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COMPACT EMBEDDINGS OF THE SPACES $A^p_{w,\omega}\left(R^d\right)$

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Abstract. For $1 \leq p \leq \infty$, $A^p_{w,\omega}\left(R^d\right)$ denotes the space (Banach space) of all functions in $L^1_w\left(R^d\right)$ a weighted L^1-space (Beurling algebra) with Fourier transforms $\hat{f}$ in $L^p_\omega\left(R^d\right)$ which is equipped with the sum norm

$$\|f\|^p_{w,\omega} = \|f\|_{1,w} + \left\|\hat{f}\right\|_{p,\omega},$$

where w and ω are Beurling weights on R^d.This space was defined in [5] and generalized in [6].

The present paper is a sequal to these works.In this paper we are going to discuss compact embeddings between the spaces $A^p_{w,\omega}\left(R^d\right)$.

0.1. NOTATION

In this paper we will work on R^d with Lebesgue measure dx. We denote by $C_c\left(R^d\right)$ the spaces of complex-valued, continuous functions with compact support. Also the translation and modulation operators L_y, M_t are given by $L_y f(x) = f(x-y)$ and $M_t f(x) = e^{2\pi i t x} f(x)$ for all $x, y, t \in R^d$. In this paper we will also use the Beurling's weight functions, i.e real valued, measurable and locally bounded functions w on R^d which satisfy

$$w(x) \geq 1, w(x+y) \leq w(x).w(y) \text{ for all } x, y \in R^d.$$

For $1 \leq p < \infty$, we set

$$L^p_w\left(R^d\right) = \left\{f : fw \in L^p\left(R^d\right)\right\}.$$

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It is known that $L_w^p\left(R^d\right)$ is a Banach space under the norm $\|f\|_{p,w} = \|fw\|_p$. Particularly $\left(L_w^1\left(R^d\right), \|f\|_{1,w}\right)$ is a Banach convolution algebra.It is called as Beurling algebra.For two weight functions w_1 and w_2 we write $w_1 \precsim w_2$ if there exists $C > 0$ such that $w_1(x) \leq Cw_2(x)$ for all $x \in R^d$. We write $w_1 \approx w_2$ if and only if $w_1 \precsim w_2$ and $w_2 \precsim w_1$.The main tool in this work is the Fourier transform denoted by the symbol $(\overset{\wedge}{.})$. One can find more informations about these notations in $[11, 12]$.

We will denote by $A_p\left(R^d\right)$ the vector spaces of all functions in $L^1\left(R^d\right)$ whose Fourier transforms $\hat{f}$ belong to $L^p\left(R^d\right)$. $A_p\left(R^d\right)$ is a Banach convolution algebra with the norm

$$\|f\|^p = \|f\|_1 + \left\|\hat{f}\right\|_p.$$

Research on $A^p\left(R^d\right)$ was initiated by Larsen, Liu and Wang [10] and a number of authors such as Martin and Yap $[13]$, Gürkanlı [8] worked on these spaces.Later some generalization to the weighted case was given by Feichtinger and Gürkanlı $[5]$, Fisher, Gürkanlı and Liu $[6]$.

2. Main Results

Definition 1. Let w, ω be Beurling weights on R^d.For $1 \leq p \leq \infty$, we set

$$A_{w,\omega}^p\left(R^d\right) = \left\{f \in L_w^1\left(R^d\right) : \hat{f} \in L_\omega^p\left(R^d\right)\right\}$$

and equip it with the norm

$$\|f\|_{w,\omega}^p = \|f\|_{1,w} + \left\|\hat{f}\right\|_{p,\omega}.$$

This space is a Banach space under this norm see $[5]$, $[6]$. In the mentioned papers some properties of this space has been discussed.

Lemma 2. *Let $(f_n)_{n\in N}$ be a sequence in $A_{w,\omega}^p\left(R^d\right)$. If $(f_n)_{n\in N}$ converges to zero in $A_{w,\omega}^p\left(R^d\right)$, then $(f_n)_{n\in N}$ also converges to zero in the vague topology(which means that*

$$\int_{R^d} f_n(x)\, k(x)\, dx \to 0,$$

for $n \to \infty$ for all $k \in C_c\left(R^d\right)$. See [2]*).*

Proof. Let $k \in C_c\left(R^d\right)$. We write

$$\left|\int_{R^d} f_n(x)\, k(x)\, dx\right| \le \|k\|_\infty \|f_n\|_{L^1} \le \|k\|_\infty \|f_n\|^p_{w,\omega}. \tag{1}$$

Hence by (1) the sequence $(f_n)_{n\in N}$ converges to zero in vague topology. ∎

Theorem 3. *Let w, ω, v be Beurling weight functions on R^d.If $v \preccurlyeq w$ and $\frac{v(x)}{w(x)}$ doesn't tend to zero in R^d as $x \to \infty$ then the embedding of the space $A^p_{w,\omega}\left(R^d\right)$ into $L^1_v\left(R^d\right)$ is never compact.*

Proof. Firstly we assume that $w(x) \to \infty$ as $x \to \infty$. Since $v \preccurlyeq w$, there exists $C_1 > 0$ such that $v(x) \le C_1 w(x)$.This implies $A^p_{w,\omega}\left(R^d\right) \subset L^1_v\left(R^d\right)$. Let $(t_n)_{n\in N}$ be a sequence with $t_n \to \infty$ in R^d. Also since $\frac{v(x)}{w(x)}$ doesn't tend to zero as $x \to \infty$ then there exists $\delta > 0$ such that $\frac{v(x)}{w(x)} \ge \delta > 0$ for $x \to \infty$. For the proof of the embedding of the space $A^p_{w,\omega}\left(R^d\right)$ into $L^1_v\left(R^d\right)$ is never compact, for any fixed $f \in A^p_{w,\omega}\left(R^d\right)$ define a sequence of functions $(f_n)_{n\in N}$, where $f_n = w(t_n)^{-1} L_{t_n} f$. This sequence is bounded in $A^p_{w,\omega}\left(R^d\right)$. Indeed we write

$$\|f_n\|^p_{w,\omega} = \left\|w(t_n)^{-1} L_{t_n} f\right\|^p_{w,\omega} = w(t_n)^{-1} \|L_{t_n} f\|^p_{w,\omega}. \tag{2}$$

By Theorem 1.9 in [6] ,we know $\|L_x f\|^p_{w,\omega} \approx w(x)$. Hence there exists $M > 0$ such that $\|L_x f\|^p_{w,\omega} \le M.w(x)$. By using (2) we write

$$\|f_n\|^p_{w,\omega} = w(t_n)^{-1} \|L_{t_n} f\|^p_{w,\omega} \le M.w(t_n)\, w(t_n)^{-1} = M.$$

Now we will prove that there wouldn't exist norm convergence of subsequence of $(f_n)_{n\in N}$ in $L^1_v\left(R^d\right)$. The sequence obtained above certainly converges to zero in the vague topology .Indeed for all $k \in C_c\left(R^d\right)$ we write

$$\begin{aligned} \left|\int_{R^d} f_n(x)\, k(x)\, dx\right| &\le \frac{1}{w(t_n)} \int_{R^d} |L_{t_n} f(x)|\, |k(x)|\, dx \\ &= \frac{1}{w(t_n)} \|k\|_\infty \|L_{t_n} f\|_{L^1} = \frac{1}{w(t_n)} . \|k\|_\infty \|f\|_{L^1}. \end{aligned} \tag{3}$$

Since right hand side of (3) tends zero for $n \to \infty$ then we have

$$\int_{R^d} f_n(x)\, k(x)\, dx \to 0.$$

Finally by Lemma 2 the only possible limit of (f_n) in $L^1_v\left(R^d\right)$ is zero. It is known by Lemma2.2 in [5] that $\|L_x f\|_{L^1_v} \approx v(x)$. Hence there exists $C_2 > 0$ and $C_3 > 0$ such that

$$C_2 v(x) \le \|L_x f\|_{1,v} \le C_3 v(x). \tag{4}$$

From (4) and the equality below

$$\|f_n\|_{1,v} = \left\| w(t_n)^{-1} L_{t_n} f \right\|_{1,v} = w(t_n)^{-1} \|L_{t_n} f\|_{1,v} \tag{5}$$

we obtain

$$\|f_n\|_{1,v} = w(t_n)^{-1} \|L_{t_n} f\|_{1,v} \geq C_2 w(t_n)^{-1} v(t_n). \tag{6}$$

Since $\frac{v(t_n)}{w(t_n)} \geq \delta > 0$ for all t_n, by using (6) we write

$$\|f_n\|_{1,v} \geq C_2 w(t_n)^{-1} v(t_n) \geq C_2 \delta.$$

It means that there would not be possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $L_v^1(R^d)$.

Now we assume that w is a constant or a bounded weight function.Since $v \preccurlyeq w$ then $\frac{v(x)}{w(x)}$ is also constant or bounded and doesn't tend to zero as $x \to \infty$. We take a function $f \in A_{w,\omega}^p(R^d)$ with compactly support and define the sequence $(f_n)_{n\in N}$ as in (2).Thus $(f_n)_{n\in N} \subset A_{w,\omega}^p(R^d)$. It is easy to show as in (2) that $(f_n)_{n\in N}$ is bounded in $A_{w,\omega}^p(R^d)$ and converges to zero in the vague topology.Then there would not be possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $L_v^1(R^d)$. This implies that the embedding $A_{w,\omega}^p(R^d) \hookrightarrow L_v^1(R^d)$ is never compact. ■

Theorem 4. *Let* w_1, w_2 *and* ω_1, ω_2 *be Beurling weight functions on* R^d*. If* $w_2 \preccurlyeq w_1$*,* $\omega_2 \preccurlyeq \omega_1$ *and* $\frac{w_2(x)}{w_1(x)}$ *or* $\frac{\omega_2(x)}{\omega_1(x)}$ *don't tend to zero in* R^d *then the embedding* $i : A_{w_1,\omega_1}^p(R^d) \hookrightarrow A_{w_2,\omega_2}^p(R^d)$ *is never compact.*

Proof. Firstly we assume that $w_1(x) \to \infty, \omega_1(x) \to \infty$ as $x \to \infty$. Since $w_2 \preccurlyeq w_1$ and $\omega_2 \preccurlyeq \omega_1$ then $A_{w_1,\omega_1}^p(R^d) \subset A_{w_2,\omega_2}^p(R^d)$ by Theorem 1.19 in [6]. It is also known by Lemma 1.18 in [6] that the unit map from $A_{w_1,\omega_1}^p(R^d)$ into $A_{w_2,\omega_2}^p(R^d)$ is continuous. Assume that $\frac{w_2(x)}{w_1(x)}$ doesn't tend to zero. We are going to show that the unit map from $A_{w_1,\omega_1}^p(R^d)$ into $A_{w_2,\omega_2}^p(R^d)$ is never compact.Take any bounded sequence $(f_n)_{n\in N}$ in $A_{w_1,\omega_1}^p(R^d)$. If there exists norm convergent subsequence of $(f_n)_{n\in N}$ in $A_{w_2,\omega_2}^p(R^d)$,this subsequence also converges in $L_{w_2}^1(R^d)$. But this is a not possible by Theorem 3 because the embedding of the space $A_{w_1,\omega_1}^p(R^d)$ into $L_{w_2}^1(R^d)$ is never compact. Now assume that $\frac{\omega_2(x)}{\omega_1(x)}$ doesn't tend to zero. Let $(t_n)_{n\in N}$ be a sequence with $t_n \to \infty$ in R^d.For any fixed $f \in A_{w_1,\omega_1}^p(R^d)$ define a sequence of functions $(f_n)_{n\in N}$, where $f_n = \omega_1(t_n)^{-1} M_{t_n} f$. This sequence is bounded in $A_{w,\omega}^p(R^d)$. Indeed we write

$$\|f_n\|_{w_1,\omega_1}^p = \left\| \omega_1(t_n)^{-1} M_{t_n} f \right\|_{w_1,\omega_1}^p = \omega_1(t_n)^{-1} \|M_{t_n} f\|_{w_1,\omega_1}^p. \tag{7}$$

Since by Theorem 1.9 in [6], $\|M_{t_n}f\|^p_{w_1,\omega_1} \approx \omega_1(t_n)$ hence there exists $C > 0$ such that $\|M_{t_n}f\|^p_{w_1,\omega_1} \leq C.\omega_1(t_n)$. Then we write

$$\begin{aligned} \|f_n\|^p_{w_1,\omega_1} &= \left\|\omega_1(t_n)^{-1} M_{t_n}f\right\|^p_{w_1,\omega_1} = \omega_1(t_n)^{-1}\|M_{t_n}f\|^p_{w_1,\omega_1} \\ &\leq C.\omega_1(t_n)\,\omega_1(t_n)^{-1} = C. \end{aligned}$$

Now we will prove that there wouldn't exist norm convergent subsequence of $(f_n)_{n\in N}$ in $A^p_{w_2,\omega_2}(R^d)$. Above sequence certainly converges to zero in the vague topology. Indeed for all $k \in C_c(R^d)$ we write

$$\begin{aligned} \left|\textstyle\int_{R^d} f_n(x)\,k(x)\,dx\right| &\leq \frac{1}{\omega_1(t_n)}\int_{R^d}|M_{t_n}f|\,|k(x)|\,dx \leq \frac{\|k\|_\infty}{\omega_1(t_n)}.\|f\|_{L^1} \\ &\leq \frac{\|k\|_\infty}{\omega_1(t_n)}.\|f\|^p_{w_2,\omega_2}. \end{aligned} \tag{8}$$

Since right hand side of (8) tends zero for $n \to \infty$, then we have

$$\int_{R^d} f_n(x)\,k(x)\,dx \to 0.$$

Finally the only possible limit in $A^p_{w_2,\omega_2}(R^d)$ is zero. It is known by Theorem 1.19 in [6] that $\|M_x f\|^p_{w_2,\omega_2} \approx \omega_2(x)$. Hence there exists $C_1 > 0$ and $C_2 > 0$ such that

$$C_1\omega_2(x) \leq \|M_x f\|^p_{w_2,\omega_2} \leq C_2\omega_2(x). \tag{9}$$

From (9) and the inequality

$$\|f_n\|^p_{w_2,\omega_2} = \left\|\omega_1(t_n)^{-1}M_{t_n}f\right\|^p_{w_2,\omega_2} = \omega_1(t_n)^{-1}\|M_{t_n}f\|^p_{w_2,\omega_2} \tag{10}$$

we obtain

$$\|f_n\|^p_{w_2,\omega_2} = \omega_1(t_n)^{-1}\|M_{t_n}f\|^p_{w_2,\omega_2} \geq C_1\omega_1(t_n)^{-1}\omega_2(t_n). \tag{11}$$

Since $\frac{\omega_2(x)}{\omega_1(x)}$ doesn't tend to zero then there exists $\delta > 0$ such that $\frac{\omega_2(x)}{\omega_1(x)} \geq \delta > 0$. Thus we write

$$\|f_n\|^p_{w_2,\omega_2} = \omega_1(t_n)^{-1}\|M_{t_n}f\|^p_{w_2,\omega_2} \geq C_1\omega_1(t_n)^{-1}\omega_2(t_n) \geq C_1\delta. \tag{12}$$

That means there would not be possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $A^p_{w_2,\omega_2}(R^d)$ This completes the proof. ∎

Now we assume that $w_1(x)$ or $\omega_1(x)$ is constant or bounded. Since $w_2 \preccurlyeq w_1$, $\omega_2 \preccurlyeq \omega_1$ then $\frac{w_2(x)}{w_1(x)}$ or $\frac{\omega_2(x)}{\omega_1(x)}$ is constant or bounded and hence $\frac{w_2(x)}{w_1(x)}$ or $\frac{\omega_2(x)}{\omega_1(x)}$

don't tend to zero in R^d. Let $w_1(x)$ is constant or bounded. Take a fixed function $f \in A^p_{w_1,\omega_1}(R^d)$ with compactly support and define the sequence $(f_n)_{n\in N}$ as in Theorem 3. Thus $(f_n)_{n\in N} \subset A^p_{w_1,\omega_1}(R^d)$. It is easy to show that $(f_n)_{n\in N}$ is bounded in $A^p_{w_1,\omega_1}(R^d)$ and converges to zero in the vague topology. Then there would not be possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $L^1_{w_2}(R^d)$ by Theorem 3. Then there would not be possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $A^p_{w_2,\omega_2}(R^d)$. This implies that the unite map $i: A^p_{w_1,\omega_1}(R^d) \to A^p_{w_2,\omega_2}(R^d)$ is never compact. Now let $\omega_1(x)$ be constant or bounded. Again take a fixed function $f \in A^p_{w_1,\omega_1}(R^d)$ with compactly support and define the sequence $(f_n)_{n\in N} \subset A^p_{w_1,\omega_1}(R^d)$ as in (7). The sequence $(f_n)_{n\in N}$ is bounded in $A^p_{w_1,\omega_1}(R^d)$ and converges to zero in the vague topology. But it is not possible to find norm convergent subsequence of $(f_n)_{n\in N}$ in $A^p_{w_2,\omega_2}(R^d)$ from (11) and (12). Hence the unite map $i: A^p_{w_1,\omega_1}(R^d) \to A^p_{w_2,\omega_2}(R^d)$ is never compact. This completes the proof. ■

Definition 5. Let w, ω be Beurling weights on R^d. For $1 \leq p \leq \infty$, we set

$$\Lambda^p_{w,\omega}\left(R^d\right) = \left\{ f \in L^1_w(R^n) : \hat{f} \in L^1_\omega(R^n) \cap L^p_\omega\left(R^d\right) \right\}$$

and equip it with the norm

$$\|f\|_{\Lambda^p_{w,\omega}(R^d)} = \|f\|_{1,w} + \left\|\hat{f}\right\|_{1,\omega} + \left\|\hat{f}\right\|_{p,\omega}.$$

It is easy to prove that $\Lambda^p_{w,\omega}(R^d)$ is a Banach space under this norm. It is a subspace of $A^p_{w,\omega}(R^d)$.

Lemma 6. *Let $w_1, w_2, \omega_1, \omega_2$ be Beurling's weight functions on R^d. Then the embedding $i: \Lambda^p_{w_1,\omega_1}(R^d) \hookrightarrow \Lambda^p_{w_2,\omega_2}(R^d)$ is continuous if and only if $w_2 \preccurlyeq w_1$, $\omega_2 \preccurlyeq \omega_1$.*

Proof. Assume that $w_2 \preccurlyeq w_1$ and $\omega_2 \preccurlyeq \omega_1$. Then it is obvious that $L^1_{w_1}(R^d) \hookrightarrow L^1_{w_2}(R^d)$. Also it is known by Theorem 3.3 in [5] that $A^1_{w_1,\omega_1}(R^d) \hookrightarrow A^1_{w_2,\omega_2}(R^d)$. Hence $\Lambda^p_{w_1,\omega_1}(R^d) \hookrightarrow \Lambda^p_{w_2,\omega_2}(R^d)$.

For the converse implication assume the embedding $\Lambda^p_{w_1,\omega_1}(R^d) \hookrightarrow \Lambda^p_{w_2,\omega_2}(R^d)$. One can find $C > 0$ such that

$$\|f\|_{\Lambda^p_{w_2,\omega_2}(R^d)} \leq C \|f\|_{\Lambda^p_{w_1,\omega_1}(R^d)} \tag{13}$$

for all $f \in \Lambda^p_{w_2,\omega_2}(R^d)$. By using Lemma 2.2, Lemma 2.3 and Theorem 2.4 in [5] one can prove that the functions $x \to \|L_x f\|_{\Lambda^p_{w,\omega}(R^d)}$ and $y \to \|M_y f\|_{\Lambda^p_{w,\omega}(R^d)}$ are equivalent to weight functions $w\left(x\right)$ and $\omega\left(y\right)$ respectively. Hence from the inequality (13) we prove that $w_2 \preccurlyeq w_1$ and $\omega_2 \preccurlyeq \omega_1$. ■

Theorem 7. *Let w_1, w_2 and ω_1, ω_2 be Beurling weight functions on R^d. Assume that ω_1, ω_2 symmetric (i.e $\omega_1\left(x\right) = \omega_1\left(-x\right)$ and $\omega_2\left(x\right) = \omega_2\left(-x\right)$ for all $x \in R^d$) and $w_2 \preccurlyeq w_1$, $\omega_2 \preccurlyeq \omega_1$.Then the embedding*

$$i : \Lambda^p_{w_1,\omega_1}\left(R^d\right) \hookrightarrow \Lambda^p_{w_2,\omega_2}\left(R^d\right)$$

is compact if and only if $\frac{w_2(x)}{w_1(x)}$ and $\frac{\omega_2(x)}{\omega_1(x)}$ tend to zero.

Proof. Assume that $\frac{w_2(x)}{w_1(x)}$ and $\frac{\omega_2(x)}{\omega_1(x)}$ tend to zero. We will prove that a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in $\Lambda^p_{w_1,\omega_1}\left(R^d\right)$ has a convergent subsequence in $\Lambda^p_{w_2,\omega_2}\left(R^d\right)$. Since $\{f_n\}_{n=1}^{\infty}$ is bounded in $\Lambda^p_{w_1,\omega_1}\left(R^d\right)$ then there exists $C > 0$ such that $\|f_n\|_{\Lambda^p_{w_1,\omega_1}(R^d)} \le C$ for all $n \in N$. Also by Lemma 6, the embedding $i : \Lambda^p_{w_1,\omega_1}\left(R^d\right) \hookrightarrow \Lambda^1_{w_2,\omega_2}\left(R^d\right)$ is continuous. Hence there exists $C_1 > 0$ such that

$$\|f_n\|_{\Lambda^p_{w_2,\omega_2}(R^d)} \le C_1 \|f_n\|_{\Lambda^p_{w_1,\omega_1}(R^d)} \tag{14}$$

for all $n \in N$. From the hypothesis there are sequences of increasing balls U^1_k and U^2_k, $(k = 1, 2, ...)$ centered at origin with radius tending to $+\infty$ as $k \to \infty$ such that

$$\frac{\omega_2\left(x\right)}{\omega_1\left(x\right)} \le \frac{1}{k} \tag{15}$$

for $x \in R^d/U^1_k$ and

$$\frac{w_2\left(x\right)}{w_1\left(x\right)} \le \frac{1}{k} \tag{16}$$

for $x \in R^d/U^2_k$. We let $U^1_k \cup U^2_k = B_k$. Thus

$$\frac{\omega_2\left(x\right)}{\omega_1\left(x\right)} \le \frac{1}{k}, \quad \frac{w_2\left(x\right)}{w_1\left(x\right)} \le \frac{1}{k} \tag{17}$$

for $x \in R^d/B_k$. Now let $\{t_n\}_{n=1}^{\infty}$ be any sequence which is dense in B_1. By using (14) we write

$$\left\|\hat{f_n}\right\|_{\infty} \le \|f_n\|_{1,\omega_2} \le \|f_n\|_{\Lambda^p_{w_2,\omega_2}(R^d)} \le C.C_1 = C_0, \tag{18}$$

for all $n \in N$. Hence there exists a subsequence $\{f_{n_i}\}_{i=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that the sequence $\left\{\hat{f}_{n_i}(t_1)\right\}_{i=1}^{\infty}$ converges in the complex plane. By extracting a subsequence from $\{f_{n_i}\}_{i=1}^{\infty}$ we find a subsequence $\left\{f_{n_{i_j}}\right\}_{j=1}^{\infty}$ such that $\left\{\hat{f}_{n_{i_j}}(t_2)\right\}_{j=1}^{\infty}$ converges. By this process and choosing a suitable diagonal sequence we can find a subsequence $\{g_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\left\{\hat{g}_n\right\}_{n=1}^{\infty}$ converges on whole B_1. By extracting a subsequence from $\{g_n\}_{n=1}^{\infty}$ we find a subsequence $\{u_n\}_{n=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$such that $\left\{\hat{u}_n\right\}_{n=1}^{\infty}$ converges on whole B_2. Repeating this process we obtain a subsequence $\{h_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that $\left\{\hat{h}_n\right\}_{n=1}^{\infty}$ converges on all B_k and hence on R^d. Also by (14) and (18) we have $\hat{h}_n \in L^1_{\omega_2}\left(R^d\right)$ and

$$\begin{aligned}\|h_n\|_{\infty} &= \left\|\tilde{\hat{\hat{h}}}_n\right\|_{\infty} \leq \left\|\tilde{\hat{h}}_n\right\|_1 = \left\|\hat{h}_n\right\|_1 \leq \left\|\hat{h}_n\right\|_{1,\omega_2} \\ &\leq \|h_n\|_{\Lambda^p_{w_2,\omega_2}(R^d)} \leq C_1 \|h_n\|_{\Lambda^p_{w_1,\omega_1}(R^d)} \leq C_1 C = C_0\end{aligned} \tag{19}$$

for all $n \in N$. That means $\{h_n\}_{n=1}^{\infty}$ is bounded. Again as in the proof of first part we obtain a subsequence $\{s_n\}_{n=1}^{\infty}$ of $\{h_n\}_{n=1}^{\infty}$ such that this sequence converges on all balls B_k. To complete the proof it is enough to show that $\{s_n\}_{n=1}^{\infty}$ is a Cauhy sequence in $\Lambda^p_{w_2,\omega_2}\left(R^d\right)$. From (14) we write

$$\begin{aligned}&\|s_n - s_m\|_{\Lambda^p_{w_2,\omega_2}(R^d)} \\ &= \|s_n - s_m\|_{1,w_2} + \left\|\hat{s}_n - \hat{s}_m\right\|_{1,\omega_2} + \left\|\hat{s}_n - \hat{s}_m\right\|_{p,\omega_2} \\ &= \|s_n - s_m \mid B_k\|_{1,w_2} + \left\|s_n - s_m \mid R^d - B_k\right\|_{1,w_2} \\ &\quad + \left\|\hat{s}_n - \hat{s}_m \mid B_k\right\|_{1,\omega_2} + \left\|\hat{s}_n - \hat{s}_m \mid R^d - B_k\right\|_{1,\omega_2} \\ &\quad + \left\|\hat{s}_n - \hat{s}_m \mid B_k\right\|_{p,\omega_2} + \left\|\hat{s}_n - \hat{s}_m \mid R^d - B_k\right\|_{p,\omega_2} \\ &\leq \|s_n - s_m \mid B_k\|_{1,w_2} + \frac{1}{k}\left\|s_n - s_m \mid R^d - B_k\right\|_{1,w_1} \\ &\quad + \left\|\hat{s}_n - \hat{s}_m \mid B_k\right\|_{1,\omega_2} + \frac{1}{k}\left\|\hat{s}_n - \hat{s}_m \mid R^d - B_k\right\|_{1,\omega_1}\end{aligned} \tag{20}$$

$$
\begin{aligned}
&+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{p,\omega_2}+\frac{1}{k}\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid R^d-B_k\right\|_{p,\omega_1}\\
&\le \|s_n-s_m\mid B_k\|_{1,w_2}+\frac{2C}{k}+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{1,\omega_2}+\frac{2C}{k}\\
&+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{p,\omega_2}+\frac{2C}{k}\\
&=\|s_n-s_m\mid B_k\|_{1,w_2}+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{1,\omega_2}\\
&+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{p,\omega_2}+\frac{6C}{k}.
\end{aligned}
$$

Let $\varepsilon>0$ be given. We can choose k large enough such that $\frac{6C}{k}<\frac{\varepsilon}{4}$. Since the sequences $\{s_n\}_{n=1}^{\infty}$ and $\left\{\overset{\wedge}{s_n}\right\}_{n=1}^{\infty}$ converge on the compact set $\bar{B}_k$, then by Lebesgue's convergence theorem there exists $n_0\in N$ such that

$$
\begin{aligned}
&\|s_n-s_m\mid B_k\|_{1,w_2}<\frac{\varepsilon}{4},\ \left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{1,\omega_2}<\frac{\varepsilon}{4}\ \text{ and}\\
&\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{p,\omega_2}<\frac{\varepsilon}{4}
\end{aligned}
\tag{21}
$$

for all $m,n\ge n_0$, where $\bar{B}_k$ is the closure of B_k. Finally from (20) and (21) we have

$$
\begin{aligned}
\|s_n-s_m\|_{\Lambda^p_{w_2,\omega_2}(R^d)}\le{}&\|s_n-s_m\mid B_k\|_{1,w_2}+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{1,\omega_2}\\
&+\left\|\overset{\wedge}{s_n}-\overset{\wedge}{s_m}\mid B_k\right\|_{p,\omega_2}+\frac{6C}{k}<\varepsilon
\end{aligned}
\tag{22}
$$

for all $m,n\ge n_0$. Hence $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Lambda^p_{w_2,\omega_2}\left(R^d\right)$.

Conversely assume that $\frac{w_2(x)}{w_1(x)}$ or $\frac{\omega_2(x)}{\omega_1(x)}$ don't tend to zero. If the embedding $\Lambda^p_{w_1,\,\omega_1}\left(R^d\right)\hookrightarrow\Lambda^p_{w_2,\,\omega_2}\left(R^d\right)$ is compact then every bounded sequence $\{f_n\}_{n=1}^{\infty}\subset\Lambda^p_{w_1,\,\omega_1}\left(R^d\right)$ has a convergent subsequence $\{f_{n_k}\}_{n=1}^{\infty}$ in $\Lambda^p_{w_2,\,\omega_2}\left(R^d\right)$. Since $\Lambda^p_{w_1,\,\omega_1}\left(R^d\right)\subset A^p_{w_1,\,\omega_1}\left(R^d\right)$, then the norm $\|f\|_{\Lambda^p_{w_1,\,\omega_1}(R^d)}$ in $\Lambda^p_{w_1,\,\omega_1}\left(R^d\right)$ is finer than the norm $\|f\|^p_{w_1,\,\omega_1}$ in $A^p_{w_1,\,\omega_1}\left(R^d\right)$. Thus $\{f_n\}_{n=1}^{\infty}$ is also bounded in $A^p_{w_1,\,\omega_1}\left(R^d\right)$. Also since $\{f_{n_k}\}_{n=1}^{\infty}$ converges in $\Lambda^p_{w_2,\,\omega_2}\left(R^d\right)$ and $\Lambda^p_{w_2,\,\omega_2}\left(R^d\right)\subset A^p_{w_2,\,\omega_2}\left(R^d\right)$, then $\{f_{n_k}\}_{n=1}^{\infty}$ also converges in $A^p_{w_2,\,\omega_2}\left(R^d\right)$. This implies

$$
i: A^p_{w_1,\,\omega_1}\left(R^d\right)\to A^p_{w_2,\,\omega_2}\left(R^d\right)
\tag{23}
$$

is compact. But this is a contradiction because the Theorem 4. This completes the proof. ■

Corollary 8. *Let* $w_{1,}w_2$ *and* ω_1, ω_2 *be Beurling weight functions on* R^d. *Assume that* ω_1, ω_2 *symmetric (i.e* $\omega_1(x) = \omega_1(-x)$ *and* $\omega_2(x) = \omega_2(-x)$ *for all* $x \in R^d$*) and* $w_2 \preccurlyeq w_1$, $\omega_2 \preccurlyeq \omega_1$. *Then the embedding*

$$i : A^1_{w_1,\omega_1}\left(R^d\right) \hookrightarrow A^1_{w_2,\omega_2}\left(R^d\right)$$

is compact if and only if $\frac{w_2(x)}{w_1(x)}$ *and* $\frac{\omega_2(x)}{\omega_1(x)}$ *tend to zero.*

Proof. Since $\Lambda^1_{w_1,\omega_1}\left(R^d\right) = A^1_{w_1,\omega_1}\left(R^d\right)$ and $\Lambda^1_{w_2,\omega_2}\left(R^d\right) = A^1_{w_2,\omega_2}\left(R^d\right)$ then the proof is direct by Theorem 7. ■

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