x(t))+\int_0^t a(t,t_2)\,f_2(t,t_2,x(t_2))\,dt_2, \tag{3.1}$$

for $t\in[0,\infty)$, where a,f_1,f_2 are given real functions. By taking $t=t_1$ in (3.1) and integrating it from 0 to $t\in[0,\infty)$ we get the integral equation

$$\begin{aligned}x(t)&=x(0)+\int_0^t f_1(t_1,x(t_1))dt_1\\&\quad+\int_0^t\left(\int_0^{t_1}a(t_1,t_2)f_2(t_1,t_2,x(t_2))dt_2\right)dt_1.\end{aligned}\tag{3.2}$$

Comparing (3.2) with (1.1) when $n=2$, we have $h(t)=x(0)$, $a_1(t,t_1)=1$, $a_2(t,t_1,t_2)=a(t_1,t_2)$, $b_1(t,t_1)=0$, $b_2(t,t_1,t_2)=0$. It is easy to observe that the hypotheses $(H_1),(H_2)$ and (H_6) are satisfied.

In conclusion, we note that Krasnoselskii's fixed point theorem [5] and its variants given in [8] are very useful in establishing existence theorems for perturbed operator equations. For some other applications, see [8,10].

References

1. S. Asirov and Ja. D. Mamedov, Investigation of solutions of nonlinear Volterra-Fredholm operator equations, *Dokal. Akad. Nauk SSSR*, **229** (1976), 982-986.
2. C. Corduneanu, *Integral equations and stability of feedback systems*, Academic Press, New York, 1973.
3. Sh. G. Gamidov, Integral inequalities for boundary value problems for differential equations, *Diff. Eqs.*, **5** (1969), 463-475 (English translation).
4. M. A. Krasnoselskii, Two remarks on the method of successive approximations, *Usephi Mat. Nauk.*, **10** (1955), 123-127.
5. M. A. Krasnoselskii, *Topological methods in the theory of nonlinear integral equations*, Macmillan, New York, 1964.
6. R. K. Miller, *Nonlinear Volterra integral equations*, W.A.Benjamin, Menlo Park, Calfornia, 1971.
7. R. K. Miller, J. A. Nohel and J. S. W. Wong, A stability theorem for nonlinear mixed integral equations, *J. Math. Anal. Appl.*, **25** (1969), 446-449.
8. M. Z. Nashed and J. S. W. Wong, Some variants of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations, *J. Math. Mech.*, **18** (1969), 767-777.
9. B. G. Pachpatte, *Integral and Finite Difference Inequalities and Applications*, Elsevier, North-Holland Mathematics Studies, Vol. 205, 2006.
10. B. G. Pachpatte, Fixed point method for studying the perturbed differential equations, *J. Indian Math. Soc.*, **37** (1973), 315-321.
11. B. G. Pachpatte, On the existence and uniqueness of solutions of Volterra-Fredholm integral equations, *Math. Seminar Notes, Kobe Univ. Japan*, **10** (1982), 733-742.
12. B. G. Pachpatte, Global existence of solutions of certain higher order differential equations, *Taiwanese Jour. Math.*, **1(2)** (1997), 135-142.

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