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ON GLOBAL EXISTENCE AND BLOW-UP OF SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH STRONG DAMPING

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Abstract. The initial boundary value problem for an integro-differential equation with strong damping in a bounded domain is considered. The existence, asymptotic behavior and blow-up of solutions are discussed under some conditions. The decay estimates of the energy function and the estimates of the lifespan of blow-up solutions are given.

1. INTRODUCTION

In this paper we consider the initial boundary value problem for the following nonlinear integro-differential equation:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u), \tag{1.1}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{1.2}$$

and boundary condition

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{1.3}$$

where $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\Omega \subset R^N$, $N \geq 1$, is a bounded domain with a smooth boundary $\partial\Omega$ so that Divergence theorem can be applied. Here, g represents the kernel of the memory term which is assumed to decay exponentially (see assumption (A1)), $h(u_t) = -\Delta u_t$, f is a nonlinear function like $f(u) = |u|^{p-2}u$, $p > 2$ and

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$M(s)$ is a positive locally Lipschitz function like $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$ and $s \geq 0$.

When $g \equiv 0$, for the case that $M \equiv 1$, the equation (1.1) becomes a nonlinear wave equation which has been extensively studied and several results concerning existence and nonexistence have been established $[1, 3, 4, 8, 10, 11]$. When M is not a constant function, a special case of equation (1.1) is Kirchhoff equation which has been introduced in order to describe the nonlinear vibrations of an elastic string. More precisely, we have

$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f, \tag{1.4}$$

for $0 < x < L, t \geq 0$; where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. Kirchhoff [9] was the first one to study the oscillations of stretched strings and plates. In this case the existence and nonexistence of solutions have been discussed by many authors and the references cited therein $[5, 6, 16, 17, 18, 19]$.

When g is not trivial on R, for the case that $M \equiv 1$, (1.1) becomes a semilinear viscoelastic equation. Cavalcanti et al. [2] treated (1.1) for $h(u_t) = a(x)u_t$, here $a(x)$ may be null on a part of the domain. By assuming the kernel g in the memory term decays exponentially, they obtained an exponentially decay rate of the energy. This work extended the result of Zuazua [22] in which he considered (1.1) with $g = 0$ and the damping is localized. On the other hand, when $h = 0$, Jiang and Rivera [8] proved, in the framework of nonlinear viscoelasticity, the exponential decay of the energy provided that the kernel g decays exponentially. Recently, Wu and Tsai [20] discuss the global solution as well as energy decay, and blow-up of solutions for h and f are power-like functions. In the case that M is not a constant function, the equation (1.1) is a model to describe the motion of deformable solids as hereditary effect is incorporated. The equation (1.1) was first studied by Torrejon and Young [21] who proved the existence of weakly asymptotic stable solution for large analytical datum. Later, Rivera [14] showed the existence of global solutions for small datum and the total energy decays to zero exponentially under some restrictions.

In this paper we show that under some conditions the solution is global in time and the energy decays exponentially. In this way, we can extend the result of [14] to nonzero external force term $f(u)$ and the result of [20] to nonconstant $M(s)$. We also obtain the new results for blow-up properties of local solution with small positive initial energy by using the direct method [13]. The content of this paper is organized as follows. In section 2, we give some lemmas and assumptions which will be used later. In section 3, we first use Faedo-Galerkin method to study the existence of the simpler problem $(3.1)-(3.3)$. Then, we obtain the local existence Theorem 3.2 by using contraction mapping principle. Moreover, the uniqueness of solution is also given. In section 4, we first define an energy function $E(t)$ in (4.7) and show that it is a non-increasing function of t. We obtain global existence and decay properties of the solutions of $(1.1)-(1.3)$ given in Theorem 4.4. Finally, the blow-up properties of $(1.1)-(1.3)$ and the estimates for the blow-up time T^* are also given.

2. Preliminary Results

In this section, we shall give some lemmas and assumptions which will be used throughout this work.

Lemma 2.1. (Sobolev-Poincaré inequality [12]) *If* $2 \leq p \leq \frac{2N}{N-2}$, *then*

$$\|u\|_p \leq B_1 \|\nabla u\|_2 \, ,$$

for $u \in H_0^1(\Omega)$ *holds with some constant* B_1, *where* $\|\cdot\|_p$ *denotes the norm of* $L^p(\Omega)$.

Lemma 2.1. [13] *Let* $\delta > 0$ *and* $B(t) \in C^2(0, \infty)$ *be a nonnegative function satisfying*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0. \tag{2.1}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.2}$$

then

$$B'(t) > K_0$$

for $t > 0$, *where* K_0 *is a constant*, $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ *is the smallest root of the equation*

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 2.3. [13] *If* $J(t)$ *is a non-increasing function on* $[t_0, \infty)$, $t_0 \geq 0$ *and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}} \text{ for } t_0 \geq 0, \tag{2.3}$$

where $a > 0, b \in R$, *then there exists a finite time* T^* *such that*

$$\lim_{t \to T^{*-}} J(t) = 0$$

and the upper bound of T^* is estimated respectively by the following cases:

(i) *If $b < 0$ and $J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$ then*

$$T^* \le t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}.$$

(ii) *If $b = 0$, then*

$$T^* \le t_0 + \frac{J(t_0)}{\sqrt{a}}.$$

(iii) *If $b > 0$, then*

$$T^* \le \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \le t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\},$$

where $c = (\frac{b}{a})^{\frac{\delta}{2+\delta}}$.

Lemma 2.4. [15] *Let $\phi(t)$ be a non-increasing and nonnegative function on $[0, T]$, $T > 1$, such that*

$$\phi(t)^{1+r} \le \omega_0 \left(\phi(t) - \phi(t+1)\right) \text{ on } [0, T],$$

where ω_0 is a positive constant and r is a nonnegative constant. Then we have

(i) *if $r > 0$, then*

$$\phi(t) \le \left(\phi(0)^{-r} + \omega_0^{-1} r [t-1]^+\right)^{-\frac{1}{r}},$$

where $[t-1]^+ = \max\{t-1, 0\}$.

(ii) *if $r = 0$, then*

$$\phi(t) \le \phi(0) e^{-\omega_1 [t-1]^+} \text{ on } [0, T],$$

where $\omega_1 = ln(\frac{\omega_0}{\omega_0 - 1})$, here $\omega_0 > 1$.

Now, we state the general hypotheses:

(A1) $g : R^+ \to R^+$ is a bounded C^1 function satisfying

$$m_0 - \int_0^\infty g(s)ds = l > 0, \tag{2.4}$$

and there exist positive constants ξ_1, ξ_2, and ξ_3 such that

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t). \tag{2.5}$$

(A2) $f(0) = 0$ and there is a positive constant k_1 such that

$$|f(u) - f(v)| \leq k_1 |u - v| \left(|u|^{p-2} + |v|^{p-2} \right),$$

for $u, v \in R$ and $2 < p \leq \frac{2(N-1)}{N-2}$; $(\infty,\ \text{if } N \leq 2)$.

3. Local Existence

In this section, we shall discuss the local existence of solutions for integro-differential equations $(1.1) - (1.3)$ by using contraction mapping principle.

An important step in the proof of local existence Theorem 3.2 below is the study of the following simpler problem :

$$u_{tt} - \mu(t)\Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = f_1(x,t) \text{ on } \Omega \times (0,T), \tag{3.1}$$

with initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{3.2}$$

and Dirichlet boundary condition

$$u(x,t) = 0,\ x \in \partial\Omega,\ t > 0. \tag{3.3}$$

Here, $T > 0$, f is a fixed forcing term on $\Omega \times (0,T)$, and μ is a positive locally Lipschitz function on $[0,\infty)$ with $\mu(t) \geq m_0 > 0$ for $t \geq 0$.

Lemma 3.1. *Suppose that* (A1) *holds, and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$ *and* $f_1 \in L^2([0,T]; L^2(\Omega))$. *Then the problem* $(3.1) - (3.3)$ *admits a unique solution* u *such that*

$$\begin{aligned} u &\in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), \\ u_t &\in C([0,T]; L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)), \\ u_{tt} &\in L^2([0,T]; L^2(\Omega)). \end{aligned}$$

Proof. Let $(w_n)_{n\in N}$ be a basis in $H_0^1(\Omega) \cap H^2(\Omega)$ and V_n be the space generated by $w_1, \cdots, w_n$, $n = 1, 2, \cdots$.
Let us consider

$$u_n(t) = \sum_{i=1}^{n} r_{in}(t) w_i$$

be the weak solution of the following approximate problem corresponding to $(3.1)-(3.3)$

$$\begin{aligned}&\int_\Omega u_n''(t)wdx+\mu(t)\int_\Omega \nabla u_n(t)\cdot\nabla wdx\\ &-\int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla wdxd\tau+\int_\Omega \nabla u_n'(t)\cdot\nabla wdx\\ =&\int_\Omega f_1(x,t)wdx \text{ for } w\in V_n,\end{aligned}\tag{3.4}$$

with initial conditions

$$u_n(0)=u_{0n}\equiv\sum_{i=1}^n p_{in}w_i\to u_0 \text{ in } H_0^1(\Omega)\cap H^2(\Omega),\tag{3.5}$$

and

$$u_n'(0)=u_{1n}\equiv\sum_{i=1}^n q_{in}w_i\to u_1 \text{ in } L^2(\Omega),\tag{3.6}$$

where $p_{in}=\int_\Omega u_0w_idx$, $q_{in}=\int_\Omega u_1w_idx$ and $u'=\frac{\partial u}{\partial t}$.

By standard methods in differential equations, we prove the existence of solutions to $(3.4)-(3.6)$ on some interval $[0,t_n), 0<t_n<T$. In order to extend the solution of $(3.4)-(3.6)$ to the whole interval $[0,T]$, we need following a prior estimate.

Step 1. Setting $w=u_n'(t)$ in (3.4), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\left\|u_n'(t)\right\|_2^2+\frac{\mu(t)}{2}\left\|\nabla u_n(t)\right\|_2^2\right)+\left\|\nabla u_n'(t)\right\|_2^2\\ =&\int_\Omega f_1(x,t)u_n'(t)dx+\int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau\\ &+\frac{\mu'(t)}{2}\left\|\nabla u_n(t)\right\|_2^2.\end{aligned}$$

Noting that, by Hölder inequality and Young's inequality, we have

$$\begin{aligned}&\int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau\\ \le&\ \frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2+\frac{\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\left\|\nabla u_n(\tau)\right\|_2^2d\tau,\end{aligned}\tag{3.8}$$

and

$$\int_\Omega f_1(x,t)u_n'(t)dx \le \frac{1}{2}\|f_1\|_2^2 + \frac{1}{2}\left\|u_n'(t)\right\|_2^2. \tag{3.9}$$

Then, by using (3.8) and (3.9), we obtain from (3.7)

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\left\|u_n'(t)\right\|_2^2 + \frac{\mu(t)}{2}\|\nabla u_n(t)\|_2^2\right) + \frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2 \\
&\le \frac{1}{2}\|f_1\|_2^2 + \frac{\|g\|_{L^1}}{2}\int_0^t g(t-\tau)\|\nabla u_n(\tau)\|_2^2 d\tau \\
&\quad + \frac{\mu'(t)}{2}\|\nabla u_n(t)\|_2^2 + \frac{1}{2}\left\|u_n'(t)\right\|_2^2.
\end{aligned} \tag{3.10}$$

By integrating (3.10), we get

$$\begin{aligned}
&\left\|u_n'(t)\right\|_2^2 + \mu(t)\|\nabla u_n(t)\|_2^2 + \int_0^t \left\|\nabla u_n'(t)\right\|_2^2 dt \\
&\le c_1 + \int_0^t \left[1 + \frac{1}{\mu(t)}\left(\left|\mu'(t)\right| + \|g\|_{L^1}^2\right)\right]\left[\left\|u_n'(t)\right\|_2^2 + \mu(t)\|\nabla u_n(t)\|_2^2\right] dt.
\end{aligned} \tag{3.11}$$

where $c_1 = \|u_{1n}\|_2^2 + \mu(0)\|\nabla u_{0n}\|_2^2 + \int_0^t \|f_1\|_2^2 dt$.

Thus, by employing Gronwall's Lemma, we see that

$$\left\|u_n'(t)\right\|_2^2 + \mu(t)\|\nabla u_n(t)\|_2^2 + \int_0^t \left\|\nabla u_n'(t)\right\|_2^2 dt \le L_1, \tag{3.12}$$

for $t \in [0,T]$ and L_1 is a positive constant independent of $n \in N$.

Step 2. Setting $w = u_n''(t)$ in (3.4), we have

$$\begin{aligned}
&\left\|u_n''(t)\right\|_2^2 + \frac{d}{dt}\left(\mu(t)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx + \frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2\right) \\
&= \mu'(t)\int_\Omega \nabla u_n(t)\cdot x\nabla u_n'(t)dx + \mu(t)\left\|\nabla u_n'(t)\right\|_2^2 \\
&\quad + \frac{d}{dt}\left(\int_0^t g(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau\right) - g(0)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx \\
&\quad - \int_0^t g'(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau + \int_\Omega f_1(x,t)u_n''(t)dx.
\end{aligned} \tag{3.13}$$

Noting that, by (2.5), Hölder inequality and Young's inequality, we have

$$\begin{aligned}
&-\int_0^t g'(t-\tau)\int_\Omega \nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau \\
&\le \eta\left\|\nabla u_n'(t)\right\|_2^2 + \frac{\xi_1^2\|g\|_{L^1}}{4\eta}\int_0^t g(t-\tau)\|\nabla u_n(\tau)\|_2^2 d\tau.
\end{aligned} \tag{3.14}$$

By Hölder inequality and Young's inequality again, we get

$$g(0)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx \le \eta\left\|\nabla u_n'(t)\right\|_2^2+\frac{g(0)^2}{4\eta}\left\|\nabla u_n(t)\right\|_2^2, \tag{3.15}$$

and

$$\left|\mu'(t)\int_\Omega \nabla u_n(t)\cdot\nabla u_n'(t)dx\right| \le \eta\left\|\nabla u_n'(t)\right\|_2^2+\frac{M_1^2}{4\eta}\left\|\nabla u_n(t)\right\|_2^2, \tag{3.16}$$

where $0<\eta\le\frac{1}{4}$ is some positive constant and $M_1=\sup\limits_{0\le t\le T}\{|\mu'(t)|\}$.

Thus, integrating (3.13) over $(0,t)$, and using $(3.14)-(3.16)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\left\|\nabla u_n'(t)\right\|_2^2+\frac{1}{2}\int_0^t\left\|u_n''(t)\right\|_2^2dt\\
\le\ &\frac{M_1^2+\xi_1^2\left\|g\right\|_{L^1}^2+g(0)^2}{4\eta}\int_0^t\left\|\nabla u_n(\tau)\right\|_2^2d\tau+\mu(t)\int_0^t\left\|\nabla u_n'(t)\right\|_2^2dt\\
&+3\eta\int_0^t\left\|\nabla u_n'(t)\right\|_2^2dt+\frac{1}{2}\int_0^t\left\|f_1\right\|_2^2dt\\
&+\int_0^t g(t-\tau)\int_\Omega\nabla u_n(\tau)\cdot\nabla u_n'(t)dxd\tau\\
&+\mu(t)\left|\int_\Omega\nabla u_n(t)\cdot\nabla u_n'(t)dx\right|+\mu(0)\left|\int_\Omega\nabla u_{0n}\cdot\nabla u_{1n}dx\right|.
\end{aligned} \tag{3.17}$$

By using Hölder inequality and Young's inequality on the fifth and sixth term in (3.17) and by (3.12), we deduce

$$\begin{aligned}
&\left(\frac{1}{2}-2\eta\right)\left\|\nabla u_n'(t)\right\|_2^2+\frac{1}{2}\int_0^t\left\|u_n''(t)\right\|_2^2dt\\
\le\ &c_2+(M_2+3\eta)\int_0^t\left\|\nabla u_n'(\tau)\right\|_2^2d\tau,
\end{aligned}$$

where $c_2=\mu(0)\left\|\nabla u_{0n}\right\|_2\left\|\nabla u_{1n}\right\|_2+\frac{\left[\left(M_1^2+\xi_1^2\|g\|_{L^1}^2+g(0)^2+\|g\|_{L^1}\|g\|_{L^\infty}\right)T+M_2^2\right]L_1}{4\eta m_0}$ $+\frac{1}{2}\int_0^t\left\|f_1\right\|_2^2dt$ and $M_2=\sup\limits_{0\le t\le T}\{|\mu(t)|\}$.

Then, by Gronwall's Lemma, we have

$$\left\|\nabla u_n'(t)\right\|_2^2+\int_0^t\left\|u_n''(t)\right\|_2^2dt\le L_2, \tag{3.18}$$

for all $t\in[0,T]$ and L_2 is a positive constant independent of $n\in N$.

Step 3. Setting $w = -\Delta u_n$ in (3.4), we deduce

$$\begin{aligned} &\frac{d}{dt}\left(-\int_\Omega u_n'(t)\Delta u_n(t)dx + \frac{1}{2}\|\Delta u_n(t)\|_2^2\right) \\ &- \left\|\nabla u_n'(t)\right\|_2^2 + \mu(t)\|\Delta u_n\|_2^2 \\ \le\ & \frac{1}{4\eta}\|f_1\|_2^2 + \eta\|\Delta u_n\|_2^2 + \int_0^t g(t-\tau)\int_\Omega \Delta u_n(\tau)\Delta u_n(t)dxd\tau, \end{aligned} \tag{3.19}$$

where $0 < \eta \le \frac{m_0}{2}$ is some positive constant.
Since

$$\begin{aligned} &\int_0^t g(t-\tau)\int_\Omega \Delta u_n(\tau)\Delta u_n(t)dxd\tau \\ \le\ & \eta\|\Delta u_n(t)\|_2^2 + \frac{\|g\|_{L^1}}{4\eta}\int_0^t g(t-\tau)\|\Delta u_n(\tau)\|_2^2 d\tau, \end{aligned} \tag{3.20}$$

then by integrating (3.19) and using (3.20) and (3.18), we obtain

$$\begin{aligned} &\frac{1}{4}\|\Delta u_n\|_2^2 + (m_0 - 2\eta)\int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau \\ \le\ & c_3 + \frac{\|g\|_{L^1}^2}{4\eta}\int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau, \end{aligned}$$

where $c_3 = \|u_{1n}\|_2\|\Delta u_{0n}\|_2 + \frac{1}{2}\|\Delta u_{0n}\|_2^2 + \frac{1}{4\eta}\int_0^t \|f_1\|_2^2 dt + L_1 + L_2T$.

Thus, by Gronwall's Lemma, we have

$$\|\Delta u_n\|_2^2 + \int_0^t \|\Delta u_n(\tau)\|_2^2 d\tau \le L_{3,} \tag{3.21}$$

for all $t \in [0, T]$ and L_3 is a positive constant independent of $n \in N$.

Step 4. Let $j \ge n$ be two natural numbers and consider $z_n = u_j - u_n$. Then, applying the same way as in the estimate step 1 and step3 and observing that $\{u_{0n}\}$ and $\{u_{1n}\}$ are Cauchy sequence in $H_0^1(\Omega) \cap H^2(\Omega)$ and $L^2(\Omega)$, respectively, we deduce

$$\left\|z_n'(t)\right\|_2^2 + \mu(t)\|\nabla z_n(t)\|_2^2 + \int_0^t \left\|\nabla z_n'(t)\right\|_2^2 dt \to 0, \tag{3.22}$$

and

$$\|\Delta z_n\|_2^2 + \int_0^t \|\Delta z_n(\tau)\|_2^2 d\tau \to 0, \text{ as } n \to \infty, \tag{3.23}$$

for all $t \in [0, T]$.

Therefore, from (3.12), (3.18), (3.21), (3.22) and (3.23), we see that

$$u_i \to u \ \text{ strongly in } \ C(0,T;H_0^1(\Omega)), \tag{3.24}$$

$$u_i' \to u' \ \text{ strongly in } \ C(0,T;L^2(\Omega)), \tag{3.25}$$

$$u_i' \to u' \ \text{ strongly in } \ L^2(0,T;H_0^1(\Omega)), \tag{3.26}$$

$$u_i'' \to u'' \ \text{ weakly in } L^2(0,T;L^2(\Omega)). \tag{3.27}$$

Then $(3.24)-(3.27)$ are sufficient to pass the limit in (3.4) to obtain

$$u_{tt} - \mu(t)\Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = f_1(x,t) \ \text{ in } \ L^2(0,T;H^{-1}(\Omega)).$$

Next, we want to show the uniqueness of $(3.1)-(3.3)$. Let $u^{(1)}, u^{(2)}$ be two solutions of $(3.1)-(3.3)$. Then $z = u^{(1)} - u^{(2)}$ satisfies

$$\begin{aligned} &\int_\Omega z''(t)wdx + \mu(t)\int_\Omega \nabla z(t)\cdot\nabla wdx - \int_0^t g(t-\tau)\int_\Omega \nabla z(\tau)\cdot\nabla wdxd\tau \\ &\quad + \int_\Omega \nabla z'(t)\cdot\nabla wdx = 0 \text{ for } w \in H_0^1(\Omega), \\ &\quad z(x,0) = 0, z'(x,0) = 0, \quad x \in \Omega, \end{aligned} \tag{3.28}$$

and

$$z(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0.$$

Setting $w = z'(t)$ in (3.28), then as in deriving (3.12), we see that

$$\begin{aligned} &\left\|z'(t)\right\|_2^2 + \mu(t)\left\|\nabla z(t)\right\|_2^2 + \int_0^t \left\|\nabla z'(t)\right\|_2^2 dt \\ \leq &\int_0^t \left[1 + \frac{1}{\mu(s)}\left(\left|\mu'(s)\right| + \|g\|_{L^1}^2\right)\right]\left[\left\|z'(s)\right\|_2^2 + \mu(s)\left\|\nabla z(s)\right\|_2^2\right] dt. \end{aligned}$$

Thus, employing Gronwall's Lemma, we conclude that

$$\left\|z'(t)\right\|_2 = \|\nabla z(t)\|_2 = 0 \text{ for all } t \in [0,T]. \tag{3.29}$$

Therefore, we have the uniqueness.

Now, we are ready to to show the local existence of the problem $(1.1)-(1.3)$.

Theorem 3.2. *Suppose that* (A1) *and* (A2) *hold, and that* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, *then there exists a unique solution* u *of* $(1.1) - (1.3)$ *satisfying*

$$u \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)) \textit{ and } u_t \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)).$$

Moreover, at least one of the following statements holds true :

$$\begin{aligned} &(i) T = \infty, \\ &(ii) e(u(t)) \equiv \|u_t(t)\|_2^2 + \|\Delta u(t)\|_2^2 \to \infty \textit{ as } t \to T^-. \end{aligned} \tag{3.30}$$

Proof. Define the following two-parameter space: $X_{T,R_0} =$

$$\left\{ \begin{array}{c} v \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega)), v_t \in C([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)) : \\ e(v(t)) \leq R_0^2,\ t \in [0,T],\ \text{with } v(0) = u_0 \text{ and } v_t(0) = u_1. \end{array} \right\},$$

for $T > 0,\ R_0 > 0$. Then X_{T,R_0} is a complete metric space with the distance

$$d(y,z) = \sup_{0 \leq t \leq T} e(y(t) - z(t))^{\frac{1}{2}}. \tag{3.31}$$

where $y,\ z \in X_{T,R_0}$.

Given $v \in X_{T,R_0}$, we consider the following problem

$$u_{tt} - M(\|\nabla v\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s) ds - \Delta u_t = f(v), \tag{3.32}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{3.33}$$

and boundary condition

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0. \tag{3.34}$$

First of all, we observe that

$$\begin{aligned} \frac{d}{dt} M(\|\nabla v\|_2^2) &= 2M'(\|\nabla v\|_2^2) \int_\Omega \nabla v \cdot \nabla v_t dx \\ &\leq 2M_3 \|\Delta v\|_2 \|v_t\|_2 \\ &\leq 2M_3 R_0^2, \end{aligned} \tag{3.35}$$

where $M_3 = \sup\{|M'(s)|; 0 \leq s \leq B_1^2 R_0^2\}$. And by (A2), we see that $f \in L^2([0,T]; L^2(\Omega))$.

Thus, by Lemma 3.1, there exists a unique solution u of $(3.32) - (3.34)$. We define the nonlinear mapping $Sv = u$, and then, we shall show that there exist $T > 0$ and $R_0 > 0$ such that

(i) $S : X_{T,R_0} \to X_{T,R_0}$,

(ii) S is a contraction mapping in $X_{T,R_0}$with respect to the metric $d(\cdot,\cdot)$ defined in (3.31).

(i) Multiplying (3.32) by $2u_t$, and then integrating it over $\Omega \times (0,t)$, we obtain

$$\begin{aligned}(3.36)\qquad &\frac{d}{dt}\left[\|u_t\|_2^2 + \left(M(\|\nabla v\|_2^2) - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\right]\\ &+2\|\nabla u_t\|_2^2 - (g' \diamond \nabla u)(t) + g(t)\|\nabla u(t)\|_2^2\\ &= I_1 + I_2,\end{aligned}$$

where

$$I_1 = \left(\frac{d}{dt}M(\|\nabla v\|_2^2)\right)\|\nabla u(t)\|_2^2,$$

and

$$I_2 = 2\int_\Omega f(v)u_t dx.$$

The equality in (3.36) is obtained, because

$$\begin{aligned}(3.37)\qquad &-\int_0^t\int_\Omega g(t-\tau)\nabla u(\tau)\cdot\nabla u_t(t)dxd\tau\\ &= \frac{1}{2}\frac{d}{dt}\left[(g\diamond\nabla u)(t) - \int_0^t g(\tau)\|\nabla u(t)\|_2^2 d\tau\right]\\ &-\frac{1}{2}(g'\diamond\nabla u)(t) + \frac{1}{2}g(t)\|\nabla u(t)\|_2^2,\end{aligned}$$

where

$$(3.38)\qquad (g\diamond\nabla u)(t) = \int_0^t g(t-\tau)\int_\Omega |\nabla u(\tau) - \nabla u(t)|^2 dxd\tau.$$

Noting that by using (3.35) and (3.30), we have

$$(3.39)\qquad |I_1| \le 2M_3 B_1^2 R_0^2 e(u(t)),$$

and by (A2), Hölder inequality and Poincaré inequality, we get

$$\begin{aligned}(3.40)\qquad |I_2| &\le 2k_1\int_\Omega |v|^{p-1}|u_t|dx\\ &\le 2k_1 B_1^{2(p-1)}\|\Delta v\|_2^{p-1}\|u_t\|_2\\ &\le 2k_1 B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}}.\end{aligned}$$

Then, by (3.39), (3.40) and (A1), we have from (3.36)

$$
\begin{aligned}
& \frac{d}{dt}\left[\|u_t\|_2^2 + \left(M(\|\nabla v\|_2^2) - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t)\right] \\
(3.41) \qquad & +2\|\nabla u_t\|_2^2 \\
\leq\ & 2M_3 B_1^2 R_0^2 e(u(t)) + 2k_1 B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}}.
\end{aligned}
$$

On the other hand, multiplying (3.32) by $-2\Delta u$, and integrating it over Ω, we get

$$
\begin{aligned}
& \frac{d}{dt}\left\{\|\Delta u\|_2^2 - 2\int_\Omega u_t \Delta u dx\right\} + 2M\left(\|\nabla v\|_2^2\right)\|\Delta u(t)\|_2^2 \\
\leq\ & 2\int_\Omega u_t \Delta u_t dx - 2\int_\Omega f(v)\Delta u dx + 2\int_0^t g(t-\tau)\int_\Omega \Delta u(\tau)\Delta u(t) dx d\tau.
\end{aligned}
$$

Using similar arguments as for (3.20) and (3.40), we deduce

$$
\begin{aligned}
& \frac{d}{dt}\left\{\|\Delta u\|_2^2 - 2\int_\Omega u_t \Delta u dx\right\} + 2\left(M\left(\|\nabla v\|_2^2\right) - \eta\right)\|\Delta u(t)\|_2^2 \\
(3.42) \qquad \leq\ & 2k_1 B_1^{2(p-1)} R_0^{p-1} e(u)^{\frac{1}{2}} + \frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau \\
& +2\|\nabla u_t\|_2^2,
\end{aligned}
$$

where $0 < \eta \leq \frac{\|g\|_{L^1}}{2}$ is some constant.

Multiplying (3.42) by ε, $0 < \varepsilon \leq 1$, and adding (3.41) together, we obtain

$$
\begin{aligned}
& \frac{d}{dt}e^*(u(t)) + 2(1-\varepsilon)\|\nabla u_t\|_2^2 + 2\varepsilon\left(M\left(\|\nabla v\|_2^2\right) - \eta\right)\|\Delta u(t)\|_2^2 \\
(3.43) \qquad \leq\ & 2M_3 B_1^2 R_0^2 e(u(t)) + 2k_1(1+\varepsilon) B_1^{2(p-1)} R_0^{p-1} e(u(t))^{\frac{1}{2}} \\
& +\varepsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\|\Delta u(\tau)\|_2^2 d\tau,
\end{aligned}
$$

where

$$
\begin{aligned}
& e^*(u(t)) \\
=\ & \|u_t\|_2^2 + \left(M(\|\nabla v\|_2^2) - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) \\
& -2\varepsilon\int_\Omega u_t \Delta u dx + \varepsilon\|\Delta u\|_2^2.
\end{aligned}
$$

By Young's inequality, we get

$$\left|2\varepsilon \int_\Omega u_t \Delta u dx\right| \leq 2\varepsilon \|u_t\|_2^2 + \frac{\varepsilon}{2}\|\Delta u\|_2^2.$$

Hence

$$\begin{aligned} e^*(u(t)) \ \geq \ & (1-2\varepsilon)\|u_t\|_2^2 + \frac{\varepsilon}{2}\|\Delta u\|_2^2 + \left(M(\|\nabla v\|_2^2) - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 \\ & + (g \diamond \nabla u)(t). \end{aligned}$$

Choosing $\varepsilon = \frac{2}{5}$ and by (2.4), we have

$$e^*(u(t)) \geq \frac{1}{5} e(u(t)), \tag{3.44}$$

and

$$\begin{aligned} e^*(u_0) &\leq (1+2\varepsilon)\|u_1\|_2^2 + \tfrac{3\varepsilon}{2}\|\Delta u_0\|_2^2 + M\left(\|\nabla u_0\|_2^2\right)\|\nabla u_0\|_2^2 \\ &\leq c_2, \end{aligned} \tag{3.45}$$

where

$$c_2 = 2\|u_1\|_2^2 + \|\Delta u_0\|_2^2 + M\left(\|\nabla u_0\|_2^2\right)\|\nabla u_0\|_2^2.$$

Integrating (3.43) over $(0,t)$, we get

$$\begin{aligned} & e^*(u(t)) + \frac{4}{5}\left(m_0 - \eta - \frac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t \|\Delta u(s)\|_2^2 ds \\ & \leq e^*(u_0) + \int_0^t [10 M_3 B_1^2 R_0^2 e^*(u(s)) \\ & + \frac{14\sqrt{5}}{5} k_1 B_1^{2(p-1)} R_0^{p-1} e^*(u(s))^{\frac{1}{2}}] ds \end{aligned} \tag{3.46}$$

Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (3.46), then from (2.4), we deduce

$$e^*(u(t)) \leq e^*(u_0) + \int_0^t \left(10 M_3 B_1^2 R_0^2 e^*(u(s)) + \frac{14\sqrt{5}}{5} k_1 B_1^{2(p-1)} R_0^{p-1} e^*(u(s))^{\frac{1}{2}}\right) ds.$$

Thus, by Gronwall's Lemma and using (3.45), we have

$$e^*(u(t)) \leq \left(\sqrt{c_2} + \frac{7\sqrt{5}}{20} k_1 B_1^{2(p-1)} R_0^{p-1} T\right)^2 \mathrm{e}^{10 M_3 B_1^2 R_0^2 T}. \tag{3.47}$$

Then, by (3.44), we obtain

$$e(u(t)) \leq \chi(u_0, u_1, R_0, T)^2 \mathrm{e}^{10M_3B_1^2R_0^2T}, \tag{3.48}$$

for any $t \in (0, T]$ and

$$\chi(u_0, u_1, R_0, T) = \sqrt{5c_2} + \frac{7}{4}k_1 B_1^{2(p-1)} R_0^{p-1} T.$$

We see that if parameters T and R_0 satisfy

$$\chi(u_0, u_1, R_0, T)^2 \mathrm{e}^{20M_3B_1^2R_0^2T} \leq R_0^2. \tag{3.49}$$

Moreover, by Lemma 3.1, $u \in C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$. On the other hand, it follows from (3.41) and (3.48) that $u_t \in L^2((0,T); H_0^1(\Omega))$. Thus, S maps X_{T,R_0} into itself.

Next, we will show that S is a contraction mapping with respect to the metric $d(\cdot,\cdot)$. Let $v_i \in X_{T,R_0}$ and $u^{(i)} \in X_{T,R_0}$, $i = 1, 2$ be the corresponding solution to $(3.32) - (3.34)$.

Let $w(t) = (u^{(1)} - u^{(2)})(t)$, then w satisfy the following system:

$$\begin{aligned} & w_{tt} - M\left(\|\nabla v_1\|_2^2\right)\Delta w + \int_0^t g(t-\tau)\Delta w(\tau)d\tau - \Delta w_t \\ = \; & f(v_1) - f(v_2) + \left[M\left(\|\nabla v_1\|_2^2\right) - M\left(\|\nabla v_2\|_2^2\right)\right]\Delta u^{(2)}, \end{aligned} \tag{3.50}$$

with initial conditions

$$w(0) = 0, \; w_t(0) = 0, \tag{3.51}$$

and boundary condition

$$w(x,t) = 0, \; x \in \partial\Omega \text{ and } t \geq 0. \tag{3.52}$$

Multiplying (3.50) by $2w_t$, and integrating it over Ω, we have

$$\begin{aligned} & \frac{d}{dt}\left[\|w_t\|_2^2 + \left(M(\|\nabla v_1\|_2^2) - \int_0^t g(s)ds\right)\|\nabla w(t)\|_2^2 + (g \diamond \nabla w)(t)\right] \\ & +2\|\nabla w_t\|_2^2 - (g' \diamond \nabla w)(t) + g(t)\|\nabla w(t)\|_2^2 \\ = \; & I_3 + I_4 + I_5, \end{aligned} \tag{3.53}$$

where

$$I_3 = 2\left[M\left(\|\nabla v_1\|_2^2\right) - M\left(\|\nabla v_2\|_2^2\right)\right]\int_\Omega \Delta u^{(2)} w_t dx,$$

$$I_4 = 2\int_\Omega \left(f(v_1) - f(v_2)\right) w_t dx,$$

and

$$I_5 = \left(\frac{d}{dt} M(\|\nabla v_1\|_2^2)\right) \|\nabla w(t)\|_2^2 .$$

To proceed the estimates of $I_i, i = 3, 4, 5$, we observe that

$$\begin{aligned} |I_3| &\le 2L\left(\|\nabla v_1\|_2 + \|\nabla v_2\|_2\right) \|\nabla v_1 - \nabla v_2\|_2 \left\|\Delta u^{(2)}\right\|_2 \|w_t\|_2 \\ &\le 4LB_1^2 R_0^2 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{3.54}$$

$$|I_4| \le 4k_1 B_1^{2(p-1)} R_0^{p-2} e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \tag{3.55}$$

and

$$|I_5| \le 2M_3 B_1^2 R_0^2 e(w(t)), \tag{3.56}$$

where $L = L(R_0)$ is the Lipschitz constant of $M(r)$ in $[0, R_0]$.

Thus, by using $(3.54) - (3.56)$ in (3.53), we get

$$\begin{aligned} &\frac{d}{dt}\left[\|w_t\|_2^2 + \left(M(\|\nabla v_1\|_2^2) - \int_0^t g(s)ds\right) \|\nabla w(t)\|_2^2 + (g \diamond \nabla w)(t)\right] \\ &+2\|\nabla w_t\|_2^2 \\ &\le 2M_3 B_1^2 R_0^2 e(w(t)) + c_3 e(v_1 - v_2)^{\frac{1}{2}} e(w(t))^{\frac{1}{2}}, \end{aligned} \tag{3.57}$$

where $c_3 = 4\left(LB_1^2 R_0^2 + k_1 B_1^{2(p-1)} R_0^{p-2}\right)$.

On the other hand, multiplying (3.50) by $-2\Delta w$, and as in deriving (3.42), (3.54) and (3.56), we deduce

$$\begin{aligned} &\frac{d}{dt}\left\{\|\Delta w\|_2^2 - 2\int_\Omega w_t \Delta w dx\right\} + 2\left(M\left(\|\nabla v_1\|_2^2\right) - \eta\right) \|\Delta w(t)\|_2^2 \\ \le\ & c_3 e(v_1 - v_2)^{\frac{1}{2}} e(w)^{\frac{1}{2}} + \frac{\|g\|_{L^1}}{2\eta} \int_0^t g(t-\tau) \|\Delta w(\tau)\|_2^2 d\tau \\ &+2\|\nabla w_t\|_2^2 , \end{aligned} \tag{3.58}$$

where $0 < \eta \le \frac{\|g\|_{L^1}}{2}$.

Multiplying (3.58) by ε, $0<\varepsilon\leq 1$, and adding (3.57) together, we obtain

$$\begin{aligned}
&\frac{d}{dt}e_*(w(t))+2(1-\varepsilon)\left\|\nabla w_t\right\|_2^2+2\varepsilon\left(M\left(\left\|\nabla v_1\right\|_2^2\right)-\eta\right)\left\|\Delta w\right\|_2^2\\
\leq\ & 2M_3B_1^2R_0^2e(w(t))+(1+\varepsilon)c_3e(v_1-v_2)^{\frac{1}{2}}e(w(t))^{\frac{1}{2}}\\
&+\varepsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\left\|\Delta w(\tau)\right\|_2^2 d\tau,
\end{aligned}\tag{3.59}$$

where

$$\begin{aligned}
&e_*(w(t))\\
=\ &\left\|w_t\right\|_2^2+\left(M(\|\nabla v_1\|_2^2)-\int_0^t g(s)ds\right)\left\|\nabla w\right\|_2^2+(g\diamond\nabla w)(t)\\
&-2\varepsilon\int_\Omega w_t\Delta w dx+\varepsilon\left\|\Delta w\right\|_2^2.
\end{aligned}\tag{3.60}$$

By using Young's inequality on the fourth term of right hand side of (3.60), we get

$$\begin{aligned}
e_*(w(t))\ \geq\ &(1-2\varepsilon)\left\|w_t\right\|_2^2+\frac{\varepsilon}{2}\left\|\Delta w\right\|_2^2\\
&+\left(M(\|\nabla v_1\|_2^2)-\int_0^t g(s)ds\right)\left\|\nabla w(t)\right\|_2^2+(g\diamond\nabla w)(t).
\end{aligned}$$

Choosing $\varepsilon=\frac{2}{5}$ and by (2.4), we have

$$e_*(w(t))\geq\frac{1}{5}e(w(t)),\tag{3.61}$$

and by $(3.51)-(3.52)$, we also see that

$$e_*(w(0))=0.\tag{3.62}$$

Then, applying the same way as in obtaining (3.46) and then taking $\eta=\frac{\|g\|_{L^1}}{2}$, we deduce

$$\begin{aligned}
e_*(w(t))\ \leq\ &e_*(w(0))+\int_0^t[10M_3B_1^2R_0^2e_*(w(s))\\
&+\frac{7\sqrt{5}c_3}{5}e(v_1-v_2)^{\frac{1}{2}}e_*(w(s))^{\frac{1}{2}}]ds.
\end{aligned}$$

Thus, by Gronwall's Lemma, we obtain

$$e_*(w(t))\leq\left(\frac{7\sqrt{5}c_3}{20}B_1^{2(p-1)}R_0^{p-2}\right)^2T^2\mathrm{e}^{10M_3B_1^2R_0^2T}\sup_{0\leq t\leq T}e(v_1-v_2).$$

By (3.61) and (3.31), we have

$$d(u^1, u^2) \leq C(T, R_0)^{\frac{1}{2}} d(v_1, v_2), \tag{3.63}$$

where

$$C(T, R_0) = 5\left(\frac{7\sqrt{5}c_3}{20} B_1^{2(p-1)} R_0^{p-2}\right)^2 T^2 \mathrm{e}^{10M_3 B_1^2 R_0^2 T}.$$

Hence, under inequality (3.49), S is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 sufficient large and T sufficient small so that (3.49) and (3.63) are satisfied at the same time. By applying Banach fixed point theorem, we obtain the local existence result.

The second statement of the theorem is proved by a standard continuation argument. The proof of Theorem 3.2 is now completed.

4. Global Existence and Energy Decay

In this section, we consider the global existence and energy decay of solutions for a kind of the problem $(1.1) - (1.3)$:

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p-2} u, \tag{4.1}$$

with initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \tag{4.2}$$

and boundary condition

$$u(x,t) = 0, x \in \partial\Omega, t \geq 0, \tag{4.3}$$

where $2 < p \leq \frac{2(N-1)}{N-2}$ and $M(s) = m_0 + bs^\gamma$, $m_0 > 0$, $b \geq 0$, $\gamma \geq 1$ and $s \geq 0$.

Let

$$\begin{aligned} I_1(t) \equiv I_1(u(t)) = &\left(m_0 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + (g \diamond \nabla u)(t) \\ &-\|u(t)\|_p^p, \end{aligned} \tag{4.4}$$

$$\begin{aligned} I_2(t) \equiv I_2(u(t)) = &\left(m_0 - \int_0^t g(s)ds\right) \|\nabla u(t)\|_2^2 + b\,\|\nabla u(t)\|_2^{2(\gamma+1)} \\ &+(g \diamond \nabla u)(t) - \|u(t)\|_p^p, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} J(t) \equiv \quad & J(u(t)) = \frac{1}{2}\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) \\ & + \frac{b}{2(\gamma+1)}\|\nabla u(t)\|_2^{2(\gamma+1)} - \frac{1}{p}\|u(t)\|_p^p, \end{aligned} \tag{4.6}$$

for $u(t) \in H_0^1(\Omega)$, $t \geq 0$, and $(g \diamond \nabla u)(t)$ is given in (3.38).

We define the energy of the solution u of $(4.1) - (4.3)$ by

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + J(t). \tag{4.7}$$

Lemma 4.1. $E(t)$ *is a non-increasing function on* $[0, \infty)$ *and*

$$E'(t) = -\|\nabla u_t\|_2^2 + \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \tag{4.8}$$

Proof. By using Divergence theorem, $(4.1) - (4.3)$ and (3.37), we see that (4.8) follows at once.

Lemma 4.2. *Let* u *be the solution of* $(4.1) - (4.3)$. *Assume the conditions of Theorem* 3.2 *hold. If* $I_1(u_0) > 0$ *and*

$$\alpha = \frac{B_1^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} < 1, \tag{4.9}$$

then $I_2(t) > 0$, *for all* $t \geq 0$.

Proof. Since $I_1(u_0) > 0$, it follows from the continuity of $u(t)$ that

$$I_1(t) > 0, \tag{4.10}$$

for some interval near $t = 0$. Let $t_{\max} > 0$ be a maximal time (possibly $t_{\max} = T$), when (4.10) holds on $[0, t_{\max})$.

From (4.6) and (4.4), we have

$$\begin{aligned} J(t) \geq \quad & \frac{1}{2}\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + \frac{1}{2}(g \diamond \nabla u)(t) - \frac{1}{p}\|u\|_p^p \\ \geq \quad & \frac{p-2}{2p}\left[\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u\|_2^2 + (g \diamond \nabla u)(t)\right] + \frac{1}{p}I_1(t). \end{aligned} \tag{4.11}$$

By using (4.11), (4.7) and Lemma 4.1, we get

$$
\begin{aligned}
l\,\|\nabla u\|_2^2 &\le \left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \le \frac{2p}{p-2}J(t) \\
&\le \frac{2p}{p-2}E(t) \le \frac{2p}{p-2}E(0).
\end{aligned}
\tag{4.12}
$$

Then, from Poincaré inequality and (4.9), we obtain from (4.12)

$$
\begin{aligned}
\|u\|_p^p &\le B_1^p\|\nabla u\|_2^p \le \frac{B_1^p}{l}\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}} l\,\|\nabla u\|_2^2 \\
&= \alpha l\,\|\nabla u\|_2^2 < \left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2 \text{ on } [0,t_{\max}).
\end{aligned}
\tag{4.13}
$$

Thus

$$
I_1(t)=\left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+(g\diamond\nabla u)(t)-\|u\|_p^p>0 \text{ on } [0,t_{\max}).
\tag{4.14}
$$

This implies that we can take $t_{\max}=T$. But, from (4.4) and (4.5), we see that

$$
I_2(t)\ge I_1(t), t\in[0,T].
$$

Therefore, we have $I_2(t)>0, t\in[0,T]$.

Next, we want to show that $T=\infty$. Multiplying (4.1) by $-2\Delta u$, and integrating it over Ω, we get

$$
\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u dx\right\}+2M\left(\|\nabla v\|_2^2\right)\|\Delta u\|_2^2 \\
&\le 2\,\|\nabla u_t\|_2^2-2\int_\Omega |u|^{p-2}\,u\Delta u dx+2\int_0^t g(t-\tau)\int_\Omega \Delta u(\tau)\Delta u(t)dxd\tau.
\end{aligned}
$$

Applying the same arguments as in (3.42), we have

$$
\begin{aligned}
&\frac{d}{dt}\left\{\|\Delta u\|_2^2-2\int_\Omega u_t\Delta u dx\right\}+\left(2M\left(\|\nabla u\|_2^2\right)-2\eta\right)\|\Delta u\|_2^2 \\
&\le 2\,\|\nabla u_t\|_2^2+\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\,\|\Delta u(\tau)\|_2^2\,d\tau \\
&\quad -2\int_\Omega |u|^{p-2}\,u\Delta u dx,
\end{aligned}
\tag{4.15}
$$

where $0<\eta\le\frac{\|g\|_{L^1}}{2}$.

Multiplying (4.15) by ε, $0<\varepsilon\leq 1$, and multiplying (4.8) by 2, and then adding them together, we obtain

$$\begin{aligned}&\frac{d}{dt}E^*(t)+2(1-\varepsilon)\left\|\nabla u_t\right\|_2^2+2\varepsilon\left(M\left(\left\|\nabla u\right\|_2^2\right)-\eta\right)\left\|\Delta u\right\|_2^2\\ \le& -2\varepsilon\int_\Omega |u|^{p-2}u\Delta u dx+\varepsilon\frac{\|g\|_{L^1}}{2\eta}\int_0^t g(t-\tau)\left\|\Delta u(\tau)\right\|_2^2 d\tau,\end{aligned}\tag{4.16}$$

where

$$E^*(t)=2E(t)-2\varepsilon\int_\Omega u_t\Delta u dx+\varepsilon\left\|\Delta u\right\|_2^2.\tag{4.17}$$

By Young's inequality, we get

$$\left|2\varepsilon\int_\Omega u_t\Delta u dx\right|\le 2\varepsilon\left\|u_t\right\|_2^2+\frac{\varepsilon}{2}\left\|\Delta u\right\|_2^2.$$

Hence, choosing $\varepsilon=\frac{2}{5}$ and by (4.14), we see that

$$E^*(t)\ge\frac{1}{5}\left(\left\|u_t\right\|_2^2+\left\|\Delta u\right\|_2^2\right).\tag{4.18}$$

Moreover, we note that

$$\begin{aligned}2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right|\le&\ 2(p-1)\textstyle\int_\Omega |u|^{p-2}\left|\nabla u\right|^2 dx\\ \le&\ 2(p-1)\left\|u\right\|_{(p-2)\theta_1}^{p-2}\left\|\nabla u\right\|_{2\theta_2}^2,\end{aligned}\tag{4.19}$$

where $\frac{1}{\theta_1}+\frac{1}{\theta_2}=1$, so that, we put $\theta_1=1$ and $\theta_2=\infty$, if $N=1$; $\theta_1=1+\varepsilon_1$ (for arbitrary small $\varepsilon_1>0$), if $N=2$; and $\theta_1=\frac{N}{2}$, $\theta_2=\frac{N}{N-2}$, if $N\ge 3$.

Then, by Poincaré inequality, (4.12) and (4.18), we have

$$\begin{aligned}2\left|\int_\Omega |u|^{p-2}u\Delta u dx\right|\le&\ 2B_1^p(p-1)\left\|\nabla u\right\|_2^{p-2}\left\|\Delta u\right\|_2^2\\ \le&\ c_1E^*(t),\end{aligned}\tag{4.20}$$

where $c_1=10B_1^p(p-1)\left(\frac{2p}{l(p-2)}E(0)\right)^{\frac{p-2}{2}}$.

Substituting (4.20) into (4.16), and then integrating it over $(0,t)$, we obtain

$$\begin{aligned}&E^*(t)+\frac{4}{5}\left(m_0-\eta-\frac{\|g\|_{L^1}^2}{4\eta}\right)\int_0^t\left\|\Delta u(s)\right\|_2^2 ds\\ \le&\ E^*(0)+\int_0^t c_1E^*(s)ds\end{aligned}\tag{4.21}$$

Taking $\eta = \frac{\|g\|_{L^1}}{2}$ in (4.21), and then by Gronwall's Lemma, we deduce

$$E^*(t) \leq E^*(0)\exp(c_1 t),$$

for any $t \geq 0$. Therefore by Theorem 3.2, we have $T = \infty$.

Lemma 4.3. *If u satisfies the assumptions of Lemma 4.2, then there exists $0 < \eta_1 < 1$ such that*

$$\|u(t)\|_p^p \leq (1-\eta_1)\left(m_0 - \int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \text{ on } [0,\infty), \tag{4.22}$$

where $\eta_1 = 1 - \alpha$.

Proof. From (4.11), we get

$$\|u\|_p^p \leq \alpha l \,\|\nabla u\|_2^2 .$$

Let $\eta_1 = 1 - \alpha$, then we have (4.22).

Theorem 4.4. (Global existence and Energy decay) *Suppose that* (A1) *holds. Assume $I_1(u_0) > 0$ and (4.9) holds, then the problem $(4.1)-(4.3)$ admits a global solution u if $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_1 \in L^2(\Omega)$.*
Furthermore, we have the following decay estimates:

$$E(t) \leq E(0)e^{-\tau_1 t} \text{ on } [0,\infty),$$

where τ_1 is given in (4.38).

Proof. By integrating (4.8) over $[t, t+1]$, we get

$$E(t) - E(t+1) \equiv D(t)^2, \tag{4.23}$$

where

$$D(t)^2 = \int_t^{t+1} \|\nabla u_t\|_2^2\, dt - \frac{1}{2}\int_t^{t+1} (g' \diamond \nabla u)(t)dt + \frac{1}{2}\int_t^{t+1} g(t)\,\|\nabla u(t)\|_2^2\, dt$$

Hence, by (A1), there exist $t_1 \in [t, t+\frac{1}{4}], t_2 \in [t+\frac{3}{4}, t+1]$ such that

$$\|\nabla u_t(t_i)\|_2^2 \leq 4D(t)^2, \quad i = 1, 2. \tag{4.24}$$

Next, multiplying (4.1) by u and integrating it over $\Omega \times [t_1, t_2]$, we have

$$\begin{aligned} & \int_{t_1}^{t_2} \left[\left(m_0 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 + b \|\nabla u\|_2^{2(\gamma+1)} - \|u\|_p^p \right] dt \\ = & -\int_{t_1}^{t_2} \int_\Omega u_{tt} u dx dt - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u dx dt \\ & + \int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot [\nabla u(s) - \nabla u(t)] \, ds dx dt. \end{aligned}$$

Then, by (4.5), we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} I_2(t) dt \\ = & -\int_{t_1}^{t_2} \int_\Omega u_{tt} u dx dt - \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u dx dt + \int_{t_1}^{t_2} (g \diamond \nabla u)(t) dt \\ & + \int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot [\nabla u(s) - \nabla u(t)] \, ds dx dt. \end{aligned}$$

By using Hölder inequality and Young's inequality, we have

$$\left| \int_{t_1}^{t_2} \int_\Omega \nabla u_t \cdot \nabla u dx dt \right| \le \int_{t_1}^{t_2} \|\nabla u_t\|_2 \|\nabla u\|_2 \, dt, \tag{4.25}$$

and

$$\begin{aligned} & \int_{t_1}^{t_2} \int_\Omega \int_0^t g(t-s) \nabla u(t) \cdot [\nabla u(s) - \nabla u(t)] \, ds dx dt \\ \le & \ \delta \int_{t_1}^{t_2} \int_0^t g(t-s) \|\nabla u\|_2^2 \, ds dt + \frac{1}{4\delta} \int_{t_1}^{t_2} (g \diamond \nabla u)(t) dt, \end{aligned} \tag{4.26}$$

where δ is some positive constant to be chosen later.

Note that by integrating by parts, Hölder inequality and Poincaré inequality, we get

$$\begin{aligned} & \left| \int_{t_1}^{t_2} \int_\Omega u_{tt} u dx dt \right| \\ \le & \ B_1^2 \sum_{i=1}^{2} \|\nabla u_t(t_i)\|_2 \|\nabla u(t_i)\|_2 + B_1^2 \int_t^{t+1} \|\nabla u_t\|_2^2 \, dt. \end{aligned} \tag{4.27}$$

Then, by $(4.25)-(4.27)$, we deduce

$$\begin{aligned}&\int_{t_1}^{t_2} I_2(t)dt\\ \leq\ & B_1^2\sum_{i=1}^{2}\|\nabla u_t(t_i)\|_2\|\nabla u(t_i)\|_2+B_1^2\int_t^{t+1}\|\nabla u_t\|_2^2\,dt\\ &+\int_{t_1}^{t_2}\|\nabla u_t\|_2\|\nabla u\|_2\,dt+(\frac{1}{4\delta}+1)\int_{t_1}^{t_2}(g\diamond\nabla u)(t)dt\\ &+\delta\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u(t)\|_2^2\,dsdt.\end{aligned}$$

Furthermore, by (4.24) and (4.12), we have

$$\|\nabla u_t(t_i)\|_2\|\nabla u(t_i)\|_2\leq c_2D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}, \tag{4.28}$$

and

$$\int_{t_1}^{t_2}\|\nabla u_t\|_2\|\nabla u\|_2\,dt\leq\frac{c_2}{2}D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}, \tag{4.29}$$

where $c_2=2\left(\frac{2p}{l(p-2)}\right)^{\frac{1}{2}}$.

Thus, by using (4.28) and (4.29), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2} I_2(t)dt\\ \leq\ & c_3D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}+B_1^2D(t)^2+(\frac{1}{4\delta}+1)\int_{t_1}^{t_2}(g\diamond\nabla u)(t)dt\\ &+\delta\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u(t)\|_2^2\,dsdt,\end{aligned} \tag{4.30}$$

where $c_3=\left(2B_1^2+\frac{1}{2}\right)c_2$.

On the other hand, from (2.5) and (4.23), we get

$$\begin{aligned}&\int_{t_1}^{t_2}(g\diamond\nabla u)(t)dt\\ \leq\ & -\frac{1}{\xi_2}\int_{t_1}^{t_2}(g'\diamond\nabla u)(t)dt\\ \leq\ & \frac{2}{\xi_2}D(t)^2,\end{aligned} \tag{4.31}$$

and by (2.4) and Lemma 4.3, we have

$$
\begin{aligned}
\int_{t_1}^{t_2}\int_0^t g(t-s)\|\nabla u(t)\|_2^2\,dsdt &\leq -\frac{1}{\xi_2}\int_{t_1}^{t_2}\int_0^t g'(t-s)\,\|\nabla u(t)\|_2^2\,dsdt \\
&= \frac{1}{\xi_2}\int_{t_1}^{t_2}[g(0)-g(t)]\,\|\nabla u(t)\|_2^2\,dt \\
&\leq \frac{1}{\xi_2}\int_{t_1}^{t_2} g(0)\,\|\nabla u(t)\|_2^2\,dt \\
&\leq \frac{g(0)}{\eta_1 l\xi_2}\int_{t_1}^{t_2} I_2(t)dt,
\end{aligned}
\tag{4.32}
$$

where the last inequality is derived by (4.22), because

$$
\left(m_0-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2 \leq \frac{1}{\eta_1}I_1(t) \leq \frac{1}{\eta_1}I_2(t) \text{ for } t\geq 0. \tag{4.33}
$$

Hence, by choosing δ such that $\frac{\delta g(0)}{\eta_1 l\xi_2}=\frac{1}{2}$ and by $(4.31)-(4.32)$, we obtain from (4.30)

$$
\int_{t_1}^{t_2} I_2(t)dt \leq 2c_3D(t)\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}+c_4D(t)^2, \tag{4.34}
$$

where $c_4=4\left[B_1^2+(\frac{g(0)}{2\eta_1 l\xi_2}+1)\frac{1}{\xi_2}\right]$.

Moreover, from (4.7), (4.4) and using (4.14), we see that

$$
\begin{aligned}
E(t)\leq\ & \frac{1}{2}\|u_t\|_2^2+c_5\left(m_0-\int_0^t g(s)ds\right)\|\nabla u\|_2^2+c_5(g\diamond\nabla u)(t) \\
&+c_6I_2(t),
\end{aligned}
\tag{4.35}
$$

where $c_5=\frac{1}{2}-\frac{1}{p}$ and $c_6=\left(\frac{1}{p}+\frac{1}{2(\gamma+1)}\right)$.

By integrating (4.35) over (t_1,t_2), we obtain

$$
\begin{aligned}
\int_{t_1}^{t_2}E(t)dt \leq\ & \frac{1}{2}\int_{t_1}^{t_2}\|u_t\|_2^2\,dt+c_6\int_{t_1}^{t_2}I_2(t)dt+c_5\int_{t_1}^{t_2}(g\diamond\nabla u)(t)dt \\
&+c_5\int_{t_1}^{t_2}\left(m_0-\int_0^t g(s)ds\right)\|\nabla u(t)\|_2^2\,dt.
\end{aligned}
$$

Thus, by Poincaré inequality, (4.23), (4.31) and (4.33), we have

$$
\int_{t_1}^{t_2}E(t)dt \leq \frac{B_1^2}{2}D(t)^2+(c_6+\frac{c_5}{\eta_1})\int_{t_1}^{t_2}I_2(t)dt+\frac{2c_5}{\xi_2}D(t)^2. \tag{4.36}
$$

By multiplying (4.1) by u_t and then integrating it over $[t, t_2] \times \Omega$, we obtain

$$\begin{aligned} E(t) =\ & E(t_2) + \int_t^{t_2} \|\nabla u(t)\|_2^2 \, dt - \frac{1}{2} \int_t^{t_2} (g' \diamond \nabla u)(t) dt \\ & + \frac{1}{2} \int_t^{t_2} g(s) \|\nabla u(s)\|_2^2 \, ds. \end{aligned}$$

Since $t_2 - t_1 \geq \frac{1}{2}$, we get

$$E(t_2) \leq 2 \int_{t_1}^{t_2} E(t) dt.$$

Then, thanks to (4.23), we have

$$\begin{aligned} E(t) \ \leq\ & 2 \int_{t_1}^{t_2} E(t) dt + \int_t^{t+1} \|\nabla u(t)\|_2^2 \, dt - \frac{1}{2} \int_t^{t+1} (g' \diamond \nabla u)(t) dt \\ & + \frac{1}{2} \int_t^{t+1} g(s) \|\nabla u(s)\|_2^2 \, ds \\ =\ & 2 \int_{t_1}^{t_2} E(t) dt + D(t)^2. \end{aligned}$$

Thus, by using (4.36) and (4.34), we obtain

$$E(t) \leq c_7 D(t)^2 + c_8 D(t) \sup_{t_1 \leq s \leq t_2} E(s)^{\frac{1}{2}},$$

where $c_7 = B_1^2 + \frac{4c_5}{\xi_2} + 2(c_6 + \frac{c_5}{\eta_1})c_4 + 1$ and $c_8 = 4c_3(c_6 + \frac{c_5}{\eta_1})$.

Hence, by Young's inequality, we deduce

$$E(t) \leq c_9 D(t)^2, \tag{4.37}$$

where c_9 is some positive constant.

Therefore, we have the following decay estimates:

From (4.37) and (4.23), we have

$$E(t) \leq c_{10} \left[E(t) - E(t+1) \right] \text{ for } t \geq 0,$$

$c_{10} = \max\{c_9, 1\}$.

Thus, by Lemma 2.4, we obtain

$$E(t) \leq E(0) e^{-\tau_1 t}, \text{ on } [0, \infty), \tag{4.38}$$

where $\tau_1 = \ln \frac{c_{10}}{c_{10} - 1}$.

5. Blow-up Property

In this section, we shall discuss the blow up phenomena of problem $(1.1)-(1.3)$;

$$u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = f(u). \tag{5.1}$$

In order to state our results, we make further assumptions on f, M and g:

(A3) there exists a positive constant δ such that

$$sf(s) \geq (2+4\delta)F(s), \text{ for all } s \in R,$$

where

$$F(s) = \int_0^s f(r)dr,$$

and

$$(2\delta+1)\overline{M}(s) - (M(s) + 2\delta m_0)\, s \geq, \text{ for all } s \geq 0,$$

where

$$\overline{M}(s) = \int_0^s M(r)dr.$$

(A4) We make the following extra assumption on g

$$\int_0^\infty g(s)ds < \frac{4\delta m_0}{1+4\delta},$$

here δ is the constant appeared in (A3).

Remark. (1) In this case, we define the energy function of the solution u of (5.1), (1.2) and (1.3) by

$$\begin{aligned} E(t) = \ & \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\overline{M}(\|\nabla u(t)\|_2^2) + \frac{1}{2}(g \diamond \nabla u)(t) \\ & -\frac{1}{2}\int_0^t g(s)ds\, \|\nabla u(t)\|_2^2 - \int_\Omega F(u(t))dx, \end{aligned} \tag{5.2}$$

for $t \geq 0$. Then we have

$$\begin{aligned} E(t) = \ & E(0) - \int_0^t \|\nabla u_t(t)\|_2^2\, dt + \frac{1}{2}\int_0^t (g' \diamond \nabla u)(t)dt \\ & -\frac{1}{2}\int_0^t g(t)\, \|\nabla u(t)\|_2^2\, dt. \end{aligned} \tag{5.3}$$

We note that the energy function $E(t)$ defined by (5.2) is the same as in (4.7).

(2) It is clear that $f(u) = |u|^{p-2}\, u,\ p > 2$ satisfies (A3) with $\delta = \frac{p-2}{4}$ and $M(s) = m_0 + bs^\gamma$ satisfies (A3) for $m_0 > 0,\ b \geq 0,\ \gamma \geq 1,\ s \geq 0$.

Definition. A solution u of $(5.1), (1.2), (1.3)$ is called blow-up if there exists a finite time T^* such that

$$\lim_{t \to T^{*-}} \left(\int_\Omega |\nabla u|^2 \, dx \right)^{-1} = 0.$$

Now, let u be a solution of (5.1) and define

$$a(t) = \int_\Omega u^2 dx + \int_0^t \int_\Omega |\nabla u|^2 \, dx dt,\ t \geq 0. \tag{5.4}$$

Lemma 5.1. *Assume that* (A1)$-$(A4) *hold, then we have*

$$a''(t) - 4(\delta + 1) \int_\Omega u_t^2 dx \geq (-4 - 8\delta)\, E(0) + (4 + 8\delta) \int_0^t \|\nabla u_t\|_2^2 \, dt. \tag{5.5}$$

Proof. Form (5.4), we have

$$a'(t) = 2 \int_\Omega u u_t dx + \|\nabla u\|_2^2 . \tag{5.6}$$

By (5.1) and Divergence theorem, we get

$$\begin{aligned} a''(t) = {} & 2\, \|u_t\|_2^2 - 2M\left(\|\nabla u\|_2^2\right) \|\nabla u\|_2^2 + 2 \int_\Omega f(u) u dx \\ & +2 \int_0^t \int_\Omega g(t-s) \nabla u(s) \cdot \nabla u(t) dx ds. \end{aligned} \tag{5.7}$$

By (5.2) and (5.3), we have from (5.7)

$$\begin{aligned} & a''(t) - 4(\delta + 1)\, \|u_t\|_2^2 \\ \geq {} & (-4-8\delta)\, E(0) + (4+8\delta) \int_0^t \|\nabla u_t(t)\|_2^2 \, dt + \int_\Omega 2\left[f(u)u - (2+4\delta) F(u)\right] dx \\ & + \left\{ (2+4\delta)\, \overline{M}\left(\|\nabla u(t)\|_2^2\right) - \left[2M\left(\|\nabla u(t)\|_2^2\right) + (2+4\delta) \int_0^t g(s) ds \right] \|\nabla u(t)\|_2^2 \right\} \\ & +2 \int_0^t \int_\Omega g(t-s) \nabla u(s) \cdot \nabla u(t) dx ds - (2+4\delta) \int_0^t (g' \diamond \nabla u)(t) dt \\ & +(2+4\delta)(g \diamond \nabla u)(t). \end{aligned}$$

By using Hölder inequality and Young's inequality, we have

$$
\begin{aligned}
&\int_\Omega\int_0^t g(t-s)\nabla u(s)\cdot\nabla u(t)dsdx \\
(5.8)\quad &= \int_\Omega\int_0^t g(t-s)\nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\,dsdx+\int_0^t g(t-s)ds\,\|\nabla u(t)\|_2^2 \\
&\geq -\left[\frac{1}{2}(g\diamond\nabla u)(t)+\frac{1}{2}\int_0^t g(s)ds\,\|\nabla u(t)\|_2^2\right]+\int_0^t g(s)ds\,\|\nabla u(t)\|_2^2\,.
\end{aligned}
$$

Then by (5.8), we get

$$
\begin{aligned}
& a''(t)-4(\delta+1)\|u_t\|_2^2 \\
\geq\ & (-4-8\delta)E(0)+(4+8\delta)\int_0^t\|\nabla u_t(t)\|_2^2\,dt \\
& +\int_\Omega 2\left[f(u)u-(2+4\delta)F(u)\right]dx \\
& +\Big\{(2+4\delta)\,\overline{M}\left(\|\nabla u(t)\|_2^2\right)-\Big[2M\left(\|\nabla u(t)\|_2^2\right) \\
& +\ (1+4\delta)\int_0^t g(s)ds\Big]\,\|\nabla u(t)\|_2^2\Big\} \\
& +2\int_0^t\int_\Omega g(t-s)\nabla u(s)\cdot\nabla u(t)dxds-(2+4\delta)\int_0^t(g'\diamond\nabla u)(t)dt \\
& +(1+4\delta)(g\diamond\nabla u)(t).
\end{aligned}
$$

Therefore by (A3), (A4) and (A1), we obtain (5.5).

Now, we consider three different cases on the sign of the initial energy $E(0)$.

(1) If $E(0)<0$, then from (5.5), we have

$$a'(t)\geq a'(0)-4(1+2\delta)E(0)\,t,\ \ t\geq 0.$$

Thus we get $a'(t)>\|\nabla u_0\|_2^2$ for $t>t^*$, where

$$t^*=\max\left\{\frac{a'(0)-\|\nabla u_0\|_2^2}{4(1+2\delta)E(0)},\ 0\right\}. \tag{5.9}$$

(2) If $E(0)=0$, then $a''(t)\geq 0$ for $t\geq 0$.
Furthermore, if $a'(0)>\|\nabla u_0\|_2^2$, then $a'(t)>\|\nabla u_0\|_2^2,\ t\geq 0$

(3) For the case that $E(0) > 0$, we first note that

$$2\int_0^t \int_\Omega \nabla u \cdot \nabla u_t dxdt = \|\nabla u(t)\|_2^2 - \|\nabla u_0\|_2^2 . \tag{5.10}$$

By using Hölder inequality and Young's inequality, we have from (5.10)

$$\|\nabla u(t)\|_2^2 \leq \|\nabla u_0\|_2^2 + \int_0^t \|\nabla u(t)\|_2^2 \, dt + \int_0^t \|\nabla u_t(t)\|_2^2 \, dt. \tag{5.11}$$

By Hölder inequality and Young's inequality in (5.6) and by (5.11), we get

$$a'(t) \leq a(t) + \|\nabla u_0\|_2^2 + \|u_t\|_2^2 + \int_0^t \|\nabla u_t(t)\|_2^2 \, dt. \tag{5.12}$$

Hence by (5.5) and (5.12), we obtain

$$a''(t) - 4(\delta+1) a'(t) + 4(\delta+1) a(t) + K_1 \geq 0,$$

where

$$K_1 = (4+8\delta) E(0) + 4(\delta+1) \|\nabla u_0\|_2^2 \; .$$

Let

$$b(t) = a(t) + \frac{K_1}{4(1+\delta)}, \; t > 0.$$

Then $b(t)$ satisfies (2.1). By (2.2), we see that if

$$a'(0) > r_2 \left[a(0) + \frac{K_1}{4(1+\delta)} \right] + \|\nabla u_0\|_2^2 , \tag{5.13}$$

then $a'(t) > \|\nabla u_0\|_2^2$, $t > 0$.

Consequently, we have

Lemma 5.2. *Assume that* (A1)$-$(A4) *hold and that either one of the following conditions is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \|\nabla u_0\|_2^2$,

(iii) $E(0) > 0$ *and* (5.13) *holds, then* $a'(t) > \|\nabla u_0\|_2^2$ *for* $t > t_0$, *where* $t_0 = t^*$ *is given by* (5.9) *in case* (i) *and* $t_0 = 0$ *in cases* (ii) *and* (iii).

Now, we will find the estimate for the life span of $a(t)$.

Let

$$J(t) = (a(t) + (T_1 - t) \|\nabla u_0\|_2^2)^{-\delta}, \text{ for } t \in [0, T_1], \tag{5.14}$$

where $T_1 > 0$ is a certain constant which will be specified later.

Then we have

$$J'(t) = -\delta J(t)^{1+\frac{1}{\delta}} \left(a'(t) - \|\nabla u_0\|_2^2 \right)$$

and

$$J''(t) = -\delta J(t)^{1+\frac{2}{\delta}} V(t), \tag{5.15}$$

where

$$V(t) = a''(t) \left(a(t) + (T_1 - t) \|\nabla u_0\|_2^2 \right) - (1+\delta) \left(a'(t) - \|\nabla u_0\|_2^2 \right)^2. \tag{5.16}$$

For simplicity of calculation, we denote

$$\begin{aligned} P &= \int_\Omega u^2 dx, \\ Q &= \int_0^t \|\nabla u(t)\|_2^2 \, dt, \\ R &= \int_\Omega u_t^2 dx, \\ S &= \int_0^t \|\nabla u_t(t)\|_2^2 \, dt. \end{aligned}$$

From (5.6), by (5.10) and Hölder inequality, we get

$$\begin{aligned} a'(t) &= 2 \int_\Omega u_t u dx + \|\nabla u_0\|_2^2 + 2 \int_0^t \int_\Omega \nabla u \cdot \nabla u_t dx dt \\ &\leq 2(\sqrt{RP} + \sqrt{QS}) + \|\nabla u_0\|_2^2 . \end{aligned} \tag{5.17}$$

By (5.5), we have

$$a''(t) \geq (-4 - 8\delta) E(0) + 4(1+\delta)(R+S). \tag{5.18}$$

Thus, by using (5.17) and (5.18) in (5.16), we obtain

$$\begin{aligned} V(t) \ \geq \ & [(-4 - 8\delta) E(0) + 4(1+\delta)(R+S)] (a(t) + (T_1 - t) \|\nabla u_0\|_2^2) \\ & -4(1+\delta)(\sqrt{RP} + \sqrt{QS})^2. \end{aligned}$$

And by (5.14), we have

$$\begin{aligned} V(t) \ \geq \ & (-4 - 8\delta) E(0) J(t)^{-\frac{1}{\delta}} + 4(1+\delta)(R+S)(T_1 - t) \|\nabla u_0\|_2^2 \\ & +4(1+\delta) \left[(R+S)(P+Q) - (\sqrt{RP} + \sqrt{QS})^2 \right]. \end{aligned}$$

By Schwarz inequality, the last term in the above inequality is nonnegative. Hence we have

$$V(t) \geq (-4-8\delta) E(0) J(t)^{-\frac{1}{\delta}}, \ t \geq t_0. \tag{5.19}$$

Therefore by (5.15) and (5.19), we get

$$J''(t) \leq \delta(4+8\delta) E(0) J(t)^{1+\frac{1}{\delta}}, \ t \geq t_0. \tag{5.20}$$

Note that by Lemma 5.2, $J'(t) < 0$ for $t > t_0$. Multiplying (5.20) by $J'(t)$ and integrating it from t_0 to t, we have

$$J'(t)^2 \geq \alpha + \beta J(t)^{2+\frac{1}{\delta}} \ \text{ for } t \geq t_0,$$

where

$$\alpha = \delta^2 J(t_0)^{2+\frac{2}{\delta}} \left[(a'(t_0) - \|\nabla u_0\|_2^2)^2 - 8E(0) J(t_0)^{\frac{-1}{\delta}} \right] \tag{5.21}$$

and

$$\beta = 8\delta^2 E(0). \tag{5.22}$$

We observe that

$$\alpha > 0 \ \text{ iff } \ E(0) < \frac{\left(a'(t_0) - \|\nabla u_0\|_2^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\|\nabla u_0\|_2^2\right]}.$$

Then by Lemma 2.3, there exists a finite time T^* such that $\lim_{t \to T^*-} J(t) = 0$ and the upper bounds of T^* are estimated respectively according to the sign of $E(0)$. This will imply that

$$\lim_{t \to T^*-} \left\{ \int_\Omega u^2 dx + \int_0^t \|\nabla u\|_2^2 \, dt \right\}^{-1} = 0.$$

Thus by Poincaré inequality, we deduce

$$\lim_{t \to T^*-} \left(\int_\Omega |\nabla u|^2 \, dx \right)^{-1} = 0. \tag{5.23}$$

Theorem 5.3. *Assume that* (A1)-(A4) *hold and that either one of the following conditions is satisfied:*

(i) $E(0) < 0$,

(ii) $E(0) = 0$ *and* $a'(0) > \|\nabla u_0\|_2^2$,

(iii) $0 < E(0) < \frac{\left(a'(t_0) - \|\nabla u_0\|_2^2\right)^2}{8\left[a(t_0) + (T_1 - t_0)\|\nabla u_0\|_2^2\right]}$ *and* (5.13) *holds, then the solution* u *blows up at finite time* T^* *in the sense of* (5.23).

In case (i),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{5.24}$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\alpha}{-\beta}}\right\}$, *then we have*

$$T^* \leq t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}. \tag{5.25}$$

In case (ii),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)} \tag{5.26}$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\alpha}}. \tag{5.27}$$

In case (iii),

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}} \tag{5.28}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\alpha}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}, \tag{5.29}$$

where $c = \left(\frac{\beta}{\alpha}\right)^{\frac{\delta}{2+\delta}}$, *here* α *and* β *are in* (5.21) *and* (5.22) *respectively.*

Note that in case (i), $t_0 = t^*$ is given in (5.9) and $t_0 = 0$ in case (ii) and (iii).

Remark. The choice of T_1 in (5.14) is possible under some conditions. We shall discuss it as follows :

(i) for the case $E(0) = 0$,

First, we note that the condition $a'(0) > \|\nabla u_0\|_2^2$ implies $\int_\Omega u_0 u_1 dx > 0$.

By (5.26), we choose

$$T_1 \geq -\frac{J(0)}{J'(0)}.$$

Then, by Hölder inequality, Poincaré inequality and Young's inequality, we have

$$\|u_0\|_2^2 + T_1 \|\nabla u_0\|_2^2 \leq \delta \left(\varepsilon B_1^2 \|\nabla u_0\|_2^2 + \frac{1}{\varepsilon} \|u_1\|_2^2 \right) T_1.$$

where ε is some positive constant.

Choosing $\varepsilon = \frac{1}{\delta B_1^2}$, we get

$$T_1 \geq \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}.$$

In particular, we choose T_1 as

$$T_1 = \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}.$$

We then get

$$T^* \leq \frac{\|u_0\|_2^2}{\delta^2 B_1^2 \|u_1\|_2^2}.$$

(ii) for the case $E(0) < 0$,

(1) If $\int_\Omega u_0 u_1 dx > 0$, then $a'(t) > \|\nabla u_0\|_2^2$ and $t^* = 0$. Thus T_1 can be chosen as in **(i)**.

(2) If $\int_\Omega u_0 u_1 dx \leq 0$, then $t^* = \frac{a'(0) - \|\nabla u_0\|_2^2}{4(1+2\delta)E(0)}$. Thus, by (5.24), we choose $T_1 \geq t^* - \frac{J(t^*)}{J'(t^*)}$.

(iii) for the case $E(0) > 0$. Under the condition

$$E(0) < \min\{\kappa_1, \kappa_2\},$$

where

$$\kappa_1 = \frac{(1+\delta)\left[a'(0) - r_2 a(0) - (r_2+1)\|\nabla u_0\|_2^2\right]}{r_2(1+2\delta)},$$

and

$$\kappa_2 = \frac{\left[4\left(\int_\Omega u_0 u_1 dx\right)^2 - 1\right]\left[\delta - \|\nabla u_0\|_2^2\right]}{8\delta \|\nabla u_0\|_2^2}.$$

If $\|\nabla u_0\|_2^2 < \delta$, T_1 is chosen to satisfy

$$\kappa_3 \leq T_1 \leq \kappa_4,$$

here

$$\kappa_3 = \frac{\|u_0\|_2^2}{\delta - \|\nabla u_0\|_2^2},$$

$$\kappa_4 = \frac{4\left(\int_\Omega u_0 u_1 dx\right)^2 - 8E(0)\|u_0\|_2^2 - 1}{8E(0)\|\nabla u_0\|_2^2}.$$

Therefore we have

$$T \leq T^* \leq \frac{\kappa_3}{\sqrt{4(\int_\Omega u_0 u_1 dx)^2 - 8E(0)\kappa_3}}.$$

References

1. J. Ball, Remarks on blow up and nonexistence theorems for nonlinear evolution equations, *Quart. J. Math. Oxford*, **28** (1977), 473-486.
2. M. M. Cavalcanti, Domingos Cavalcanti, V. N. and Soriano, J. A., Exponential decay for the solution of semilinear viscoleastic wave equation with localized damping, *Electronic J. Diff. Eqns.*, **44** (2002), 1-14.
3. R. T. Glassey, Blow-up theorems for nonlinear wave equations, *Math. Z.*, **132** (1973), 183-203.
4. A. Haraux and E. Zuazua, Decay estimates for some semilinear damped hyperbolic problems, *Arch. Rational Mech. Anal.*, **100** (1988), 191-206.
5. M. Hosoya and Y. Yamada, On some nonlinear wave equations II : global existence and energy decay of solutions, *J. Fac. Sci. Univ. Toyko Sect. IA Math.*, **38** (1991), 239-250.
6. R. Ikehata, A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms, *Differential and Integral Equations*, **8** (1995), 607-616.
7. S. Jiang and J. E. Munoz Rivera, A global existence theorem for the Dirichlet problem in nonlinear n-dimensional viscoelastic, *Differential and Integral Equations*, **9** (1996), 791-810.
8. V. K. Kalantarov and O. A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, *J. Soviet Math.*, **10** (1978), 53-70.
9. G. Kirchhoff, *Vorlesungen über Mechanik*, Leipzig, Teubner, 1883.

10. M. Kopackova, Remarks on bounded solutions of a semilinear dissipative hyperbolic equation, *Comment. Math. Univ. Carloin.*, **30** (1989), 713-719.

11. H. A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $Du_{tt} = Au + F(u)$, *Trans. Amer. Math. Soc.*, **192** (1974), 1-21.

12. T. Matsutama and R. Ikehata, On global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping, *J. Math. Anal. and Appli.*, **204** (1996). 729-753.

13. M. R. Li and L. Y. Tsai, Existence and nonexistence of global solutions of some systems of semilinear wave equations, *Nonlinear Anal., Theory, Methods & Applications,* **54** (2003), 1397-1415.

14. J. E. Munoz Rivera, Global solution on a quasilinear wave equation with memory, *Bolletino U.M.I.*, **7(8B)** (1994), 289-303.

15. M. Nako, A difference inequality and its application to nonlinear evolution equations, *J. Math. Soc. Japan*, **30** (1978), 747-762.

16. K. Nishihara and Y. Yamada, On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms, *Funkcial Ekvac.*, **33** (1990), 151-159.

17. K. Ono, On global existence, asymptotic stability and blowing up of solutions for some degenerate nonlinear wave equations of Kirchhoff type with a strong dissipation, *Math. Methods in The Appl. Sci.*, **20** (1997), 151-177.

18. K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. and Appli.*, **216** (1997), 321-342.

19. S. T. Wu and L. Y. Tsai, Blow-up of solutions for some nonlinear wave equations of Kirchhoff type with some dissipation, to appear in *Nonlinear Anal., Theory, Methods & Applications.*

20. S. T. Wu and L. Y. Tsai, On global solutions and blow-up solutions for a nonlinear viscoelastic wave equation with nonlinear damping, National Chengchi University, preprint, 2004.

21. R. M. Torrejón and J. Young, On a quasilinear wave equation with memory, *Nonlinear Anal., Theory, Methods & Applications*, **16** 1991), 61-78.

22. E. Zuazua, Exponential decay for the semilinear wave equation with locally distributed damping, *Comm. PDE.*, **15** (1990), 205-235.

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