TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 4, pp. 881-890, June 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

REGULAR ELEMENTS WHICH IS A SUM OF AN IDEMPOTENT AND A LEFT CANCELLABLE ELEMENT

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Abstract. Let M be a right R-module, and let $a \in End_RM$ be unit-regular. If $End_R(Ima)$ is an exchange ring and $End_R(Kera)$ has stable rank one, it is shown that there exist an idempotent $e \in End_RM$ and a left cancellable $u \in End_RM$ such that $a = e + u$ and $aM \cap eM = 0$.

1. INTRODUCTION

A ring R is an exchange ring if for every right R-module A and two decompositions $A = M \oplus N = \oplus_{i\in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\oplus_{i\in I} A_i')$. It is well known that a ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$. Clearly, regular rings, π-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero (cf. [2, Theorem 7.2]) are all exchange rings. We say that a right R-module M has the finite exchange property if and only if End_RM is an exchange ring. A ring R has stable rank one in case $aR + bR = R$ with $a, b \in R$ implies that there exists $y \in R$ such that $a + by$ is a unit of R. We know that a right R-module M can be cancelled from direct sums if and only if End_RM has stable rank one. Also we know that every strongly π-regular ring has stable rank one.

Recall that an element $x \in R$ is clean provided that it is a sum of an idempotent and a unit. We say that a ring R is clean if every element in R is clean. Many author investigated clean rings such as [1],[4-7] and [10-16]. Answering a question of Nilcholson, Camillo and Yu [5, Theorem 5] claimed that every unit-regular ring is clean. But there was a gap in their proof. Camillo and Khurana proved this result

Received April 1, 2004; accepted September 14, 2004.
Communicated by Shun-Jen Cheng.
2000 *Mathematics Subject Classification*: 15E50, 19B10.
Key words and phrases: Exchange ring, Stable rank one, Idempotent.

by a new route and gave a characterization of unit regular rings. They proved a ring R is unit-regular if and only if for any $a \in R$ there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$. In this paper, we extend Camillo and Khurana's result to exchange rings and get a new characterization of a regular element which is a sum of an idempotent and a left cancellable.

Throughout the paper, every ring is associative with an identity. An element $x \in R$ is regular if there exists $y \in R$ such that $x = xyx$. If y can be chosen to be a unit, we say that x is unit-regular. A ring R is (unit) regular in case every element in R is (unit) regular. An element $u \in R$ is said to be left(right) cancellable in case for any $x, y \in R$, $ax = ay(xa = ya)$ implies $x = y$. We use $U(R)$ to denote the set of all units in R.

Theorem 1. *Let M be a right R-module, and let $a \in End_RM$ be unit-regular. If $End_R(Ima)$ is an exchange ring and $End_R(Kera)$ has stable rank one, then there exist an idempotent $e \in End_RM$ and a left cancellable $u \in End_RM$ such that $a = e + u$ and $aM \cap eM = 0$.*

Proof. Set $E = End_RM$. Since $a \in E$ is regular, we have $x \in E$ such that $a = axa$. So $M = Ima \oplus (1_M - ax)M = xaM \oplus Kera$. As $End_R(Ima)$ is an exchange ring, there exist right R-modules X_1, Y_1 such that $M = Ima \oplus X_1 \oplus Y_1$ with $X_1 \subseteq Kera$ and $Y_1 \subseteq xaM$. Clearly, $Kera = Kera \cap (X_1 \oplus Ima \oplus Y_1) = X_1 \oplus X_2$, where $X_2 = Kera \cap (Ima \oplus Y_1)$. Likewise, we have a right R-module Y_2 such that $xaM = Y_1 \oplus Y_2$. Since $a \in E$ is unit-regular, we get $Kera \cong M/Ima$; hence, $X_1 \oplus X_2 \cong Kera \cong Cokera \cong X_1 \oplus Y_1$. So we have an isomorphism $k : X_1 \oplus X_2 \to X_1 \oplus Y_1$. As $End_R(Kera)$ has stable rank one, so has End_RX_1. Thus X_1 can be cancelled from direct sums, so we get a right R-module isomorphism $\psi : X_2 \to Y_1$.

Let $h : M = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = M$ given by $h(x_1 + x_2 + y_1 + y_2) = k(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v : M = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = M$ given by $v(x_1 + y_1 + x_2 + y_2) = k^{-1}(x_1 + y_1) + \psi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. For any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$, we have $hvh(x_1 + x_2 + y_1 + y_2) = hv\big(k(x_1 + x_2) + y_1\big) = h\big(x_1 + x_2 + k^{-1}(y_1)\big) = k(x_1 + x_2) + y_1 = h(x_1 + x_2 + y_1 + y_2)$; hence $h = hvh$. Set $e = hv$. Then $e \in E$ is an idempotent.

Assume that $(a - hv)(x_1 + y_1 + x_2 + y_2) = 0$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. Then $a(y_1 + y_2) = x_1 + y_1 + \psi(x_2) \in Ima \cap (X_1 \oplus Y_1) = 0$, and then $x_1 = -y_1 - \psi(x_2) \in X_1 \cap Y_1 = 0$. It follows from $a(y_1 + y_2) = 0$ that $y_1 + y_2 = (1 - xa)(y_1 + y_2) = Kera \in (X_1 \oplus X_2) \cap (Y_1 \oplus Y_2) = 0$; hence $y_1 + y_2 = 0$. This infers that $y_1 = -y_2 \in Y_1 \cap Y_2 = 0$, and then $y_1 = y_2 = 0$. Furthermore, we get $\psi(x_2) = -y_1 = 0$. As ψ is an isomorphism, we have $x_2 = 0$.

Thus $x_1 + y_1 + x_2 + y_2 = 0$. This means that $a - e \in R$ is left cancellable. Let $u = a - e$. Then $a = e + u$. Furthermore, we get $aM \cap eM \subseteq aM \cap (X_1 \oplus Y_1) = 0$. This implies that $aM \cap eM = 0$. ■

Let F be a field of characteristic 2. For any $a \in F[x]/(x^2)$, we have $b \in F[x]/(x^2)$ such that $a^2 = ba^3$. Hence $F[x]/(x^2)$ is strongly π-regular, and then $F[x]/(x^2)$ is an exchange ring having stable rank one. In addition, it is easy to show that every left cancellable element in a strongly strongly ring is a unit. It follows by Theorem 1 that for any regular $c \in F[x]/(x^2)$, there exist an idempotent $e \in F[x]/(x^2)$ and a unit $u \in F[x]/(x^2)$ such that $c = e + u$ and $c\big(F[x]/(x^2)\big) \cap e\big(F[x]/(x^2)\big) = 0$. But we note that $F[x]/(x^2)$ is not regular because $J\big(F[x]/(x^2)\big) = \big(x + (x^2)\big) \neq 0$. This means that Theorem 1 is a nontrivial generalization of [4, Theorem 1].

Corollary 2. *Let V be a right vector space over a division ring, and let $R = End_D V$. If $x \in R$ is congruent modulo $Soc(R)$ to a unit, then there exist an idempotent $e \in R$ and a left invertible $u \in R$ such that $a = e + u$ and $aV \cap eV = 0$.*

Proof. Since $x \in R$ is congruent modulo $Soc(R)$ to a unit, by [3, Lemma 3.3], $dim_D(Kerx) = dim_D(Cokerx) < \infty$. It follows from $dim_D(Kerx) = dim_D(Cokerx)$ that $x \in R$ is unit-regular. It follows from $dim_D(Kerx) < \infty$ that $End_D(Kerx)$ has stable rank one. In view of Theorem 1, there exist an idempotent $e \in R$ and a left cancellable element $u \in R$ such that $a = e + u$ and $aV \cap eV = 0$. Since R is a regular ring, we have a $v \in R$ such that $u = uvu$; hence, $vu = 1$. That is, $u \in R$ is left invertible. Therefore we complete the proof. ■

Let V be a right vector space over a division ring, and let $R = End_D V$. Very recently, Nicholson et al. proved that for any $a \in R$, there exist an idempotent $e \in R$ and an invertible $u \in R$ such that $a = e + u$(see [16, Lemma 1]). But we claim that $aV \bigcap eV = 0$ may be not true. Let V be an infinitely dimensional vector space over a division ring D with a basis $\{x_1, x_2, \cdots, x_n, \cdots\}$. Define $\sigma : V \to V$ given by $\sigma(x_i) = x_{i+1} (i = 1, 2, \cdots)$ and $\tau : V \to V$ given by $\tau(x_1) = 0, \tau(x_i) = x_{i-1} (i = 2, 3, \cdots)$. Clearly, $\tau\sigma = 1_V$ and $\sigma\tau \neq 1_V$. By [16, Lemma 1], there exist an idempotent $e \in R$ and an invertible $u \in R$ such that $\sigma = e + u$. If $\sigma V \bigcap eV = 0$, then $\sigma u^{-1} e = (e + u)u^{-1} e = eu^{-1} e + e \in aV \bigcap eV = 0$; hence, $\sigma u^{-1}(\sigma - u) = 0$. This implies that $\sigma = u \in U(R)$, a contradiction. Therefore $aV \bigcap eV \neq 0$.

Recall that an ideal I of a ring R is of bounded index if there is a positive integer n such that $x^n = 0$ for any nilpotent $x \in I$. Let $a \in R$. We use a_L to denote the right R-module homomorphism from R to R given by $a_L(r) = ar$ for any $r \in R$.

Corollary 3. *Let I be a bounded ideal of an exchange ring R. Then the following hold:*

(1) *For any unit-regular* $a \in 1 + I$*, there exist an idempotent* $e \in R$ *and a left cancellable* $u \in R$ *such that* $a = e + u$ *and* $aR \cap eR = 0$*.*

(2) *For any unit-regular* $a \in 1 + I$*, there exist an idempotent* $e \in R$ *and a right cancellable* $u \in R$ *such that* $a = e + u$ *and* $Ra \cap Re = 0$*.*

Proof. (1) Let $a \in 1 + I$ be unit-regular. Then we have a unit $x \in 1 + I$ such that $a = axa$. Hence $a_L \in End_R R$ is unit-regular. Clearly, $End_R(Ima_L)$ is an exchange ring. On the other hand, $End_R(Kera_L) = (1 - xa)R(1 - xa)$. Since I is a bounded ideal of R, $(1 - xa)R(1 - xa)$ is an exchange ring of bounded index. By [18, Corollary 4], $End_R(Kera_L)$ has stable rank one. It follows by Theorem 1 that there exist an idempotent $e_L \in End_R R$ and a left cancellable $u_L \in End_R R$ such that $a_L = e_L + u_L$ and $a_L R \cap e_L R = 0$. Let $e = e_L(1)$ and $u = u_L(1)$. Then $e \in R$ is an idempotent and $u \in R$ is left cancellable, as required.

(2) Let R^{op} be the opposite ring of R. Then I^{op} is a bounded ideal of the exchange ring R^{op}. Applying (1) to $a^{op} \in R^{op}$, we obtain the result. ■

Let I be an ideal of a ring R. We say that I has stable rank one provided that $aR + bR = R$ with $a \in 1 + I$ and $b \in R$ implies that there exists $y \in R$ such that $a + by$ is a unit of R. An ideal I of an exchange ring R has stable rank one if and only if for any regular $a \in 1 + I$, there exists a unit $u \in R$ such that $a = aua$(See [7, Proposition 2.3]). It is well known that every bounded ideal of a regular ring has stable rank one. We note that an ideal I has stable rank one only depends on the ring structure of I and doesn't depend on the choice of R. In other words, I has stable rank one as an ideal of R if and only if I has stable rank one as a non-unital ring.

Theorem 4. *Let* I *be an ideal of an exchange ring* R*. If* I *has stable rank one, then for any regular* $a \in 1 + I$*, there exist an idempotent* $e \in I$ *and a left cancellable* $u \in 1 + I$ *such that* $a = e + u$ *and* $aR \cap eR = 0$*.*

Proof. Let $a \in 1 + I$ be regular. Then $a = axa$ for some $x \in R$. Since I has stable rank one, it follows by [7, Proposition 2.3] that $a \in R$ is unit-regular. This means that a_L is unit-regular. Obviously, $End_R(Ima_L)$ is an exchange ring and $End_R(Kera_L)$ has stable rank one. Similarly to Theorem 1, we get $R = a_L R \oplus (1_R - a_L x_L)R = x_L a_L R \oplus (1_R - x_L a_L)R$. Since R is an exchange ring, we have right R-modules X_1, Y_1 such that $R = aR \oplus X_1 \oplus Y_1$ with $X_1 \subseteq (1 - xa)R$ and $Y_1 \subseteq xaR$. Furthermore, we have right R-modules X_2 and Y_2 such that $(1 - xa)R = X_1 \oplus X_2$ and $xaR = Y_1 \oplus Y_2$. Also we have $k : X_1 \oplus Y_1 \cong Cokera \cong Kera \cong X_1 \oplus X_2$ and $\psi : X_2 \cong Y_1$. Let $h : R = X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 \to X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 = R$ given by $h(x_1 + x_2 + y_1 + y_2) = k^{-1}(x_1 + x_2) + y_1$ for any $x_1 \in X_1, x_2 \in X_2, y_1 \in Y_1, y_2 \in Y_2$. Let $v : R = X_1 \oplus Y_1 \oplus X_2 \oplus Y_2 \to$

$X_1 \oplus X_2 \oplus Y_1 \oplus Y_2 = R$ given by $v(x_1 + y_1 + x_2 + y_2) = k(x_1 + y_1) + \psi(x_2)$ for any $x_1 \in X_1, y_1 \in Y_1, x_2 \in X_2, y_2 \in Y_2$. Let $e = h(1)v(1)$. Analogously to Theorem 1, we get $a = e + u$ and $aR \cap eR = 0$.

Assume that $1 = a_1 + b_1 + a_2 + b_2$ with $a_1 \in X_1, b_1 \in Y_1, a_2 \in X_2, b_2 \in Y_2$. Clearly, $a_2 = a_2^2$. Then we have $h(1)v(1) = hv(a_1+b_1+a_2+b_2) = a_1+b_1+\psi(a_2)$. So we have some $r \in R$ such that $a_1 + b_1 = k^{-1}\big((1 - xa)r\big) = k^{-1}(1 - xa)(1 - xa)r \in I$. Also we have some $t \in R$ such that $a_2 = (1 - xa)t \in I$; hence $\psi(a_2) = \psi(a_2)(1 - xa)t \in I$. This shows that $e \in I$, as desired. ■

Let I be an ideal of a exchange ring R. Since R is an exchange ring, so is the opposite ring R^{op}. Also we know that if I has stable rank one then so does I^{op}. Applying Theorem 4 to the ideal I^{op} of the ring R^{op}, we prove that for any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a right cancellable $u \in 1 + I$ such that $a = e + u$ and $Ra \cap Re = 0$. We note that the matrix $[[\delta_{i2j}]] \in \mathbb{CFM}_{\mathbb{N}}(\mathbb{R})$ is left cancellable, while it is not right cancellable. We don't know whether "a left cancellable $u \in 1 + I$" could be replaced by "a unit $u \in 1 + I$ in the proceeding theorem. A ring R is cohopfian if any injective right R-module homomorphism from R to R is an isomorphism. As a consequence of Theorem 4, we now derive the following.

Corollary 5. *Let I be an ideal of a cohopfian exchange ring R. Then the following are equivalent:*

(1) *I has stable rank one.*

(2) *For any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in 1 + I$ be regular. By Theorem 4, there exist an idempotent $e \in I$ and a left cancellable $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$. Let $u_L : R \to R$ given by $u_L(r) = ur$ for any $r \in R$. Since $u \in R$ is cancellable, u_L is injective. As R is a cohopfian ring, u_L is an isomorphism. Assume that $u_L v = 1 = v u_L$ for a $v \in End_R R$. This infers that $u = v(1)^{-1} \in U(R)$, as required.

(2) $\Rightarrow$ (1) For any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$. Hence $au^{-1}e = (e + u)u^{-1}e = eu^{-1}e + e \in aR \cap eR = 0$, and then $au^{-1}(a - u) = 0$. This gives $a = au^{-1}a$. So I has stable rank one by [7, Proposition 2.3]. ■

Recall that a ring R is said to be strongly π-regular in case for any $x \in R$ there exist a positive integer n and a $y \in R$ such that $x^n = x^{n+1}y$. A right R-module M is said to satisfy Fitting's lemma if, for all $f \in End_R M$, there exists a positive integer n such that $M = f^n(M) \oplus Ker(f^n)$. It is well known that a module satisfies Fitting's lemma if and only if its endomorphism ring is a strongly π-regular ring. Also we know that every strongly π-regular ring is a cohopfian exchange ring having stable rank one. Let R be a strongly π-regular ring. Using Corollary 5, we prove that $x \in R$ is regular if and only if there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$.

Let $R = M_2\big(F[x]/(x^2)\big)$, where F is a field. Then R is strongly π-regular, so it is a clean ring. Let $a = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{x} \end{pmatrix} \in R$, and let $u = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix}$. Then $a = aua$ with $u \in U(R)$; hence, a is unit-regular. Thus we have an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $aR \cap eR = 0$. But a^2 can not be written in the form above. This is because a^2 is not regular. In other words, some elements in a ring R can be written in this form, while the other elements can not be written in this form.

A ring R is a π-regular ring in case for any $a \in R$ there exists a positive integer $n(x)$ such that $a^{n(x)} = a^{n(x)}ca^{n(x)}$ for a $c \in R$. Clearly, every π-regular ring is an exchange ring.

Corollary 6. *Let I be an ideal of a π-regular ring R. Then the following are equivalent:*

(1) *I has stable rank one.*

(2) *For any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$.*

Proof. $(1) \Rightarrow (2)$ Let $a \in 1 + I$ be regular. By Theorem 4, there exist an idempotent $e \in I$ and a left cancellable $u \in 1 + I$ such that $a = e + u$ and $aR \cap eR = 0$. Since R is π-regular ring, we have a positive integer n such that $u^n = u^n v u^n$ for a $v \in R$. Hence $u^n(1 - vu^n) = 0$. As u is left cancellable, we deduce that $vu^n = 1$. Clearly, $v \in 1 + I$. From $vu^n + 0 = 1$, we can find a $y \in R$ such that $v = v + 0 \times y \in U(R)$ because I has stable rank one. This means that $u \in U(R)$.

$(2) \Rightarrow (1)$ is analogous to Corollary 5. ■

Let I be an ideal of a π-regular ring R. Analogously, we prove that I has stable rank one if and only if for any regular $a \in 1 + I$, there exist an idempotent $e \in I$ and a unit $u \in 1 + I$ such that $a = u - e$ and $aR \cap eR = 0$. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0 \text{ and } 3 \nmid b \text{ and } 5 \nmid b\}$. By [1, Proposition 16], each element $a \in R$ can be written in the form $a = u + e$ or $a = u - e$ where $u \in U(R)$ and $e \in R$ is an idempotent. But R is not a clean ring. In other words, there exists an element $a \in R$ which is not a sum of an idempotent and a unit can be written in the form $a = u - e$ where $u \in U(R)$ and $e \in R$ is an idempotent.

Corollary 7. *Let R be a regular ring, and let $a \in R$. If RaR has stable rank one, then there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = e + u$ and $(1-a)R \cap (1-e)R = 0$.*

Proof. Let $I = RaR$ and $b = 1 - a$. Then I has stable rank one and $b \in 1 + I$. By Theorem 4, there exist an idempotent $f \in I$ and a left cancellable $v \in 1 + I$ such that $b = f + v$ and $bR \cap fR = 0$. As R is regular, there exists a $w \in R$ such that $v = vwv$. So we see that $v \in 1 + I$ is left invertible. On the other hand, I has stable rank one. Hence $v \in 1+I$ is a unit. Let $e = 1-f$. Then $e \in R$ is an idempotent. In addition, we have $a = 1 - b = e + (-u)$. Set $u = -v$. Then $v \in R$ is a unit and $a = e + u$. Furthermore, we have $(1-a)R \cap (1-e)R = 0$, as required. ■

Let R be a regular ring, and let $A = (a_{ij}) \in M_n(R)$. If every $Ra_{ij}R$ has stable rank one, we claim that there exist an idempotent $E \in M_n(R)$ and an invertible $U \in M_n(R)$ such that $A = E + U$ and $(I_n - A)M_n(R) \cap (I_n - E)M_n(R) = 0$. Set $I = \sum\limits_{1 \le i,j \le n} Ra_{ij}R$. One easily checks that I has stable rank one. Clearly, $M_n(R)$ is regular. It follows from $M_n(R)AM_n(R) \subseteq M_n(I)$ that $M_n(R)AM_n(R)$ has stable rank one. In view of Corollary 7, we are done.

Let $LTM_n(R)\big(UTM_n(R)\big)$ be the ring of all lower(upper) triangular matrices over a ring R. We note that $LTM_2(R)$ is not a regular ring even if R is regular. The reason is that $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ is not a regular element in $LTM_2(R)$. Now we investigate the conditions under which a triangular matrix can be written in the form above.

Theorem 8. *Let R be regular, and let $A = (a_{ij}) \in LTM_n(R)$. If every $Ra_{ii}R$ has stable rank one, then there exist an idempotent $E \in LTM_n(R)$ and an invertible $U \in LTM_n(R)$ such that $A = E + U$ and $(I_n - A)LTM_n(R) \cap (I_n - E)LTM_n(R) = 0$.*

Proof. If $n = 1$, then the result follows by Corollary 7. Assume that the result holds for $n = k (k \ge 1)$. Let $n = k + 1$. Given any $A = \begin{pmatrix} A_1 & 0 \\ * & a_{nn} \end{pmatrix}$ with any $Ra_{ii}R$ has stable rank one, by the hypothesis, we can find an idempotent $E_1 \in LTM_k(R)$ and an invertible $U_1 \in LTM_k(R)$ such that $A = E_1 + U_1$ and $(I_k - A_1)LTM_k(R) \cap (I_k - E)LTM_k(R) = 0$. Similarly, we can find an idempotent $e_2 \in R$ and an invertible $u_2 \in R$ such that $a_{nn} = e_2 + u_2$ and $(1 - a_{nn})R \cap (1 - e)R = 0$. One easily checks that $A = diag(E_1, e_2) + \begin{pmatrix} U_1 & 0 \\ * & u_2 \end{pmatrix}$. Clearly, $diag(E_1, e_2) \in M_n(R)$ is an idempotent matrix and $\begin{pmatrix} U_1 & 0 \\ * & u_2 \end{pmatrix} \in M_n(R)$ is an invertible triangular matrix. Furthermore, we verify that $(I_n - A)LTM_n(R) \cap (I_n - E)LTM_n(R) = 0$. By induction, we complete the proof. ■

Corollary 9. *Let R be unit-regular, and let $A \in LTM_n(R)$. Then there exist an idempotent $E \in LTM_n(R)$ and an invertible $U \in LTM_n(R)$ such that $A = E + U$ and $(I_n - A)LTM_n(R) \cap (I_n - E)LTM_n(R) = 0$.*

Proof. Since R is unit-regular, it is shown that every $Ra_{ii}R$ has stable rank one. Therefore the result follows by Theorem 8. ■

Let R be unit-regular, and let $A \in UTM_n(R)$. Analogously, we deduce that there exist an idempotent $E = (e_{ij}) \in UTM_n(R)$ and an invertible $U = (u_{ij}) \in UTM_n(R)$ such that $A = E+U$ and $(I_n-A)UTM_n(R)\cap(I_n-E)UTM_n(R) = 0$. Define $QM_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a+c = b+d, a, b, c, d \in R \right\}$.

Corollary 10. *Let $A = (a_{ij})$ be a 2×2 matrix over a unit-regular ring R. If $a_{11} + a_{21} = a_{12} + a_{22}$, then there exist an idempotent $E = (e_{ij}) \in M_2(R)$ and an invertible $U = (u_{ij}) \in M_2(R)$ such that*

(1) $A = E + U$.
(2) $e_{11} + e_{21} = e_{12} + e_{22}$.
(3) $u_{11} + u_{21} = u_{12} + u_{22}$.

Proof. Construct a map $\psi : QM_2(R) \to TM_2(R)$ given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a+c & 0 \\ c & d-c \end{pmatrix}$ for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in QM_2(R)$. For any $\begin{pmatrix} x & 0 \\ z & y \end{pmatrix} \in TM_2(R)$, we have

$$\psi \begin{pmatrix} x-z & x-y-z \\ z & y+z \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & y \end{pmatrix}.$$

Thus ψ is an epimorphism. It is easy to verify that ψ is a monomorphism; hence, it is a ring isomorphism. Therefore we complete the proof by Theorem 8. ■

Let $A = (a_{ij})$ be a 2×2 matrix over a unit-regular ring R. If $a_{11} + a_{12} = a_{21} + a_{22}$, analogously to the consideration above, we conclude that there exist an idempotent $E = (e_{ij}) \in M_2(R)$ and an invertible $U = (u_{ij}) \in M_2(R)$ such that (1) $A = E + U$; (2) $e_{11} + e_{12} = e_{21} + e_{22}$; (3) $u_{11} + u_{12} = u_{21} + u_{22}$.

ACKNOWLEDGEMENT

The author would like to thank the referee for his/her helpful comments and suggestions, which lead to the new version of this paper.

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