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GENERALIZED LIE DERIVATIONS IN PRIME RINGS

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Abstract. We define generalized Lie derivations on rings and prove that every generalized Lie derivation on a prime ring R is a sum of a generalized derivation from R into its central closure R_C and an additive map from R into the extended centroid C sending commutators to zero.

1. Introduction

Recall that a *derivation* on a ring R is an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. This concept has been generalized in many ways. For instance, a *Lie derivation* on R is defined as an additive mapping $d : R \to R$ satisfying $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in R$. Here, as usually, $[x, y]$ denotes the commutator $xy - yx$.

Of course, all derivations are Lie derivations. In [4] Bresar described the structure of Lie derivations on prime rings:

Theorem A. *Let R be a prime ring with char$(R) \neq 2$. If $deg(R) > 2$, then every Lie derivation d on R is of the form $d = \delta + \zeta$, where δ is a derivation from R into its central closure and ζ is an additive mapping of R into the extended centroid C sending commutators to zero.*

Important examples of derivations are the so-called inner derivations, i.e. mappings of the form $\delta_a(x) = ax - xa$ where a is a fixed element in R. More generally, mappings of the form $f(x) = ax + xb$ (with $a, b \in R$ fixed elements) are called *generalized inner derivations*. In order to find a wider class of mappings that covers both derivations and generalized inner derivations, the concept of a generalized

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derivation was introduced in [3] as follows. An additive mapping $f : R \to R$ is called a *generalized derivation* if there exists an additive mapping d on R such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. Besides derivations and generalized inner derivations this also generalizes the concept of left multipliers, i.e. additive mappings satisfying $f(xy) = f(x)y,\ x, y \in R$. We refer the reader to [6-8] for some results concerning generalized derivations.

In this context we mention that in the definition of a generalized derivation one often requires that d is a derivation on R. However, Brešar proved in [3, Remark 1] that this assumption is usually unnecessary, in particular we have:

Remark A. Let R be a prime ring and let $f : R \to R$ be a generalized derivation, i.e. a map satisfying $f(xy) = f(x)y + xd(y)$ for some additive mapping $d : R \to R$ and for all $x, y \in R$. Then d is a derivation.

Let f be a generalized inner derivation given by $f(x) = ax + xb$. Note that $f([x, y]) = f(x)y - f(y)x + x\delta_{-b}(y) - y\delta_{-b}(x)$. In view of this observation we now give the following definition. An additive mapping $f : R \to R$ will be called a *generalized Lie derivation* if there exists an additive mapping $d : R \to R$ such that

$$f([x, y]) = f(x)y - f(y)x + xd(y) - yd(x) \tag{1}$$

holds for all $x, y \in R$. Of course, the class of generalized Lie derivations covers both the classes of Lie derivations and generalized derivations.

This definition of generalized Lie derivation was suggested to us by Matej Brešar. The referee pointed out that related yet somewhat different definitions of generalized Lie derivations were introduced also by Atsusi Nakajima [9].

In this work we will investigate the structure of generalized Lie derivations on prime rings and prove Theorem 1, a Lie analogue of Brešar's Theorem A. We will also discuss an alternative definition of a generalized Lie derivation where in (1) d is assumed to be a Lie derivation on R. We will get a Lie version of Remark A.

2. Preliminaries

Throughout the paper, R will be a prime ring with maximal left ring of quotients $Q_{ml}(R)$, extended centroid C and central closure $R_C = RC + C$. See [1] for details about these notions. We also recall that $\deg(R) \leq n$ if and only if R satisfies S_{2n}, the standard polynomial identity of degree $2n$.

Let A be any of the rings $R, R_C,\ Q_{ml}(R)$. Note that we have defined the notions of derivations, Lie derivations, generalized derivations and generalized Lie derivations for maps $f : R \to R$. However, these definitions can be extended to mappings $f : R \to A$ in a natural way.

Our method will be based on the theory of functional identities. We refer the reader to [5] for an introductory account on this theory. We will now state two results on functional identities due to Beidar [2]. We will use them as our main technical tool.

The first result follows from [2, Theorem 2.4]; to be precise, in [2] only the symmetric version where the mappings appear on the left hand side is treated, but it is clear that the same argument works in the proof of the following proposition.

Proposition 1. *Suppose that* $\deg(R) > 3$. *If* $F_j : R\times R \to Q_{ml}(R),\ j = 1, 2, 3$ *are mappings such that*

$$x\,F_1(y,z) + y\,F_2(x,z) + z\,F_3(x,y) \in C$$

for all $x, y, z \in R$, *then each* $F_j = 0$.

The second result is a special case of [2, Theorem 1.2].

Proposition 2. *Suppose that* $\deg(R) > 2$. *If* $E_i, F_j : R\times R \to Q_{ml}(R),\ i, j = 1, 2, 3,$ *are biadditive mappings such that*

$$E_1(y,z)x + E_2(x,z)y + E_3(x,y)z + xF_1(y,z) + yF_2(x,z) + zF_3(x,y) = 0 \tag{2}$$

for all $x, y, z \in R$. *Then there exist unique additive mappings* $p_{ij} : R \to Q_{ml}(R), i, j = 1, 2, 3,$ *and biadditive mappings* $\lambda_k : R \times R \to C,\ k = 1, 2, 3,$ *such that:*

$$\begin{aligned}
E_1(y,z) &= yp_{12}(z) + zp_{13}(y) + \lambda_1(y,z)\\
E_2(x,z) &= xp_{21}(z) + zp_{23}(x) + \lambda_2(x,z)\\
E_3(x,y) &= xp_{31}(y) + yp_{32}(x) + \lambda_3(x,y)\\
F_1(y,z) &= -(p_{21}(z)y + p_{31}(y)z + \lambda_1(y,z))\\
F_2(x,z) &= -(p_{12}(z)x + p_{32}(x)z + \lambda_2(x,z))\\
F_3(x,y) &= -(p_{13}(y)x + p_{23}(x)y + \lambda_3(x,y)).
\end{aligned}$$

3. The Structure of Generalized Lie Derivations

Theorem 1. *Let* R *be a prime ring with char*$(R) \neq 2$ *and* $deg(R) > 3$, *and let* $f : R \to R$ *be a generalized Lie derivation. Then* f *is of the form* $f = \delta + \zeta$, *where* $\delta : R \to R_C$ *is a generalized derivation and* $\zeta : R \to C$ *is an additive mapping sending commutators to zero.*

Proof. Replacing y by x^2 in $f([x,y]) = f(x)y - f(y)x + xd(y) - yd(x)$ we arrive at

$$f(x)x^2 - f(x^2)x + xd(x^2) - x^2d(x) = 0$$

for all $x \in R$. Linearizing this relation we get

$$\begin{aligned} 0 = \; & f(x)zy + f(x)yz + f(y)xz + f(y)zx + f(z)xy + f(z)yx - f(xy)z \\ & -f(yx)z - f(xz)y - f(zx)y - f(yz)x - f(zy)x + xd(yz) \\ & +xd(zy) + yd(xz) + yd(zx) + zd(xy) + zd(yx) - xyd(z) \\ & -yxd(z) - xzd(y) - zxd(y) - yzd(x) - zyd(x) \end{aligned}$$

for all $x, y, z \in R$. Defining biadditive mappings

$$\begin{aligned} E(x,y) &= f(x)y + f(y)x - f(xy) - f(yx), \\ F(x,y) &= d(xy) + d(yx) - xd(y) - yd(x), \end{aligned}$$

the relation above can be rewritten as

$$E(x,y)z + E(x,z)y + E(y,z)x + xF(y,z) + yF(x,z) + zF(x,y) = 0$$

for all $x, y, z \in R$. Apply Proposition 2 and note that $E_k = E,\ F_k = F,\ k = 1,2,3$. In view of the uniqueness of p_{ij}'s and λ_i's this implies that all p_{ij}'s as well as all λ_i's are equal. We therefore see that there exist an additive mapping $p : R \to Q_{ml}(R)$ and a biadditive mapping $\lambda : R \times R \to C$ such that

$$\begin{aligned} E(x,y) &= xp(y) + yp(x) + \lambda(x,y) \\ F(x,y) &= -p(x)y - p(y)x - \lambda(x,y) \end{aligned}$$

for all $x, y \in R$. Since E and F are symmetric mappings, λ is a symmetric biadditive mapping as well.

Recalling the definition of E we see that

$$f(xy) + f(yx) = f(x)y + f(y)x - xp(y) - yp(x) - \lambda(x,y)$$

which together with the initial relation

$$f(xy) - f(yx) = f(x)y - f(y)x + xd(y) - yd(x)$$

yields

$$2f(xy) = 2f(x)y + x\,(d(y) - p(y)) - y\,(d(x) + p(x)) - \lambda(x,y).$$

We have therefore shown that for every generalized Lie derivation f there exist additive mappings $g, h : R \to Q_{ml}(R)$ and a symmetric biadditive mapping $\mu = -\frac{1}{2}\lambda : R \times R \to C$, such that

(3) $$f(xy) = f(x)y + xg(y) + yh(x) + \mu(x, y)$$

for all $x, y \in R$. Note that

(4) $$g - h = d.$$

Let us collect some more information about the mappings g and h. First, the equality $f((xy)z) = f(x(yz))$, together with (3), gives us

$$f(xy)z + xyg(z) + zh(xy) + \mu(xy, z) = f(x)yz + xg(yz) + yzh(x) + \mu(x, yz).$$

The left hand side of this identity is equal to

$$f(x)yz + xg(y)z + yh(x)z + \mu(x, y)z + xyg(z) + zh(xy) + \mu(xy, z)$$

which implies

$$x\,(g(yz) - g(y)z - yg(z)) \;+\; y\,(zh(x) - h(x)z) \;+\; z\,(-\mu(x, y) - h(xy)) \in C$$

for all $x, y, z \in R$. We are now in a position to apply Proposition 1. Hence we get that $g : R \to Q_{ml}(R)$ is a derivation, h maps into the extended centroid C and $h(xy) = -\mu(x, y)$ for all $x, y \in R$. Moreover, from (4) it follows that g in fact maps R into R_C. Denote $\zeta = -h : R \to C$. Returning back to (3) we obtain

$$f(xy) = f(x)y + xg(y) - \zeta(x)y + \zeta(xy)$$

and finally

$$(f - \zeta)(xy) = (f - \zeta)(x)y + xg(y).$$

The mapping $f - \zeta = \delta : R \to R_C$ is therefore a generalized derivation. Besides, since $\zeta(xy) = \mu(x, y)$ and since μ is a symmetric biadditive mapping, ζ sends all commutators to zero. Write $f = \delta + \zeta$ and the theorem is proved. ■

Now we are in position to answer the question whether the more restrictive definition of a generalized Lie derivation, demanding the map d to be a Lie derivation on R, would give a smaller class of mappings. The next remark tells us that in prime rings the answer is no.

Remark 1. Let R be a prime ring with $\mathrm{char}(R) \neq 2$ and $deg(R) > 3$, and let $f : R \to R$ be a generalized Lie derivation, i.e. a map satisfying $f([x, y]) =$

$f(x)y - f(y)x + xd(y) - yd(x)$ for some additive mapping $d : R \to R$ and for all $x, y \in R$. Then d is a Lie derivation.

Proof. It follows from (4) that d is the sum of a derivation $g : R \to R_C$ and an additive mapping $\zeta = -h : R \to C$ sending commutators to zero. Therefore d satisfies $d([x, y]) = [d(x), y] + [y, d(x)]$ for all $x, y \in R$, that is d is a Lie derivation. ■

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