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ON MODULES AND MATRIX RINGS WITH SIP-EXTENDING

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Abstract. In this note we study modules with the property that the intersection of two direct summands is essential in a direct summand (SIP-extending). Amongst other results we show that the class of right SIP-extending modules is neither closed under direct sums nor Morita invariant. Further we deal with direct summands of a SIP-extending module and SIP-extending matrix rings.

1. Introduction

Throughout this paper all rings are associative with unity and R always denotes such a ring. Modules are unital and for an abelian group M, we use M_R (resp. ${}_RM$) to denote a right (resp. left) R-module. For any unexplained terminology please see [1, 5].

A module M_R has the *Summand Intersection Property*, SIP, if the intersection of every pair of direct summands of M_R is a direct summand of M_R. Modules having SIP were motivated by the following result of Kaplansky [7]: a free module over a principal ideal domain, has SIP. This property has been studied by many authors including [2, 3, 6, 8] and [13].

Recall that a module M is called an *extending module* (or, *CS-module*) if every submodule is essential in a direct summand of M. In [5] and [9], extending modules were studied in details. As a generalization of extending modules, C_{11} *modules* were investigated in [10, 11] and [14]. Recall that a module M is called C_{11}*-module* (or satisfies C_{11}) if every submodule of M has a complement which is a direct summand of M.

We will use $Mat_m(R)$ and R^m to denote the full m-by-m matrix ring over R and the direct sum of m copies of R_R for any positive integer m, respectively.

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In this paper we call a module M as an ***SIP-extending module*** if the intersection of every pair of direct summands of M is essential in a direct summand of M and we obtain basic results of this type of modules. To this end we observe that SIP-extending condition with every direct summand is the unique closure in the module of its essential submodules is inherited by direct summands but this is not the case for direct sums. For the latter case we obtain an affirmative answer for a special case. Further it is obtained that SIP-extending is not Morita invariant and a m-by-m full matrix ring over a ring S is right SIP-extending ring if and only if the free right S-module S^m is SIP-extending.

2. SIP-extending Modules

Definition 1. A module M is called an ***SIP-extending*** module provided that the intersection of every pair of direct summands of M is essential in a direct summand of M. We say a ring R is a right ***SIP-extending ring*** if the module R_R is an SIP-extending module. i.e., for every pair of idempotents e, c in R there exists $g^2 = g \in R$ such that $eR \cap cR$ is essential in gR.

Among the examples of SIP-extending modules, we can mention modules with summand intersection property i.e., SIP-modules and extending modules. Note that if the module has the property that intersection of two complements is again a complement in the module then SIP-extending and SIP are the same. However the following examples will make it clear that SIP-extending modules are proper generalizations of both SIP-modules and extending modules.

Example 2. Let F be any field and V be a F-vector space with $dimV_F \geq 2$. Let

$$R = \left\{ \begin{bmatrix} a & v \\ 0 & a \end{bmatrix} : a \in F, v \in V \right\},$$

be the trivial extension of F by V. It is clear that R is a right SIP-extending ring. Since $dimV_F \geq 2$, R is not a right extending ring.

The following example is taken from [4, Example 1.5].

Example 3. Let F be a field and

$$T = \left\{ \begin{bmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{bmatrix} : a, b, x, y \in F \right\}.$$

Let

$$e = e^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$c = c^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $eT \cap cT$ is nilpotent. Hence $eT \cap cT$ is not a direct summand of T. It follows that T does not have SIP. However it is a right SIP-extending ring.

The following proposition is an analogy of [13, Proposition 1.a] which provides a characterization of SIP-extending modules in terms of canonical projections.

Proposition 4. *The module M is an SIP-extending module if and only if for every pair of summands S and T with $\pi : M \longrightarrow S$ the projection map, the kernel of the restricted map $\pi \mid_T$ is essential in a direct summand of M.*

Proof. Suppose first that M is an SIP-extending module. For $S' = \ker \pi$, we get $M = S \oplus S'$. Now $\ker \pi \mid_T = T \cap S'$ is essential in a direct summand of M. Conversely, suppose that M has the stated property. Given S and T direct summands of M, choose S' as a complement of S and let ρ be the projection to S'. We see that $S \cap T = \ker \rho \mid_T$ is essential in a direct summand of M. ■

Given any module M, by a *closure* of a submodule N of M, we mean a maximal essential extension of N in M. Recall that for a nonsingular module M every submodule has a unique closure in M (see, for example [12]). The following lemma appeared in [11, Lemma 6] and its proof is given for completeness.

Lemma 5. *Let N be a submodule of a module M such that N has a unique closure K in M. Then K is the sum of all submodules L of M containing N such that N is essential in L.*

Proof. Let H be the sum of submodules L of M such that N is an essential submodule of L. Since N is essential in its closure K, it follows that $K \subseteq H$. Conversely, let L be any submodule of M such that N is an essential submodule of L. Let L' be any closure of L in M. Clearly, L' is a closure of N in M and so $L' = K$. Thus, $L \subseteq K$. It follows that $H \subseteq K$ and hence $H = K$. ■

Lemma 6. *Let M be an SIP-extending module, and let N be a direct summand of M. Suppose that N is the unique closure in M of any of its essential submodules. Then N is also an SIP-extending module.*

Proof. Let $M = N \oplus N'$ for some submodule N' of M. Let K, L be direct summands of N. Hence K, L are direct summands of M. By hypothesis, there exists a direct summand T_1 of M such that $K \cap L$ is essential in T_1 and $M = T_1 \oplus T_1'$ for some submodule T_1' of M. Then $K \cap L \cap N' = 0$ and hence $T_1 \cap N' = 0$. Also, $N \cap T_1$ is essential in T_1 because $K \cap L \subseteq N \cap T_1$.

Let $p : M \to T_1$ be the canonical projection along T_1'. Then we set $g : N \to T_1$ the restriction to N of the projection p. Take $K = g^{-1}(N \cap T_1)$. Now, K is an essential submodule of N because $N \cap T_1$ is essential in T_1. It is easy to verify that $K = (N \cap T_1) \oplus (N \cap T_1')$. Indeed, one inclusion is obvious. To see the other one, let $x \in K$. Then $x = y + z$ with $y \in T_1$, $z \in T_1'$. But then $g(x) = y \in N$ and hence $z \in N$. This shows that $x \in (N \cap T_1) \oplus (N \cap T_1')$.

Let us call $N \cap T_1' = Y$. Then K is an essential submodule of $T_1 \oplus Y$ because $N \cap T_1$ is essential in T_1. But N is the unique closure in M of K because K is essential in N. We have then that N contains $T_1 \oplus Y$. In particular $T_1 \subseteq N$. It follows then that T_1 is a direct summand of N and $K \cap L$ is essential in a direct summand of N. ■

Our next result concerns a left exact preradical r in the category of right modules over a ring R. For the definition and basic properties of left exact preradicals, see [12]. In particular, we shall need the following properties of a left exact preradical r for a ring R:

(i) $r(M)$ is submodule of M for every right R-module M,

(ii) $r(M_1 \oplus M_2) = r(M_1) \oplus r(M_2)$ for all right R-modules M_1, M_2,

(iii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M, and

(iv) $\phi(r(M)) \subseteq r(M')$ for every homomorphism $\phi : M \to M'$ for right R-modules M, M'.

Proposition 7. *Let R be a ring, r a left exact preradical for the category of right R-modules, and M a right SIP-extending R-module such that $r(M)$ has a unique closure in M. If $M = M_1 \oplus M_2$ with $r(M_1)$ essential in M_1 and $r(M_2) = 0$ then M_1 is an SIP-extending module.*

Proof. First note that $r(M) = r(M_1) \oplus r(M_2) = r(M_1)$, so M_1 is the unique closure of $r(M)$ in M. Let $\pi_1 : M \to M_1$ denote the canonical projection.

Let N, N' be any direct summands of M_1. There exist direct summands K, K' of M such that $N \cap N'$ is essential in K and $M = K \oplus K'$. Since $K \cap M_2 =$

0, it follows that $K \cong \pi_1(K)$. Note that, because r is left exact, $r(\pi_1(K)) = \pi_1(K) \cap r(M_1)$ is essential in $\pi_1(K)$. Hence, $r(K)$ is essential in K and, $r(M) = r(K) \oplus r(K')$ is essential in $K \oplus r(K')$. By Lemma 5, $K \oplus r(K') \subseteq M_1$ and hence $K \subseteq M_1$. Now $M_1 = K \oplus (M_1 \cap K')$ i.e., M_1 is an SIP-extending module. ■

Observe that Proposition 7 applies in the case that $r(M)$ is a direct summand of M. Recall that a direct summand of a C_{11}-module is not a C_{11}-module in general; see [11, Example 4]. In [14, Theorem 2.1(1)] a related result on direct summands of a C_{11}-module has been obtained. Next result generalizes [14, Theorem 2.1(1)].

Theorem 8. *Let M be a C_{11}-module and E be a submodule of M. If for every direct summand D of M, $E \cap D$ is essential in a direct summand of E then E is a C_{11}-module.*

Proof. Let A be a submodule of E. By C_{11} condition there exists a complement N_2 of A in M such that $M = N_1 \oplus N_2$ for some submodule N_1 of M. So $N_2 \cap A = 0$ and $N_2 \oplus A$ is essential in M. Thus

$$(N_2 \cap E) \cap A = E \cap (N_2 \cap A) = 0.$$

Therefore

$$(N_2 \cap E) \oplus A = E \cap (N_2 \oplus A)$$

is essential in E. By assumption there exists a direct summand T of E such that $N_2 \cap E$ is essential in T. Since $A \cap T = 0$ and $(N_2 \cap E) \oplus A$ is essential in E, $T \oplus A$ is an essential submodule of E. Hence T is a complement of A in E by [10, Lemma 2.2]. It follows that E is a C_{11}-module. ■

Corollary 9. *Let M be a C_{11}-module such that every direct summand of M is the unique closure in M of its essential submodules. If M is an SIP-extending module, then any direct summand of M is a C_{11}-module.*

Proof. Immediate Lemma 6 and Theorem 8. ■

Now we provide an example which shows that a direct sum of SIP-extending modules need not to be an SIP-extending, in general.

Example 10. Let R be as in [11, Example 4]. That is

$$S = \mathbb{R}[x, y, z] \ and \ R = S/sS, \ where \ s = x^2 + y^2 + z^2 - 1.$$

Let $M = R \oplus R \oplus R$. It is clear that M_R is nonsingular. Assume that M is an SIP-extending R-module. Then by Corollary 9, every direct summand of M is a

C_{11}-module. However from [11, Example 4] we have a contradiction. Therefore M is not an SIP-extending module.

Next result shows a case in which a direct sum of SIP-extending modules is also an SIP-extending module.

Theorem 11. *Let $M = \oplus M_i$ be a direct sum of fully invariant submodules M_i of M. If each M_i is an SIP-extending module then M is an SIP-extending module.*

Proof. Let S be any direct summand of M. Since each M_i is fully invariant, $S = \oplus (S \cap M_i)$. Now let S, T be direct summands of M. Hence,

$$S \cap T = [\oplus (S \cap M_i)] \cap [\oplus (T \cap M_i)] = \oplus [(S \cap M_i) \cap (T \cap M_i)] .$$

Since each M_i is an SIP-extending module, there exists a direct summand K_i of M_i which contains $(S \cap M_i) \cap (T \cap M_i)$ as an essential submodule. It follows that M is an SIP-extending module. ■

3. SIP-extending Matrix Rings

It is shown in [3] that the summand intersection property is not Morita invariant. In this section, we show that the SIP-extending condition is not Morita invariant either. Let R be any ring with identity, e an idempotent in R such that $R = ReR$ and S the subring eRe. Let M be a right-submodule. Then Me is a right S-module.

Lemma 12. *Let K be submodule of M_R. Then K is a direct summand of M_R if and only if Ke is a direct summand of $(Me)_S$.*

Proof. It is simple to be checked. ■

Theorem 13. *With the above notation, let M be a right R-module. The right R-module M is an SIP-extending module if and only if the right S-module Me is an SIP-extending module.*

Proof. Immediate by Lemma 12 ■

Corollary 14. *The ring R is a right SIP-extending ring if and only if the right eRe-module Re is an SIP-extending module.*

Now we let S be a ring with identity 1, m a positive integer and R the ring $Mat_m(S)$ of all $m \times m$ matrices with entries in S. Let e_{11} denote the matrix in R

with (1,1) entry 1 and all other entries 0. It is well known that e_{11} is idempotent and $S \cong e_{11}Re_{11}$ and $R = Re_{11}R$.

Thus Theorem 13 gives, without further proof, the following result which was pointed out in the introduction.

Theorem 15. *With the above notation, let $R = Mat_m(S)$. Then R is a right SIP-extending ring if and only if the free right S-module S^m is an SIP-extending module.*

Observe that clearly the ring R in Example 10 is a right SIP-extending ring. However the right R-module $M = R^3$ is not an SIP-extending module. By Theorem 15, $Mat_3(R)$ is not a right SIP-extending ring. We conclude that being SIP-extending is not Morita invariant.

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