TAIWANESE JOURNAL OF MATHEMATICS
Vol. 10, No. 2, pp. 553-561, February 2006
This paper is available online at http://www.math.nthu.edu.tw/tjm/

CONVERGENCE OF ISHIKAWA ITERATIVE SEQUENCES FOR ACCRETIVE LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we study the convergence theorems of the Ishikawa iterative sequences with mixed errors for the accretive Lipschitzian mappings in Banach spaces.

1. Introduction and Preliminaries

Throughout this paper we assume that E is a real Banach space, E^* is the dual space of E, $F(T)$ and $R(T)$ are the sets of fixed points and range of mapping T, respectively. We also assume that $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in E^*, \langle x, f\rangle = ||x||||f||, \quad ||f|| = ||x||\}, \ \ x \in E.$$

Definition 1.1. Let $T : E \to E$ be a mapping.

(i) T is said to be *accretive*, if for any $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq 0.$$

(ii) T is said to be *m-accretive*, if $R(I + rT) = E$ for all $r > 0$, where I is the identity operator on E.

Remark 1.1. By using the Kato inequality [12], we know that $T : E \to E$ is accretive if and only if for all $x, y \in E$ and for all $r > 0$ the following inequality holds:

Received November 16, 2004; Revised April 30, 2005; Accepted June 21, 2005.
2000 *Mathematics Subject Classification*: 47H06, 47H10, 47H15.
Key words and phrases: Accretive mapping, Ishikawa iterative sequences with mixed errors, Kato inequality.
This work was supported by the Kyungnam University Research Fund 2005.

$$(1.1) \qquad ||x-y|| \le ||x-y+r(Tx-Ty)||.$$

Definition 1.2. Let $T : E \to E$ be a mapping, x_0 and u_0 be two given points in E, $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in [0, 1], $\{v_n\}$ and $\{w_n\}$ be two sequences in E satisfying the following conditions:

(i) $v_n = v' + v''$, $||v_n'|| = o(\alpha_n)$ $(n \ge 0)$ and $\sum_{n=0}^{\infty} ||v_n''|| < \infty$,

(ii) $||w_n|| \to 0$ $(n \to \infty)$.

Then

(1) the sequence $\{x_n\}$ defined by

$$(1.2) \qquad \begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + v_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + w_n \ (n \ge 0) \end{cases}$$

is called the *Ishikawa iterative sequence with mixed errors* [14].

(2) the sequence $\{x_n\}$ defined by

$$(1.3) \qquad \begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n \ (n \ge 0) \end{cases}$$

is called the *Ishikawa iterative sequence* [8], where the sequences $\{\alpha_n\}$ appeared in (1.2) and (1.3) are all same.

The convergence problems for Mann iterative sequence, Ishikawa iterative sequence, Ishikawa iterative sequence with mixed errors and Ishikawa iterative sequence with errors (see, Definition 2.1) for strong accretive mappings and strongly pseudo-contractive mappings have been studied extensively by many authors, see, for example, Chang et al [2-3], Chidume [4-6], Fang et al. [7], Ishikawa [8], Kim et al. [9-11], Li et al. [13], Liu [14-15], Mann [16], Rhoades et al. [17], Sastry et al. [18], Solutz [19], Xu [20] and Zeng [21],

In this paper, we study the convergence theorems of the Ishikawa iterative sequences with mixed errors for the accretive Lipschitzian mappings in Banach spaces.

2. Main Results

First, we study the convergence results of the Ishikawa iterative sequence with mixed errors defined by (1.2) for accretive Lipschitzian mappings in Banach spaces.

In order to prove our main theorems, we need the following important lemma.

Lemma 2.1. ([14]) *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying the condition:*

$$a_{n+1} \le (1-\lambda_n)a_n + b_n + c_n \ (n \ge n_0),$$

where n_0 is some nonnegative integer and $\{\lambda_n\}$ is a sequence in [0, 1] *such that* $\sum_{n=0}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$ *and* $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $a_n \to 0$ *as* $n \to \infty$.

Theorem 2.1. *Let E be a real Banach space, $T : E \to E$ be a Lipschitzian continuous accretive mapping with a Lipschitz constant $L \geq 1$. Define an operator $S : E \to E$ by $Sx = f - Tx,\ x \in E$, where $f \in E$ is any given point. For any given $x_0 \in E$ let $\{x_n\}$ be the Ishikawa iterative sequence with mixed errors defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + v_n \\ y_n = (1-\beta_n)x_n + \beta_n S x_n + w_n \ (n \geq 0), \end{cases} \tag{2.1}$$

where $\{v_n\}$, $\{w_n\}$ are two sequences in E and $\{\alpha_n\}$, $\{\beta_n\}$ are two real sequences in [0, 1] *satisfying the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) *there exists $\eta \in (0, \frac{1}{2})$ such that*

$(1+L)\{\alpha_n(1+L)(1+L\beta_n) + L\beta_n\} < \frac{1}{2} - \eta \ (n \geq n_0)$,

where n_0 is some nonnegative integer,

(iii) (a) $v_n = v'_n + v''_n$, $||v'_n|| = o(\alpha_n)\ (n \geq 0)$ *and* $\sum_{n=0}^{\infty} ||v''_n|| < \infty$,

(b) $||w_n|| \to 0 \ (n \to \infty)$.

Then S has a unique fixed point $p \in E$ and the sequence $\{x_n\}$ defined by (2.1) *converges strongly to p.*

Proof. Since $T : E \to E$ is a Lipschitzian continuous accretive mapping, by the well-known result in Browder [1], T is m-accretive. Therefore, for given $f \in E$, the equation $x + Tx = f$ has a unique solution $p \in E$. We know that p is a fixed point of S and S is also a Lipschitzian mapping with a Lipschitz constant $L \geq 1$. Again since $(-S)$ is accretive, by (1.1) we have

$$||x - y|| \leq ||x - y - r(Sx - Sy)|| \tag{2.2}$$

for all $x, y \in E$ and $r > 0$. It follows from (2.1) that for all $n \geq 0$,

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n - v_n \\ &= x_{n+1} + \alpha_n(x_{n+1} + \alpha_n x_n - \alpha_n S y_n - v_n) - \alpha_n S y_n - v_n \\ &= (1+\alpha_n)x_{n+1} + \alpha_n(-S)x_{n+1} + \alpha_n^2(x_n - S y_n) \\ &\quad + \alpha_n(S x_{n+1} - S y_n) - (1+\alpha_n)v_n \end{aligned}$$

and so

$$\begin{aligned}||x_n - p|| &\geq ||(1+\alpha_n)(x_{n+1} - p) + \alpha_n((-S)x_{n+1} - (-S)p)|| \\ &\quad -\alpha_n^2||x_n - Sy_n|| - \alpha_n||Sx_{n+1} - Sy_n|| - (1+\alpha_n)||v_n|| \\ &= (1+\alpha_n)\left\{||x_{n+1} - p + \frac{\alpha_n}{1+\alpha_n}((-S)x_{n+1} - (-S)p)||\right\} \\ &\quad -\alpha_n^2||x_n - Sy_n|| - \alpha_n||Sx_{n+1} - Sy_n|| - (1+\alpha_n)||v_n|| \\ &= (1+\alpha_n)\left\{||x_{n+1} - p - \frac{\alpha_n}{1+\alpha_n}(Sx_{n+1} - Sp)||\right\} \\ &\quad -\alpha_n^2||x_n - Sy_n|| - \alpha_n||Sx_{n+1} - Sy_n|| - (1+\alpha_n)||v_n||\end{aligned}$$

By using (2.2), we know that

$$\begin{aligned}||x_n - p|| &\geq (1+\alpha_n)||x_{n+1} - p|| - \alpha_n^2||x_n - Sy_n|| \\ &\quad -\alpha_n||Sx_{n+1} - Sy_n|| - (1+\alpha_n)||v_n||\end{aligned} \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}||x_n - Sx_n|| &\leq ||x_n - p|| + L||x_n - p|| \\ &\leq (1+L)||x_n - p||.\end{aligned} \tag{2.4}$$

It follows from (2.1), by using (2.4), we have

$$\begin{aligned}||x_n - y_n|| &\leq \beta_n||x_n - Sx_n|| + ||w_n|| \\ &\leq \beta_n(1+L)||x_n - p|| + ||w_n||.\end{aligned} \tag{2.5}$$

Now we consider the second and the third term on the right side of (2.3). By using (2.4) and (2.5), we have

$$\begin{aligned}||x_n - Sy_n|| &= ||x_n - Sx_n|| + ||Sx_n - Sy_n|| \\ &\leq (1+L)||x_n - p|| + L||x_n - y_n|| \\ &\leq (1+L)||x_n - p|| + L\{\beta_n(1+L)||x_n - p|| + ||w_n||\} \\ &= (1+L)(1+L\beta_n)||x_n - p|| + L||w_n||.\end{aligned} \tag{2.6}$$

By using (2.5) and (2.6), we have

$$
\begin{aligned}
||Sx_{n+1} - Sy_n|| &\le L||x_{n+1} - y_n|| \\
&\le L(||x_{n+1} - x_n|| + ||x_n - y_n||) \\
&= L(\alpha_n||x_n - Sy_n|| + ||v_n|| + ||x_n - y_n||) \\
&\le L[\alpha_n\{(1+L)(1+L\beta_n)||x_n - p|| + L||w_n||\}] \\
&\quad + L||v_n|| + L\{\beta_n(1+L)||x_n - p|| + ||w_n||\} \\
&= L(1+L)(\alpha_n(1+L\beta_n) + \beta_n)||x_n - p|| \\
&\quad + L(L\alpha_n + 1)||w_n|| + L||v_n||.
\end{aligned} \tag{2.7}
$$

Substituting (2.6) and (2.7) into (2.3), we have

$$
\begin{aligned}
&(1+\alpha_n)||x_{n+1} - p|| \\
&\le ||x_n - p|| + \alpha_n^2||x_n - Sy_n|| + \alpha_n||Sx_{n+1} - Sy_n|| + (1+\alpha_n)||v_n|| \\
&\le ||x_n - p|| + \alpha_n^2[(1+L)(1+L\beta_n)||x_n - p|| + L||w_n||] \\
&\quad + \alpha_n[L(1+L)(\alpha_n(1+L\beta_n) + \beta_n)||x_n - p|| + L(L\alpha_n + 1)||w_n|| \\
&\quad + L||v_n||] + (1+\alpha_n)||v_n|| \\
&= [1 + \alpha_n^2(1+L)(1+L\beta_n) + \alpha_n L(1+L)\{\alpha_n(1+L\beta_n) + \beta_n\}]||x_n - p|| \\
&\quad + \alpha_n^2 L||w_n|| + \alpha_n L(L\alpha_n + 1)||w_n|| + L\alpha_n||v_n|| + (1+\alpha_n)||v_n|| \\
&= ||x_n - p|| + \alpha_n(1+L)[\alpha_n(1+L\beta_n) + L\{\alpha_n(1+L\beta_n) + \beta_n\}]||x_n - p|| \\
&\quad + \alpha_n L(\alpha_n + L\alpha_n + 1)||w_n|| + (L\alpha_n + 1 + \alpha_n)||v_n||
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(1+\alpha_n)||x_{n+1} - p|| &\le ||x_n - p|| + \alpha_n\gamma_n||x_n - p|| \\
&\quad + k_n + (L\alpha_n + 1 + \alpha_n)||v_n||,
\end{aligned}
$$

and so

$$
\begin{aligned}
||x_{n+1} - p|| &\le \frac{1}{1+\alpha_n}\{||x_n - p|| + \gamma_n\alpha_n||x_n - p|| \\
&\quad + k_n + (\alpha_n(1+L) + 1)||v_n||\},
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
\gamma_n &= (1+L)[\alpha_n(1+L\beta_n) + L\{\alpha_n(1+L\beta_n) + \beta_n\}] \\
&= (1+L)\{\alpha_n(1+L)(1+L\beta_n) + L\beta_n\}
\end{aligned}
$$

and

$$
k_n = \alpha_n L(\alpha_n(1+L) + 1)||w_n||.
$$

Since $(1+\alpha_n)^{-1} \le 1$ and $(1+\alpha_n)^{-1} \le 1 - \frac{\alpha_n}{2}$, it follows from (2.8) that

$$\begin{aligned} ||x_{n+1} - p|| &\le (1 - \frac{\alpha_n}{2})||x_n - p|| + \gamma_n \alpha_n ||x_n - p|| \\ &\quad + k_n + (\alpha_n(1+L)+1)||v_n||. \end{aligned} \tag{2.9}$$

By condition (ii), we know that there exists $\eta \in (0, \frac{1}{2})$ such that

$$\gamma_n < \frac{1}{2} - \eta \ (n \ge n_0). \tag{2.10}$$

Again, by condition (iii)-(a), since $v_n = v_n' + v_n''$ and $||v_n'|| = o(\alpha_n)$, there exists a nonnegative sequence $\{\epsilon_n\}$ with $\epsilon_n \to 0$ such that $||v_n'|| = \epsilon_n \alpha_n$ and so

$$||v_n|| \le \epsilon_n \alpha_n + ||v_n''|| \ (\ n \ge 0). \tag{2.11}$$

It follows from (2.9) - (2.11) that

$$\begin{aligned} ||x_{n+1} - p|| &\le (1 - \frac{\alpha_n}{2})||x_n - p|| + (\frac{1}{2} - \eta)\alpha_n ||x_n - p|| + k_n \\ &\quad + (\alpha_n(1+L)+1)\{\epsilon_n \alpha_n + ||v_n''||\} \\ &\le (1 - \eta\alpha_n)||x_n - p|| + b_n + c_n, \end{aligned} \tag{2.12}$$

where

$$b_n = k_n + (\alpha_n(1+L)+1)\epsilon_n \alpha_n, \quad c_n = (2+L)||v_n''||.$$

Since S is Lipschitzian continuous, therefore we know that $b_n = o(\eta\alpha_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. It follows from Lemma 2.1 that

$$||x_n - p|| \to 0 \ \ (n \to \infty).$$

Hence the Ishikawa iterative sequence $\{x_n\}$ with mixed errors defined by (2.1) converges strongly to a unique fixed point p of S in E. This completes the proof of Theorem 2.1. ■

In (2.1), if $v_n = w_n = 0$ $(n \ge 0)$, then we can obtain the following theorem:

Theorem 2.2. *Let E be a real Banach space, $T : E \to E$ be an accretive Lipschitzian mapping with a Lipschitz constant $L \ge 1$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in* [0,1] *satisfying the conditions* (i) *and* (ii) *in Theorem* 2.1. *Let $\{x_n\}$ be the Ishikawa iterative sequence defined by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n \\ \quad y_n = (1-\beta_n)x_n + \beta_n S x_n \ (n \ge 0). \end{cases} \tag{2.13}$$

Then S has a unique fixed point $p \in E$ and the sequence $\{x_n\}$ defined by (2.13) *converges strongly to p.*

Now, we also study the convergence of one kind of Ishikawa iterative sequence with errors (see ([20]).

Definition 2.1. Let $T : E \to E$ be an accretive Lipschitzian mapping. Define a mapping $S : E \to E$ by $Sx = f - Tx$ for $x \in E$, and $f \in E$ be a given point. Let $x_0 \in E$ be a given point, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in [0, 1] with $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$ and $\{s_n\}$, $\{t_n\}$ be two bounded sequences in E. Then the sequence $\{x_n\}$ defined by

$$(2.14) \qquad \begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n S y_n + \gamma_n t_n \\ \quad y_n = (1 - \beta_n - \delta_n)x_n + \beta_n S x_n + \delta_n s_n \ (n \geq 0) \end{cases}$$

is called the *Ishikawa iterative sequence with errors*, which was introduced and studied in Xu [20].

Theorem 2.3. *Let E be a real Banach space, $T : E \to E$ be an accretive Lipschitzian mapping with a Lipschitz constant $L \geq 1$. Define a mapping $S : E \to E$ by $Sx = f - Tx$ for any $x \in E$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ be four real sequences in* $[0, 1]$ *satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$,

(ii) $(1 + L)\{\alpha_n(1 + L)(1 + L\beta_n) + L\beta_n\} < \frac{1}{2} - \eta \ (n \geq n_0)$, *where* $\eta \in (0, \frac{1}{2})$ *is a constant and n_0 is some nonnegative integer,*

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(iv) $\sum_{n=0}^{\infty} \gamma_n < \infty$, $\delta_n \to 0$.

Let $\{x_n\}$ be the Ishikawa iterative sequence with errors defined by (2.14)*. If the sequence $\{x_n\}$ is bounded, then S has a unique fixed point $p \in E$ and the sequence $\{x_n\}$ converges strongly to p.*

Proof. First we rewrite (2.14) as follows:

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + \gamma_n(t_n - x_n) \\ \quad y_n = (1 - \beta_n)x_n + \beta_n S x_n + \delta_n(s_n - x_n) \ (n \geq 0). \end{cases}$$

Letting $v_n = \gamma_n(t_n - x_n)$ and $w_n = \delta_n(s_n - x_n), \ \forall n \geq 0.$ Then we have

$$(2.15) \qquad \begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + v_n \\ \quad y_n = (1 - \beta_n)x_n + \beta_n S x_n + w_n \ \ \forall\, (n \geq 0). \end{cases}$$

By the assumption, the sequences $\{x_n\}$, $\{t_n\}$ and $\{s_n\}$ are bounded in E. Again by the condition (iv) we know that $\delta_n \to 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$, therefore we have

$$\sum_{n=0}^{\infty} ||v_n|| < \infty \text{ and } ||w_n|| \to 0 \ \ (n \to \infty),$$

which imply that the iterative sequence $\{x_n\}$ defined by (2.14) is a special case of the Ishikawa iterative sequence with mixed errors defined by (2.1) and all the conditions in Theorem 2.1 are satisfied. Therefore the conclusion of Theorem 2.3 can be obtained from Theorem 2.1 immediately. This completes the proof. ■

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