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RETRIEVE A SPERNER MAP FROM A SPERNER MATROID

Shyh-Nan Lee and Mau-Hsiang Shih

Abstract. A retrievel of a Sperner map from a Sperner matroid is illustrated. As an application, a new proof of a completion theorem of Lovász's matroid version of Sperner's lemma is given.

1. Introduction

The purpose of this note is to give a simple procedure which allows us to retrieve a Sperner map from a Sperner matroid.

2. Retrieving a Sperner Map on the Basis of a Sperner Matroid

Let K be a triangulation of a d-simplex $a_0a_1\dots a_d$ in a Euclidean space, and $V(K)$ the vertex set of K. A map $\varphi : V(K) \to \{0,1,\dots,d\}$ is said to be a *Sperner map* if for each $i_0, i_1, \dots, i_k$ with $0 \le i_0 < i_1 < \dots < i_k \le d$ and for each $v \in V(K) \cap a_{i_0}a_{i_1}\dots a_{i_k}$, $\varphi(v) \in \{i_0, i_1, \dots, i_k\}$. A matroid M on $V(K)$ is called a *Sperner matroid* over K if for each $S \subset \{a_0, a_1, \dots, a_d\}$ and for each $v \in V(K) \cap conv(S)$, $v \in cl_M(S)$, where $conv(S)$ stands for the convex hull of S and $cl_M(S)$ denotes the closure of S in M. Let M be a Sperner matroid over K such that $\{a_0, a_1, \dots, a_d\}$ forms a basis of M. Put

$$F_j \equiv cl_M(\{a_0, a_1, \dots, a_j\}) \ (j = 0, 1, \dots, d). \tag{1}$$

Let us define $\varphi : V(K) \to \{0,1,\dots,d\}$ by setting

$$\varphi(v) \equiv \begin{cases} 0 & \text{if} \quad v \in F_0, \\ j & \text{if} \quad v \in F_j \setminus F_{j-1} \ (j = 1, 2, \dots, d). \end{cases} \tag{2}$$

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The problem that we consider in this paper is the following : *Under what conditions can φ become a Sperner map* ? It is clearly necessary that the $(d-1)$-face $a_1a_2\ldots a_d$ contains no loops of M. To see this, if $a_1a_2\ldots a_d$ contains a loop v of M, then $\varphi(v) = 0$ and $v \in V(K) \cap a_1a_2\ldots a_d$. As φ is a Sperner map, we have $\varphi(v) \in \{1, 2, \ldots, d\}$, in contradiction. What is perhaps surprising is that this condition is also sufficient.

We shall establish the following:

Theorem 1. *Let M be a Sperner matroid over a triangulation K of a d-simplex $a_0a_1\ldots a_d$ such that $\{a_0, a_1, \ldots, a_d\}$ forms a basis. Then the map φ given in (2) is a Sperner map if and only if the $(d-1)$-face $a_1a_2\ldots a_d$ contains no loops of M.*

To prove Theorem 1, we need the following

Lemma. *Let B be a basis of a matroid M. Suppose*

(a) $S \subset T \subset B,$

(b) $X \subset B,\ X \cap T = \emptyset,$

(c) $y \in cl_M(T) \setminus cl_M(S).$

Then $y \in cl_M(T \cup X) \setminus cl_M(S \cup X).$

Proof. Suppose $X \neq \emptyset$, and put

$$X \equiv \{x_1, x_2, \ldots, x_m\}.$$

Then $y \notin cl_M(S \cup \{x_1\})$. To see this, suppose, by contradiction, that $y \in cl_M(S \cup \{x_1\})$. Then, by MacLane-Steinitz exchange property, $x_1 \in cl_M(S \cup \{y\})$. Since $S \subset T$ and $y \in cl_M(T)$, we have

$$x_1 \in cl_M(S \cup \{y\}) \subset cl_M(T),$$

which contradicts $x_1 \notin cl_M(T)$. Thus we have shown that

(a)$'$ $S \cup \{x_1\} \subset T \cup \{x_1\} \subset B,$

(b)$'$ $\{x_2, x_3, \ldots, x_m\} \subset B,\ \{x_2, x_3, \ldots, x_m\} \cap (T \cup \{x_1\}) = \emptyset,$

(c)$'$ $y \in cl_M(T \cup \{x_1\}) \setminus cl_M(S \cup \{x_1\}).$

The same argument applies (a)$'$, (b)$'$, and (c)$'$ now yielding $y \notin cl_M(S \cup \{x_1\} \cup \{x_2\})$. Iteration of this procedure m times concludes the proof of the lemma. ∎

We now proceed to prove Theorem 1.

Part " only if " is obvious. To prove part " if ", let $v \in V(K) \cap a_{i_0}a_{i_1}\ldots a_{i_k}$ $(0 \le i_0 < i_1 < \ldots < i_k \le d)$. Since M is a Sperner matroid, we have

$$(3) \qquad v \in cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_k}\}).$$

We want to conclude that $\varphi(v) \in \{i_0, i_1, \dots, i_k\}$; it is to split the argument into two cases.

Case 1. v is a loop. Then $a_{i_0}a_{i_1}\dots a_{i_k}$ contains a loop of M. It follows from the hypothesis that $i_0 = 0$. Thus $\varphi(v) = 0 \in \{i_0, i_1, \dots, i_k\}$.

Case 2. v is not a loop. It follows from (3) and

$$cl_M(\{a_{i_0}\}) \subsetneq cl_M(\{a_{i_0}, a_{i_1}\}) \subsetneq \dots \subsetneq cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_k}\})$$

that either $v \in cl_M(\{a_{i_0}\})$ or there exists $p \in \{1, 2, \dots, k\}$ such that

$$v \in cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_p}\}) \setminus cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_{p-1}}\}).$$

If $v \in cl_M(\{a_{i_0}\})$, then $v \in cl_M(\{a_0, a_1, \dots, a_{i_0}\})$ and, since v is not a loop, $v \notin cl_M(\{a_0, a_1, \dots, a_{i_0-1}\})$ if $i_0 \geq 1$. Therefore $v_0 \in F_0$ if $i_0 = 0$; $v \in F_{i_0} \setminus F_{i_0-1}$ if $i_0 \geq 1$. It follows that $\varphi = i_0 \in \{i_0, i_1, \dots, i_k\}$. Finally, if

$$v \in cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_p}\}) \setminus cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_{p-1}}\}),$$

then, by our lemma, we have

$$v \in cl_M(\{a_0, a_1, \dots, a_{i_p}\}) \setminus cl_M(\{a_0, a_1, \dots, a_{i_{p-1}}\}).$$

Thus $F_{i_{p-1}} \subsetneq F_{\varphi(v)} \subset F_{i_p}$, so that $i_{p-1} < \varphi(v) \leq i_p$. We claim that $\varphi(v) = i_p$. Suppose $\varphi(v) < i_p$. Then we have

$$v \in cl_M(\{a_0, a_1, \dots, a_{i_{\varphi(v)}}\}) \setminus cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_{p-1}}\}),$$

so that, by our lemma,

$$v \in cl_M(\{a_0, a_1, \dots, a_{i_{\varphi(v)}}, a_{i_p}\}) \setminus cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_p}\}),$$

in contradiction to $v \in cl_M(\{a_{i_0}, a_{i_1}, \dots, a_{i_p}\})$. This contradiction shows that $\varphi(v) = i_p$, and we conclude that $\varphi(v) \in \{i_0, i_1, \dots, i_k\}$.

This completes the proof. ■

3. A Sign Function

Let K be a triangulation of a d-simplex $a_0a_1\dots a_d$ in a Euclidean space, M a Sperner matroid over K, and $B \equiv (a_0, a_1, \dots, a_d)$ an ordered basis of M. Let $\Lambda_B : K \to \{-1, 0, 1\}$. We define $\Lambda_B(v_0v_1\dots v_d) = 1$ (resp.-1) if $v_0 \in F_0$ and

$v_j \in F_j \setminus F_{j-1}$ $(j = 1, 2, \ldots, d)$, where F_j $(j = 0, 1, \ldots, d)$ are given in (1), and $\det(\alpha_{ij}) > 0$ (resp. $\det(\alpha_{ij}) < 0$), where

$$(4) \qquad \begin{pmatrix} v_0 \\ \vdots \\ v_d \end{pmatrix} = \begin{pmatrix} \alpha_{00} & \cdots & \alpha_{0d} \\ \vdots & & \vdots \\ \alpha_{d0} & \cdots & \alpha_{dd} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_d \end{pmatrix}, \ \sum_{j=0}^{d} \alpha_{ij} = 1 \ (0 \le i \le d).$$

We define $\Lambda_B(v_0 v_1 \ldots v_d) = 0$ otherwise. Let $\varphi : V(K) \to \{0, 1, \ldots, d\}$. We call a d-simplex $v_0 v_1 \ldots v_d \in K$ *positively* (*resp. negatively*) *completely labelled* if $\varphi(v_j) = j$ $(j = 0, 1, \ldots, d)$, and $\det(\alpha_{ij}) > 0$ (resp. $\det(\alpha_{ij}) < 0$), where the matrix (α_{ij}) is given in (4). A d-simplex of K is *completely labelled* if it is positively or negatively completely labelled. It is obvious that $v_0 v_1 \ldots v_d \in K$ is completely labelled if and only if $\{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_d)\} = \{0, 1, \ldots, d\}$. The celebrated Sperner lemma [7] asserts that if φ is a Sperner map then $\sharp\{\sigma \in K\ ;\ \sigma$ is completely labelled $\} \equiv 1\ (mod\ 2)$. The oriented Sperner lemma [1] states that

$$\sharp\{\sigma \in K\ ;\ \sigma \text{ is positively completely labelled } \} - \sharp\{\sigma \in K\ ;\ \sigma \text{ is negatively completely labelled } \} = 1.$$

By Thoerem 1 and the oriented Sperner lemma, we have

Theorem 2. *Let K be a triangulation of a d-simplex $a_0 a_1 \ldots a_d$ in a Euclidean space, and M a Sperner matroid over K such that the $(d-1)$-face $a_1 a_2 \ldots a_d$ contains no loops of M. If $B \equiv (a_0, a_1, \ldots, a_d)$ is an ordered basis of M, then*

(a) $\sum_{\sigma \in K} \Lambda_B(\sigma) = 1,$

(b) $\sum_{\sigma \in K} |\Lambda_B(\sigma)| \equiv 1\ (mod\ 2).$

Proof. Note that (b) is a consequence of (a). We need only prove (a). By hypothesis, we can retrieve a Sperner map φ given in (2). As for each $\sigma \in K$, $\Lambda_B(\sigma) = 1$ (resp. $\Lambda_B(\sigma) = -1$) if and only if σ is positively (resp. negatively) completely labelled, (a) follows from the oriented Sperner lemma. ∎

Theorem 2 was recently proved by the authors in [4] with a completely different proof. An example given in [4] shows that the condition " the $(d-1)$-face $a_1 a_2 \ldots a_d$ contains no loops " cannot be dispensed with. Theorem 2(b) is a generalization of Lovász's theorem. It complements Lovász's theorem in two aspects: one concerns an arbitrary matroid while the other is the assertion of oddness. Indeed, Lovász [2] proved the following.

Theorem 3. *Let K be a triangulation of a d-simplex $a_0 a_1 \ldots a_d$ in a Euclidean space, and M a Sperner matroid over K without loops. If $\{a_0, a_1, \ldots, a_d\}$ is a* *basis of M , then K has a d-simplex $v_0v_1\dots v_d$ such that $\{v_0, v_1, \dots, v_d\}$ is also a basis of M.*

Finally, let us mention that it is perhaps worth developing a general matroid version which contains multiple balanced Sperner lemma [6], combinatorial Lefschetz fixed-point formula [5], and multiple combinatorial Stokes' theorem [3].

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Shyh-Nan Lee
Department of Mathemathics,
Chung Yuan Christian University,
Chung Li 320, Taiwan.
E-mail: nan@math.cycu.edu.tw

Mau-Hsiang Shih
Department of Mathematics,
National Taiwan Normal University,
Taipei 116, Taiwan.
E-mail: mhshih@math.ntnu.edu.tw