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AN EXTENSION OF THE WEIGHTED HARDY INEQUALITIES AND ITS APPLICATION TO HALF-LINEAR EQUATIONS

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Abstract. In this paper we consider a suitable extension of the weighted Hardy inequalities and by applying to second order half-linear equations we establish some oscillation and nonoscillation results.

1. Introduction

In the literature many authors including G. H. Hardy, J. E. Littlewood and G. Pólya [9] considered the continuous Hardy inequality:
If f is a nonnegative function whose p-th power is integrable over $(0, \infty)$ for $p > 1$ then f is integrable over the interval $(0, x)$ for all $x > 0$, and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \tag{1.1}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in the inequality (1.1) is sharp in the sense that it can not be replaced by any smaller number.

The inequality (1.1) was proven by Hardy in his famous paper [7] and it has been generalized and applied in analysis and in the theory of differential equations. In 1928, G. H Hardy [8] proved the estimate for some integral operators, from which the following "weighted" modification of the inequality (1.1) is obtained:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\varepsilon dx < \left(\frac{p}{p-\varepsilon-1}\right)^p \int_0^\infty f^p(x)x^\varepsilon dx \tag{1.2}$$

for $p > 1$ and $\varepsilon < p - 1$, for all measurable nonnegative functions f (see [7], Theorem 330), where the constant $\left(\frac{p}{p-\varepsilon-1}\right)^p$ is the best possible.

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During the last decades the inequality (1.2) has been developed to the form

$$\left(\int_a^b\left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{1/q} \leq C\left(\int_a^b f^p(x)v(x)dx\right)^{1/p} \tag{1.3}$$

with a, b real numbers satisfying $-\infty \leq a < b \leq \infty$, u, v positive measurable weight functions in (a, b) and p, q real parameters, satisfying $0 < q \leq \infty$ and $1 \leq p \leq \infty$. This is sometimes called the weighted form of the continuous Hardy inequality.

In 1961, R. R. Beesack [4] connected the validity of the corresponding inequality (1.3) for the case $p = q$ with the existence of a (positive) solution y of the nonlinear ordinary differential equation

$$\frac{d}{dx}\left(v(x)\left(\frac{dy}{dx}\right)^{p-1}\right) + u(x)y^{p-1}(x) = 0 \tag{1.4}$$

which is in the fact the Euler-Lagrange equation for the functional

$$J(y) = \int_0^\infty \big[y'(x)^p v(x) - y^p(x)u(x)\big]dx.$$

Although Beesack's approach was not the variational one, his approach was extended to a class of the inequalities containing the Hardy inequality as a special case [18].

In 1969, Tomaselli [27] followed Beesack's approach via equations and he has shown that the solvability of the equation (1.4) is not only sufficient but in a certain sense even necessary for (1.3) to hold.

Note that, the Tomaselli's paper [27] plays a fundamental role in the development of the Hardy inequality. Some of these developments, generalizations and applications have been described and discussed in the books [6, 7, 10, 11, 18]. A history of developments on weighted Hardy inequalities can also be found in [10].

The main aim of this paper is to obtain a suitable extension of the weighted Hardy inequalities, namely the "three-weighted Hardy type inequality." By applying this inequality, we establish some oscillation and non-oscillation results related to half-linear second order differential equation.

Applying the results of the weighted Hardy inequality (1.3) to the question oscillatory and non-oscillatory half-linear equations are in the works [13, 16, 17].

2. Main Results

Let $I = (a, b)$, $-\infty \leq a < b \leq \infty$, $1 < p < \infty$ and $p' = \frac{p}{p-1}$. Assume that w, r, ρ and $\rho^{1-p'}$ locally summable on the interval I and $w \geq 0, r \geq 0, \rho > 0$ in I. Let $W_p^1(\rho, r; I)$ be the space of locally absolutely continuous functions f on I such that the norm

$$\|f\|_{W_p^1(\rho,r)} = \left(\int_\alpha^\beta \left(\rho|f'|^p + r|f|^p\right)dt\right)^{1/p}$$

is finite. In case $\rho \equiv 1$ and $r \equiv 1$, we put $W_p^1(1,1;I) \equiv W_p^1(I)$. Let $\mathring{AC}_p(I) = \{f \in \mathring{W}_p^1(\rho,r;I)) : \operatorname{supp} f \subset I\}$. We denote by $\mathring{W}_p^1(\rho,r;I)$ and $\mathring{W}_p^1(I)$ the closure $\mathring{AC}_p(I)$ respectively in the space $W_p^1(\rho,r;I)$ and $W_p^1(I)$.

On the interval $I_0 = (\alpha,\beta)$ such that $a \le \alpha < \beta \le b$, we consider the following inequalities

$$\int_\alpha^\beta w|f|^p dt \le C \int_\alpha^\beta \Big(\rho|f'|^p + r|f|^p\Big)dt,\ f \in \mathring{W}_p^1(\rho,r;I_0). \tag{2.1}$$

Here and the sequel $I_0 = (\alpha,\beta)$.

Equivalent criteria for the validity of inequality (2.1) follows from the results of [12, 14, 15, 19]. However, the equivalence coefficients of the best constant in (2.1) are not pointed out in these works. Here we investigate inequality (2.1) by a method that allows us to find the equivalence coefficients more precisely.

We begin with a lemma for our purpose.

Lemma 2.1. *Let $p > 1$ be real number. Let g be defined as $g(\lambda) = \dfrac{\lambda^p}{\lambda^p - 1} - \dfrac{1}{(\lambda-1)^p}$ on $(1,\infty) \subset R$. Then there exists a $\lambda_0 := \lambda_0(p)$ such that $1 < \lambda_0 < 2$ and $\dfrac{1}{(\lambda_0-1)^p} = \dfrac{\lambda_0^p}{\lambda_0^p - 1}$ and $g(\lambda) > 0$ for $\lambda > \lambda_0$ and $g(\lambda) < 0$ for $1 < \lambda < \lambda_0$.*

Proof. It is clear that $g(2) > 0$ and $\lim\limits_{\lambda\to 1^+} \dfrac{\lambda^p(\lambda-1)^p}{\lambda^p - 1} = 0$. Using the definition of limit there exists a $\delta > 0$ for $\varepsilon = 1$ such that $\dfrac{\tilde\lambda^p(\tilde\lambda-1)^p}{\tilde\lambda^p - 1} < 1$ or $\dfrac{\tilde\lambda^p}{\tilde\lambda^p - 1} < \dfrac{1}{(\tilde\lambda-1)^p}$, for every $\tilde\lambda \in (1, 1+\delta)$. Thus $g(\tilde\lambda) < 0$. Since the function g is continuous in $(1,\infty)$ there exists a $\lambda_0 \in (1,2)$ such that $g(\lambda_0) = 0$, i.e.,

$$\frac{\lambda_0^p}{\lambda_0^p - 1} = \frac{1}{(\lambda_0-1)^p} \quad \text{or} \quad \lambda_0^p(\lambda_0-1)^p = \lambda_0^p - 1.$$

Let us define the functions $g_1(\lambda) := \dfrac{1}{(\lambda-1)^p}$ and $g_2(\lambda) := \dfrac{\lambda^p}{\lambda^p-1}$ which are strongly decreasing in $(1,\infty)$. Then $g_2(\lambda) > g_1(\lambda)$ at $\lambda > \lambda_0$ and $g_1(\lambda) > g_2(\lambda)$ at $1 < \lambda < \lambda_0$. ■

Remark 1. Calculation shows: $\lambda_0(2) \approx 1.8393$, $\lambda_0(3) \approx 1.9531$, $\lambda_0(4) \approx 1.9834$, $\lambda_0(5) \approx 1.9936$.

We introduce the following functions defined in I_0 as follows;

$$\varphi^-(\alpha,x) := \varphi_r^-(\alpha,x) = \inf_{\alpha<t<x} \left\{ \left(\int_t^x \rho^{1-p'}(t)dt \right)^{1-p} + \left(\int_t^x r(t)dt \right) \right\} \tag{2.2}$$

and

$$\varphi^+(x,\beta) := \varphi_r^+(x,\beta) = \inf_{x<t<\beta}\left\{\left(\int_x^t \rho^{1-p'}(t)dt\right)^{1-p} + \left(\int_x^t r(t)dt\right)\right\}. \tag{2.3}$$

Let

$$\begin{aligned} B_{r,w} &:= B_{r,w}(\alpha,\beta) \\ &= \sup_{\alpha<c<d<\beta}\left(\int_c^d w(t)dt\right)\left(\varphi^-(\alpha,c) + \int_c^d r(t)dt + \varphi^+(d,\beta)\right)^{-1}. \end{aligned} \tag{2.4}$$

Theorem 2.1. *Let λ_0 and λ be defined as in Lemma 2.1 and let $a \le \alpha < \beta \le b$. The inequality (2.1) holds if and only if $B_{r,w}(\alpha,\beta) < \infty$. Moreover the best constant C in (2.1) satisfies*

$$B_{r,w} \le C \le \gamma_p B_{r,w}, \tag{2.5}$$

where

$$\gamma_p = \inf_{1<\lambda<\lambda_0} \frac{\lambda^p(\lambda^p - 1)}{(\lambda-1)^p}. \tag{2.6}$$

Remark 2. Calculation shows: $\gamma_2 \approx 11.0902$, $\gamma_3 \approx 54.9637$, $\gamma_4 \approx 238.802$.

Remark 3. Let us notice that the value $B_{r,w}(\alpha,\beta)$ can be obtained from the results of [12, 19].

Proof. From the hypotheses assume that the inequality (2.1) holds for all $f \in \mathring{W}_p^1(\rho, r; I_0)$. Let $\alpha < \mu < c < d < \tau < \beta$. We introduce the function $f_0(t)$ defined on I_0 as the following

$$f_0(t) = \begin{cases} \left(\int_\mu^t \rho^{1-p'}(s)ds\right)\left(\int_\mu^c \rho^{1-p'}(s)ds\right)^{-1} & \mu \le t \le c\ , \\ 1 & c < t < d, \\ \left(\int_t^\tau \rho^{1-p'}(s)ds\right)\left(\int_d^\tau \rho^{1-p'}(s)ds\right)^{-1} & d \le t \le \tau\ , \\ 0 & t \in (\alpha,\beta)\backslash(\mu,\tau). \end{cases}$$

It is clear that $f_0 \in \mathring{AC}_p(\alpha,\beta)$. Simple computations show that

$$\int_\alpha^\beta w(t)|f_0(t)|^p dt > \int_c^d w(t)dt, \tag{2.7}$$

$$\int_\alpha^\beta r(t)|f_0(t)|^p dt \le \int_\mu^\tau r(t)dt, \tag{2.8}$$

and

$$
\begin{aligned}
\int_\alpha^\beta \rho(t)|f_0'(t)|^p dt &= \int_\mu^c \rho(t)\rho^{p(1-p')}(t)dt\left(\int_\mu^c \rho^{1-p'}(t)dt\right)^{-p} \\
&\quad + \int_d^\tau \rho(t)\rho^{p(1-p')}(t)dt\left(\int_d^\tau \rho^{1-p}(t)dt\right)^{-p} \\
&= \left(\int_\mu^c \rho^{1-p'}(t)dt\right)^{1-p} + \left(\int_d^\tau \rho^{1-p'}(t)dt\right)^{1-p}.
\end{aligned} \tag{2.9}
$$

We combine (2.7), (2.8) and (2.9) to obtain

$$
\begin{aligned}
\int_c^d w(t)dt < C\Bigg[&\left(\int_\mu^c \rho^{1-p'}(t)dt\right)^{1-p} \\
&+ \int_\mu^c r(t)dt + \left(\int_d^\tau \rho^{1-p'}(t)dt\right)^{1-p} + \int_d^\tau r(t)dt + \int_c^d r(t)dt\Bigg].
\end{aligned}
$$

Since the left hand side of the above inequality are independent of μ and τ and with (2.2), (2.3), we have

$$
\int_c^d w(t)dt < C\left(\varphi^-(\alpha,c) + \int_e^d r(t)dt + \varphi^+(d,\beta)\right),
$$

or

$$
\left(\int_c^d w(t)dt\right)\left(\varphi^-(\alpha,c) + \int_c^d r(t)dt + \varphi^+(d,\beta)\right)^{-1} < C.
$$

Taking supremum of both sides for $\alpha < c < d < \beta$, we have

$$
B_{r,w} \le C. \tag{2.10}
$$

Conversely, let $B_{r,w} < \infty$. Without loss of generality we assume that $f \in \overset{\circ}{A}C_p(I_0)$ and $f \ge 0$. Our purpose for $\lambda > 1$ and for $k \in Z$ we define the set $T_k := \{t \in I_0 : f(t) > \lambda^k\}$. Since the function f is bounded, then there exists an $n = n(f) \in Z$ such that

$$
T_n \neq \emptyset\,,\ T_{n+1} = \emptyset \ \text{ and } I_0 = \bigcup_{k\in Z} T_k = \bigcup_{k \in Z} \Delta T_k, \tag{2.11}
$$

where $\Delta T_k = T_k \backslash T_{k+1}$. Let $n \ge k > -\infty$. The set T_k is open. Then it is sum of a countable number of disjoint intervals $J_j^k = (c_j^k, d_j^k)$, i.e. $T_k = \cup_j J_j^k$. We denote $M_j^k = T_{k+1} \cap J_j^k$. We see that $M_j^k \neq \emptyset$. We put $\alpha_j^k = \inf M_j^k$ and $\beta_j^k = \sup M_j^k$. Considering also the definition of α_j^k and β_j^k, we obtain

$$
T_{k+1} \subset \bigcup_{j\in Z}(\alpha_j^k, \beta_j^k)\,,\ \Delta T_k \supset \bigcup_{j \in Z}(c_j^k, \alpha_j^k) \cup (\beta_j^k, d_j^k), \tag{2.12}
$$

$$f(\alpha_j^k) = f(\beta_j^k) = \lambda^{k+1} \text{ and } f(c_j^k) = f(d_j^k) = \lambda^k.$$

By (2.12) and Hölder's inequality, we can obtain the following

$$\begin{aligned}\lambda^k(\lambda-1) = \lambda^{k+1} - \lambda^k &= f(\alpha_j^k) - f(c_j^k) = \int_{c_j^k}^{\alpha_j^k} f'(t)dt \\ &\le \left(\int_{c_j^k}^{\alpha_j^k} \rho^{1-p'}(t)dt\right)^{1/p'} \left(\int_{c_j^k}^{\alpha_j^k} \rho(t)|f'(t)|^p dt\right)^{1/p}\end{aligned}$$

or

$$\lambda^{pk}\left(\int_{c_j^k}^{\alpha_j^k} \rho^{1-p'}(t)dt\right)^{1-p} \le \frac{1}{(\lambda-1)^p}\int_{c_j^k}^{\alpha_j^k} \rho(t)|f'(t)|^p dt. \tag{2.13}$$

Similarly, we have

$$\lambda^{pk}\left(\int_{\beta_j^k}^{d_j^k} \rho^{1-p'}(t)dt\right)^{1-p} \le \frac{1}{(\lambda-1)^p}\int_{\beta_j^k}^{d_j^k} \rho(t)|f'(t)|^p dt. \tag{2.14}$$

Now, we are ready to estimate the left hand side of (2.1). By using (2.11), (2.12) and considering that $\lambda^{k+1} < f(t) \le \lambda^{k+2}$ for $t \in \Delta T_{k+1}$ and equality $\lambda^{pk} = (1 - \lambda^{-p})\sum_{i=\infty}^{k} \lambda^{pi}$, we obtain

$$\begin{aligned}\int_\alpha^\beta w|f|^p dt &= \sum_{k=-\infty}^{n-1} \int_{\Delta T_{k+1}} w|f|^p dt \\ &\le \sum_{k=-\infty}^{n-1} \lambda^{p(k+2)} \int_{\Delta T_{k+1}} wdt = \lambda^{2p}\sum_{k=-\infty}^{n-1} \lambda^{pk}\int_{\Delta T_{k+1}} wdt \\ &= \lambda^{2p}(1-\lambda^{-p})\sum_{k=-\infty}^{n-1}\left(\int_{\Delta T_{k+1}} wdt \sum_{i=-\infty}^{k}\lambda^{pi}\right) \\ &= \lambda^p(\lambda^p-1)\sum_{i=-\infty}^{n-1}\lambda^{pi}\sum_{k=i}^{n-1}\int_{\Delta T_{k+1}} wdt \\ &\le \lambda^p(\lambda^p-1)\sum_{i=-\infty}^{n-1}\lambda^{pi}\int_{T_{i+1}} wdt \\ &\le \lambda^p(\lambda^p-1)\sum_{i=-\infty}^{n-1}\lambda^{pi}\sum_j\int_{\alpha_j^i}^{\beta_j^i} wdt.\end{aligned}$$

From the definition (2.4) of $B_{r,w}$ and the definitions (2.2), (2.3) of functions φ^-, φ^+ and taking into account $\alpha < c_j^i < \alpha_j^i$, $\beta_j^i < d_j^i < \beta$, we have

$$\int_{\alpha_j^i}^{\beta_j^i} w dt \leq B_{r,w}\Big(\varphi^-(\alpha,\alpha_j^i)+\int_{\alpha_j^i}^{\beta_j^i} r(t)dt+\varphi^+(\beta_j^i,\beta)\Big), \tag{2.15}$$

$$\varphi^-(\alpha,\alpha_j^i) \leq \bigg(\int_{c_j^i}^{\alpha_j^i} \rho^{1-p'}dt\bigg)^{1-p}+\int_{c_j^i}^{\alpha_j^i} r(t)dt \tag{2.16}$$

and

$$\varphi^+(\beta_j^i,\beta) \leq \bigg(\int_{\beta_j^i}^{d_j^i} \rho^{1-p'}dt\bigg)^{1-p}+\int_{\beta_j^i}^{d_j^i} r(t)dt. \tag{2.17}$$

By using (2.15), (2.16) and (2.17) from the above inequality, we obtain

$$\begin{aligned}
&\int_\alpha^\beta w|f|^p dt\\
&\leq B_{r,w}\lambda^p(\lambda^p-1)\sum_{i=-\infty}^{n-1}\lambda^{pi}\sum_j\bigg(\varphi^-(\alpha,\alpha_j^i)+\int_{\alpha_j^i}^{\beta_j^i} r(t)dt+\varphi^+(\beta_j^i,\beta)\bigg)\\
&\leq B_{r,w}\lambda^p(\lambda^p-1)\bigg\{\sum_{i=-\infty}^{n-1}\sum_j\bigg[\lambda^{pi}\bigg(\int_{c_j^i}^{\alpha_j^i}\rho^{1-p'}dt\bigg)^{1-p}\\
&\quad+\lambda^{pi}\bigg(\int_{\beta_j^i}^{d_j^i}\rho^{1-p'}dt\bigg)^{1-p}\bigg]+\sum_{i=-\infty}^{n}\lambda^{pi}\sum_j\int_{c_j^i}^{d_j^i} r(t)dt\bigg\}.
\end{aligned} \tag{2.18}$$

On the other hand, by using (2.13), (2.14) and taking into account (2.12) we have the following estimates

$$\begin{aligned}
&\sum_{i=-\infty}^{n-1}\sum_j\bigg[\lambda^{pi}\bigg(\int_{c_j^i}^{\alpha_j^i}\rho^{1-p'}dt\bigg)^{1-p}+\lambda^{pi}\bigg(\int_{\beta_j^i}^{d_j^i}\rho^{1-p'}dt\bigg)^{1-p}\bigg]\\
&\leq\frac{1}{(\lambda-1)^p}\sum_{i=-\infty}^{n-1}\sum_j\bigg(\int_{c_j^i}^{\alpha_j^i}\rho|f'|^p dt+\int_{\beta_j^i}^{\alpha_j^i}\rho|f'|^p dt\bigg)\\
&\leq\frac{1}{(\lambda-1)^p}\sum_{i=-\infty}^{n}\int_{\Delta T_i}\rho|f'|^p dt=\frac{1}{(\lambda-1)^p}\int_\alpha^\beta\rho|f'|^p dt
\end{aligned} \tag{2.19}$$

and

$$
\begin{aligned}
\sum_{i=-\infty}^{n} \lambda^{pi} \sum_{j} \int_{c_j^i}^{d_j^i} r dt &= \sum_{i=-\infty}^{n} \lambda^{pi} \int_{T_i} r dt = \sum_{i=-\infty}^{n} \lambda^{pi} \sum_{k=i}^{n} \int_{\Delta T_k} r dt \\
&= \sum_{k=-\infty}^{n} \int_{\Delta T_k} r dt \sum_{i=-\infty}^{k} \lambda^{pi} = \frac{1}{1-\lambda^{-p}} \sum_{k=-\infty}^{n} \lambda^{pk} \int_{\Delta T_k} r dt \\
&\le \frac{\lambda^p}{\lambda^p - 1} \sum_{k=-\infty}^{n} \int_{\Delta T_k} r|f|^p dt \\
&= \frac{\lambda^p}{\lambda^p - 1} \int_{\alpha}^{\beta} r|f|^p dt.
\end{aligned} \tag{2.20}
$$

Combining (2.19) and (2.20) with (2.18), we have

$$
\int_{\alpha}^{\beta} w|f|^p dt \le B_{r,w} \lambda^p (\lambda^p - 1) \max\left\{ \frac{1}{(\lambda - 1)^p}, \frac{\lambda^p}{\lambda^p - 1} \right\} \int_{\alpha}^{\beta} (\rho|f'|^p + r|f|^p) dt.
$$

Since the first part of the above inequality is independent of $\lambda > 1$, the following inequality can be obtained using Lemma 2.1,

$$
\begin{aligned}
\int_{\alpha}^{\beta} w|f|^p dt &\le B_{r,w} \inf_{\lambda > 1} \left\{ \lambda^p(\lambda^p - 1) \max\left\{ \frac{1}{(\lambda - 1)^p}, \frac{\lambda^p}{\lambda^p - 1} \right\} \int_{\alpha}^{\beta} (\rho|f'|^p + r|f|^p) dt \right\} \\
&= B_{r,w} \min\left\{ \inf_{1<\lambda<\lambda_0} \lambda^p \frac{(\lambda^p - 1)}{(\lambda - 1)^p}, \inf_{\lambda \ge \lambda_0} \lambda^{2p} \right\} \int_{\alpha}^{\beta} (\rho|f'|^p + r|f|^p) dt \\
&= \gamma_p B_{r,w} \int_{\alpha}^{\beta} (\rho|f'|^p + r|f|^p) dt.
\end{aligned}
$$

Thus the inequality (2.1) holds with $C \le \gamma_p B_{r,w}$ where C is the best constant which together with (2.10) gives (2.5). This completes the proof. ■

Now, we consider the inequality

$$
\int_{\alpha}^{\beta} w|f|^p dt \le C \int_{\alpha}^{\beta} \rho|f'|^p dt, \; f \in \mathring{W}_p^1(\rho; I_0) \tag{2.21}
$$

where $\mathring{W}_p^1(\rho; I_0)$ is the closure of $\mathring{AC}_p(I_0)$ in the norm

$$
\|f\|_{W_p^1(\rho)} = \left(\int_{\alpha}^{\beta} \rho|f'(t)|^p dt \right)^{1/p} + |f(x_0)|,
$$

and $x_0 \in I_0$ is a fixed point. In case $r \equiv 0$ from (2.2) and (2.3) we have

$$\varphi_0^-(\alpha,x)=\left(\int_\alpha^x \rho^{1-p'}(t)dt\right)^{1-p},\ \ \varphi_0^+(x,\beta)=\left(\int_x^\beta \rho^{1-p'}(t)dt\right)^{1-p}.$$

From the proof of the Theorem 2.1 follows

Theorem 2.2. *Let $a \le \alpha < \beta \le b$. The inequality* (2.21) *holds if and only if $B_w(\alpha,\beta)<\infty$. Moreover the best constant C in* (2.3) *satisfies*

$$B_w \le C \le \widetilde{\gamma}_p B_w,$$

where

$$\begin{aligned} B_w &:= B_w(\alpha,\beta)\\ &= \sup_{\alpha<c<d<\beta}\left(\int_c^d w(t)dt\right)\left[\left(\int_\alpha^c \rho^{1-p'}(t)dt\right)^{1-p}+\left(\int_d^\beta \rho^{1-p'}(t)dt\right)^{1-p}\right]^{-1}\end{aligned}$$

and

$$\widetilde{\gamma}_p = \inf_{1<\lambda}\frac{\lambda^p(\lambda^p-1)}{(\lambda-1)^p}.$$

Inequality (2.21) is the Hardy inequality in differential form. It is well studied (see., e.g., [1, 11, 18]). Compared to previous studies, Theorem 2.2 gives a criterion for the inequality (2.21), regardless of summability or nonsummability function $\rho^{1-p'}$ at the ends of the interval I_0.

3. Applications of Weighted Hardy Type Inequalities to Oscillation Results of Half-linear Differential Equations

We consider the following second order differential equation on the interval $I=(a,b)$, $-\infty \le a < b \le +\infty$:

$$\left(\rho(t)|y'|^{p-2}y'\right)' + v(t)|y|^{p-2}y = 0, \tag{3.1}$$

where $1<p<\infty$, ρ and v are continuous functions on I. Moreover $\rho(t)>0$ for any $t\in I$. When $p=2$, the equation (3.1) becomes the linear Sturm-Liouville equation

$$\left(\rho(t)y'\right)' + v(t)y = 0. \tag{3.2}$$

The investigation on qualitative properties of the solution of this equation was started by J. Sturm [21]. When $p\neq 2$, the equation (3.1) is called half-linear because the set of its solutions has the property of homogeneity but not additivity.

By a solution of (3.1), we mean a function $y: I\subset R \to R$ such that y and $\rho|y'|^{p-2}y$, are continuously differentiable and satisfy (3.1) for $t\in I$. A nontrivial solution of equation (3.1) is called oscillatory at $t = b$ ($t = a$), if it has infinite number of zeros converging to $b(a)$, otherwise it is called nonoscillatory at $t = b$ ($t = a$). Equation (3.1) is called oscillatory (nonoscillatory), if all its nontrivial solutions are oscillatory (nonoscillatory). Since Sturm Theorems hold for equation (3.1), Eq. (3.1) is oscillatory (nonoscillatory), if one of its nontrivial solution is oscillatory(nonoscillatory) [5].

To investigate the oscillation properties of (3.1) it is proper to use the notations such as conjugacy and disconjugacy. Equation (3.1) is called disconjugate on the interval $(\alpha, \beta) \subset I$, if of its any nontrivial solution has no more than one zero on (α, β). Otherwise it is called conjugates on (α, β).

Currently, there is plenty of work devoted to the study of the oscillatory solutions of the equation (3.1) using different methods [2, 3, 22, 23, 24, 25, 26]. Many of the results on oscillatory solutions of (3.1) are related to the integrability of the coefficient functions ρ and v. Some of these results are given in terms of the global integral functions ρ, v and some of them depend on whether or not the functions $\rho^{1-p'}$ and v are integrable on the end points of the integral I (see, e.g., [5, Sections 2 and 3]).

In [20], Rehak discusses that integrability of $\rho^{1-p'}$ is not required at the ends of the interval I to study the behavior of solution of (3.1). This gave us the idea that behavior oscillation of (3.1) can be studied whether or not $\rho^{1-p'}$ and v are integrable at the ends of the interval I. On the other hand, when the function v in (3.1) is nonpositive, then equation (3.1) is nonoscillatory. Therefore the oscillation of (3.1) depends of the positive part of v. This raises the following question: What are the contributions of negative and positive part of v and integrability or nonintegrability of $\rho^{1-p'}$ at the ends of the interval I for the oscillation of (3.1)? The same question also arises in the study of perturbed equations. Our study is associated with the above issues.

Consider the equation

$$\left(\rho(t)|y'|^{p-2}y'\right)' + w(t)|y|^{p-2}y - r(t)|y|^{p-2}y = 0,\ t \in I, \tag{3.3}$$

where w, r nonnegative continuous functions on I. Equation (3.1) is a special case of equation (3.3), since $v = v_+ - v_-$, where $v_+(t) = \max(0, v(t))$ and $v_-(t) = \max(0, -v(t))$ for $t \in I$. Equation (3.3) can be considered as a perturbation of the nonoscillation Equation

$$\left(\rho(t)|y'|^{p-2}y'\right)' - r(t)|y|^{p-2}y = 0. \tag{3.4}$$

One of the fundamental results in the qualitative theory of half-linear equation is the “Roundabout theorem” [5]. Our study of the equations (3.1) and (3.3) is based on the variational principle derived from “Roundabout theorem.” According to Theorem 5.8.1 from [5] we have the following result.

Lemma 3.2. *Let $a < \alpha < \beta < b$. Equation (3.1) is disconjugate on $I_0 = (\alpha, \beta)$ if and only if*

$$F(f; \alpha, \beta) \equiv \int_\alpha^\beta \Big(\rho(t)|f'|^p - v(t)|f|^p\Big) dt \geq 0 \tag{3.5}$$

for all $f \in \overset{\circ}{W}{}_p^1(I_0)$.

Remark 4. If in Lemma 3.1 the interval (α, β) is replaced by the closed interval $[\alpha, \beta]$ then by Theorem 1.2.2 from [5] the sign $\geq$ in the inequality (3.5) is replaced by the symbol $>$.

Since $v \in C[\alpha, \beta]$ and $f \in C[\alpha, \beta]$, then $\int_\alpha^\beta v(t)|f(t)|^p dt < \infty$. Thus the inequality (3.5) is equivalent to

$$\int_\alpha^\beta v(t)|f(t)|^p dt \leq \int_\alpha^\beta \rho(t)|f'(t)|^p dt \tag{3.6}$$

for all $f \in \overset{\circ}{W}{}_p^1(I_0)$.

On the other hand, since $w \in C[\alpha, \beta]$ and $r \in C[\alpha, \beta]$ then in case $v = w - r$ the inequality (3.6) are equivalent to

$$\int_\alpha^\beta w(t)|f(t)|^p dt \leq \int_\alpha^\beta \Big(\rho(t)|f'(t)|^p + r(t)|f(t)|^p\Big) dt \tag{3.7}$$

for all $f \in \overset{\circ}{W}{}_p^1(I_0)$.

We have the following

Lemma 3.3. *Let $a < \alpha < \beta < b$. Then $\overset{\circ}{W}{}_p^1(I_0) = \overset{\circ}{W}{}_p^1(\rho, r; I_0)$ and their norms are equivalent.*

Proof. Let $\gamma_1^p = \max\{\max_{\alpha \leq t \leq \beta} \rho(t), \max_{\alpha \leq t \leq \beta} r(t)\}$. Then it is obvious that

$$\|f\|_{W_p^1(\rho, r; I_0)} \leq \gamma_1 \|f\|_{W_p^1(I_0)}, \quad f \in \overset{\circ}{W}_p(I_0)$$

and, hence

$$\overset{\circ}{W}{}_p^1(I_0) \hookrightarrow \overset{\circ}{W}{}_p^1(\rho, r; I_0). \tag{3.8}$$

Since $\rho > 0$ on I and $\rho \in C[\alpha, \beta]$ then

$$\int_\alpha^\beta |f'|^p dt \leq \max_{\alpha \leq s \leq \beta} \big(\rho(s)\big)^{-1} \int_\alpha^\beta \rho |f'|^p dt. \tag{3.9}$$

Using $f(\alpha) = f(\beta) = 0$ for $f \in \overset{\circ}{AC}_p(I_0)$ and $f(t) = \int_\alpha^t f'(s)ds$ for all $t \in I_0$ and by Hölder's inequality for every $f \in \overset{\circ}{AC}_p(I_0)$ we have

$$\begin{aligned}\int_\alpha^\beta |f(t)|^p dt &= \int_\alpha^\beta \left| \int_\alpha^t f'(s)ds \right|^p dt \\ &\le \int_\alpha^\beta \left(\int_\alpha^t \rho^{1-p'}(s)ds \right)^{p-1} dt \left(\int_\alpha^\beta \rho(t)|f'(t)|^p dt \right) \\ &\le (\beta - \alpha) \left(\int_\alpha^\beta \rho^{1-p'} \right)^{p-1} \int_\alpha^\beta \rho |f'|^p dt. \end{aligned} \tag{3.10}$$

By using (3.9) and (3.10), we have

$$\int_\alpha^\beta |f(t)|^p dt + \int_\alpha^\beta |f'(t)|^p dt \le \gamma_2 \left(\int_\alpha^\beta \rho(t)|f'(t)|^p dt \right) \tag{3.11}$$

for all $f \in \overset{\circ}{AC}_p(I_0)$, where

$$\gamma_2^p = \max \left\{ (\beta - \alpha) \left(\int_\alpha^\beta \rho^{1-p'}(t)dt \right)^{p-1}, \max_{\alpha \le s \le \beta} \left(\rho(s) \right)^{-1} \right\}.$$

Since the set $\overset{\circ}{AC}_p(I_0)$ is dense in $\overset{\circ}{W}{}_p^1(\rho, r; I_0)$, then the inequality $\|f\|_{W_p^1(I_0)} \le \gamma_2 \|f\|_{W_p^1(\rho,r;I_0)}$ holds for all $f \in W_p^1(\rho, r; I_0)$. Hence

$$\overset{\circ}{W}{}_p^1(\rho, r; I_0) \hookrightarrow \overset{\circ}{W}{}_p^1(I_0). \tag{3.12}$$

Then by (3.8) and (3.12) we have $\overset{\circ}{W}{}_p^1(\rho, r; I_0) = \overset{\circ}{W}{}_p^1(I_0)$ and their norms are equivalent. ■

By using Lemmas 3.1 and 3.2, we have the following result.

Theorem 3.1. *Let $a < \alpha < \beta < b$. Then the equation* (3.3) *is disconjugate on the interval I_0 if and only if*

$$\int_\alpha^\beta w(t)|f(t)|^p dt \le \int_\alpha^\beta \left(\rho(t)|f'(t)|^p + r(t)|f(t)|^p \right) dt \tag{3.13}$$

for all nontrivial $f \in \overset{\circ}{W}{}_p^1(\rho, r; I_0)$.

Remark 5. According to Remark 3, if in Theorem 3.1 the interval (α, β) is replaced by the closed interval $[\alpha, \beta]$ then the sign $\le$ in the inequality (3.13) is replaced by the symbol $<$.

Following Theorem 3.1, we can extend the above result to the general interval I, where the functions $\rho^{1-p'}$, r and w can not be summable on the interval I.

Theorem 3.2. *Let $a \le \alpha < \beta \le b$. The equation* (3.3) *is disconjugate on the interval I_0 if and only if the inequality* (3.13) *holds.*

Proof. We will prove this by a contradiction. Suppose that (3.13) holds but the equation (3.3) is conjugate on (α, β) i.e. if the solution y of (3.3) have conjugate points t_1, t_2 such that $\alpha < t_1 < t_2 < \beta$, then there exist $c \in (\alpha, t_1)$ and $d \in (t_2, \beta)$ such that the equation (3.3) is conjugate on $(c, d) \subset (\alpha, \beta)$. From Theorem 3.1, there exists $\tilde{f} \in \mathring{W}_p^1\big(\rho, r; (c,d)\big) = \mathring{W}_p^1(c,d)$ such that

$$\int_c^d w|\tilde{f}|^p dt > \int_c^d \Big(\rho|\tilde{f}'|^p + r|\tilde{f}|^p\Big) dt.$$

Define

$$\tilde{\tilde{f}}(t) = \begin{cases} \tilde{f}(t) & t \in (c,d)\ , \\ 0 & t \in (\alpha,\beta)\backslash(c,d). \end{cases}$$

Then $\tilde{\tilde{f}} \in \mathring{W}_p^1\big(\rho, r; (\alpha,\beta)\big)$ and $\tilde{\tilde{f}}$ satisfies the following

$$\int_\alpha^\beta w|\tilde{\tilde{f}}|^p dt > \int_\alpha^\beta \Big(\rho|\tilde{\tilde{f}}'|^p + r|\tilde{\tilde{f}}|^p\Big) dt$$

which contradicts with (3.13). Hence equation (3.3) is disconjugate on $I = (\alpha, \beta)$.

Conversely, we assume that the equation (3.3) is disconjugate on I but the inequality (3.13) does not hold. Then (3.3) is also disconjugate on all $(c,d) \subset (\alpha,\beta)$. Then for arbitrary $(c,d) \subset (\alpha,\beta)$ (but $(c,d) \neq (\alpha,\beta)$), by Theorem 3.1 we have

$$\int_c^d w|f|^p dt \le \int_c^d \Big(\rho|f'|^p + r|f|^p\Big) dt, \quad \text{for all } f \in \mathring{W}_p^1\big(\rho, r; (c,d)\big). \tag{3.14}$$

Since $\operatorname{supp} f \subset (\alpha,\beta)$ for $f \in \mathring{AC}_p(\alpha,\beta)$, then there exists $(c,d) \subset (\alpha,\beta)$ such that $\operatorname{supp} f \subset (c,d)$. Then from (3.14) we obtain that the inequality (3.13) holds for all $f \in \mathring{AC}_p(\alpha,\beta)$. Because the set $\mathring{AC}_p(\alpha,\beta)$ dense in $\mathring{W}_p^1\big(\rho, r; (\alpha,\beta)\big)$ then inequality (3.13) holds for all $f \in \mathring{W}_p^1\big(\rho, r; (\alpha,\beta)\big)$ which contradicts with our assumption. Thus the proof of Theorem 3.2 is completed. ∎

Theorems 2.1 and 3.2 give the following criterion, for the equation (3.3).

Theorem 3.3. *Let $a \le \alpha < \beta \le b$. Then,*

(i) *for the disconjugacy of equation* (3.3) *on the interval $I_0 = (\alpha,\beta)$, the necessary condition is $B_{r,w}(\alpha,\beta) \le 1$ and the sufficient condition is $\gamma_p B_{r,w}(\alpha,\beta) \le 1$.*

(ii) *for the conjugacy of equation* (3.3) *on the interval* $I_0 = (\alpha, \beta)$*, the necessary condition is* $\gamma_p B_{r,w}(\alpha, \beta) > 1$ *and the sufficient condition is* $B_{r,w}(\alpha, \beta) > 1$.

Proof. Assertions (i) and (ii) are equivalent. We prove assertion (ii). Let the Equation (3.3) is conjugate on the interval $I_0 = (\alpha, \beta)$. Then by Theorem 3.2 the inequality (3.13) does not hold. Hence $C > 1$, where the best constant C is given in (2.1). Then in view of (2.5) we obtain $\gamma_p B_{r,w}(\alpha, \beta) > 1$. Conversely, if $B_{r,w}(\alpha, \beta) > 1$, then from (2.5) we have $C > 1$ for the best constant C in (2.1). Therefore the inequality (3.13) does not hold. Then by using Theorem 3.2 the equation (3.3) is conjugate on the interval $I_0 = (\alpha, \beta)$. ■

From Theorem 3.3 we have the following:

Corollary 3.1. *Let* $a \le \alpha < \beta \le b$*. If there exist* $\alpha < c < d < \beta$ *such that*

$$\int_c^d w(t)dt > \varphi^-(\alpha, c) + \int_c^d r(s)ds + \varphi^+(d, \beta)$$

then the equation (3.3) *is conjugate on the interval* $I_0 = (\alpha, \beta)$*. If the equation* (3.3) *is conjugate on the interval* I_0 *then there exists an interval* $(c, d) \subset I_0$ *such that*

$$\int_c^d w(t)dt > \gamma_p^{-1}\Big(\varphi^-(\alpha, c) + \int_c^d r(s)ds + \varphi^+(d, \beta)\Big). \tag{3.15}$$

If the equation (3.3) *is disconjugate on the interval* I_0 *then*

$$\int_c^d w(t)dt \le \varphi^-(\alpha, c) + \int_c^d r(s)ds + \varphi^+(d, \beta) \tag{3.16}$$

for all an interval $(c, d) \subset I_0$.

Remark 6. Let $a < \alpha < \beta < b$ in Corollary 3.1. If the interval I_0 is replaced by the closed interval $[\alpha, \beta]$ then in Corollary 3.1 the $(c, d) \subset I_0$ is replaced by $[c, d] \subseteq [\alpha, \beta]$ and all sign $>$ (resp.$\le$) is replaced by symbol $\ge$ (resp.$<$).

Remark 7. Corollary 3.1 shows that the local behavior of the perturbation w can turn the disconjugate equation (3.4) to the conjugate equation (3.3). For example, let $\alpha < c < d < \beta$ and $w = \mu w_1$, where w_1 is a continuous function on $I_0 = (\alpha, \beta)$ such that $\operatorname{supp} w_1 \subset (c, d)$ and $\int_c^d w_1(t)dt = 1$. Then, for $\mu > \varphi^-(\alpha, c) + \int_c^d r(s)ds + \varphi^+(d, \beta)$ the equation (3.3) is conjugate on the interval I_0.

In the next theorem we give a criterion of oscillatory or nonoscillatory of equation (3.3) without assuming the integrability or not integrability of functions $\rho^{1-p'}$ at the end of the interval I.

Theorem 3.4. *If* $\lim_{\alpha\to b} B_{r,w}(\alpha,b) > 1$ *(*$\lim_{\beta\to a} B_{r,w}(a,\beta) > 1$*), then the equation* (3.3) *is oscillatory at* $t=b$ $(t=a)$. *If there exists an* $\alpha \in I$ *(*$\beta\in I$*) such that* $\gamma_p B_{r,w}(\alpha,b) \le 1$ *(*$\gamma_p B_{r,w}(a,\beta)\le 1$*) then the equation* (3.3) *is nonoscillatory at* $t=b$ $(t=a)$.

Proof. We prove only for $t=b$, and for $t=a$, the proof is similar. Let $\lim_{\alpha\to b} B_{r,w}(\alpha,b)>1$. Then there exists a sequence $\{\alpha_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\alpha_k = b$, $B_{r,w}(\alpha_k,b)>1$, and by Theorem 3.3 the equation (3.3) on the interval (α,b) has at least two conjugate points, i.e. there is a nontrivial solution of the equation (3.3) with two zeros in this interval. Then there exists a subsequence $\{\widetilde{\alpha}_n\}_{n=1}^{\infty}\subset\{\alpha_k\}_{k=1}^{\infty}$ such that for each interval $(\widetilde{\alpha}_n,\widetilde{\alpha}_{n+1})$ there exists a nontrivial solution of the equation (3.3) which has two zeros in it. Consequently, by Sturm theory, there is a nontrivial solution of the equation (3.3) having at least one zero $x_n\in(\widetilde{\alpha}_n,\widetilde{\alpha}_{n+1})$ at each interval $(\widetilde{\alpha}_n,\widetilde{\alpha}_{n+1})$. Since $\lim_{n\to\infty}\alpha_n=b$, then $\lim_{n\to\infty}x_n=b$. Hence this solution is oscillatory at $t=b$, and therefore, all solutions of the equation (3.3) is oscillatory, i.e. Equation (3.3) is oscillatory at $t=b$.

Now, suppose that there exists a point $\alpha\in I$ such that $\gamma_p B_{r,w}(\alpha,b)\le 1$. Then by Theorem 3.3 equation (3.3) is disconjugate on the interval (α,b), i.e. all non-trivial solutions of the equation (3.3) does not have more than one zero in the interval (α,b). Hence the equation (3.3) is nonoscillatory at $t=b$. ■

From Theorems 3.3 and 3.4 we have the following:

Corollary 3.2. *If there exist the sequences of numbers* $\alpha_k,\ c_k,\ d_k,\ k\ge 1$ *such that* $a<\alpha_k<c_k<d_k<c_{k+1}<b$, $\alpha_k\to b$ *as* $k\to\infty$ *and*

$$\int_{c_k}^{d_k} w(t)dt > \varphi_r^-(\alpha_k,c_k) + \int_{c_k}^{d_k} r(t)dt + \varphi_r^+(d_k,b)$$

for all $k\ge 1$ *then the equation* (3.3) *is oscillatory at* $t=b$*; If the equation* (3.3) *is oscillatory at* $t=b$ *then there exist the sequences of numbers* $\alpha_k,\ c_k,\ d_k,\ k\ge1$ *such that* $a<\alpha_k<c_k<d_k<c_{k+1}<b$, $\alpha_k\to b$ *as* $k\to\infty$ *and*

$$\int_{c_k}^{d_k} w(t)dt > \gamma_p^{-1}\big(\varphi_r^-(\alpha_k,c_k) + \int_{c_k}^{d_k} r(t)dt + \varphi_r^+(d_k,b)\big).$$

Remark 8. Under the conditions of Corollary 3.2, we set $w=\sum_{k=1}^{\infty}\mu_k w_k$, where $\mu_k>0$ and the function $w_k, k\ge 1$ is continuous on I and satisfy the conditions $\operatorname{supp} w_k\subset(c_k,d_k)$ and $\int_{c_k}^{d_k} w_k(t)dt=1$. If $\mu_k>\varphi_r^-(\alpha_k,c_k)+\int_{c_k}^{d_k} r(t)dt+\varphi_r^+(d_k,b)$ for all sufficiently large k, then the equation (3.3) oscillatory at $t=b$, i.e. repetitive impulse perturbations translate nonoscillatory equation (3.4) to the oscillatory equation (3.3).

In particular, from Theorems 3.3, 3.4 and Corollaries 3.1, 3.2 we have

Theorem 3.5. *Let $a \le \alpha < \beta \le b$. If $B_{v_-,v_+}(\alpha,\beta) > 1$, then equation* (3.1) *is conjugate on the interval $I_0 = (\alpha,\beta)$ and if the equation* (3.1) *is conjugate on the interval $I_0 = (\alpha,\beta)$ then $\gamma_p B_{v_-,v_+}(\alpha,\beta) > 1$.*

If $\gamma_p B_{v_-,v_+}(\alpha,\beta) \le 1$, then the equation (3.1) *is disconjugate on (α,β) and if the equation* (3.1) *is disconjugate in (α,β) then $B_{v_-,v_+}(\alpha,\beta) \le 1$.*

Theorem 3.6. *If $\lim_{\alpha\to b} B_{v_-,v_+}(\alpha,b) > 1$ ($\lim_{\beta\to a} B_{v_-,v_+}(a,\beta) > 1$), then the equation* (3.1) *is oscillatory at $t = b$ $(t = a)$. If there exists an $\alpha \in I$ ($\beta \in I$) such that $\gamma_p B_{v_-,v_+}(\alpha,b) \le 1$ ($\gamma_p B_{v_-,v_+}(a,\beta) \le 1$) then the equation* (3.1) *is nonoscillatory at $t = b$ $(t = a)$.*

Corollary 3.3. *Let $a \le \alpha < \beta \le b$.*

(i) *If there exist $\alpha < c < d < \beta$ such that*

$$\int_c^d v^+(t)dt > \varphi^-_{v^-}(\alpha,c) + \int_c^d v^-(s)ds + \varphi^+_{v^-}(d,\beta)$$

or

$$\int_c^d v(t)dt > \varphi^-_{v^-}(\alpha,c) + \varphi^+_{v^-}(d,\beta)$$

then the equation (3.1) *is conjugate on the interval $I_0 = (\alpha,\beta)$.*

(ii) *If the equation* (3.1) *is conjugate on the interval I_0 then there exists an interval $(c,d) \subset I_0$ such that*

$$\int_c^d v^+(t)dt > \gamma_p^{-1}\Big(\varphi^-(\alpha,c) + \int_c^d v^-(s)ds + \varphi^+(d,\beta)\Big).$$

(iii) *If the equation* (3.1) *is disconjugate on the interval I_0 then*

$$\int_c^d v(t)dt \le \varphi^-(\alpha,c) + \varphi^+(d,\beta)$$

for all interval $(c,d) \subset I_0$.

Remark 9. Let $a < \alpha < \beta < b$ in Theorem 3.5 and Corollary 3.3. If the interval I_0 is replaced by the closed interval $[\alpha,\beta]$ then in Theorem 3.5 and Corollary 3.3 the $(c,d) \subset I_0$ is replaced by $[c,d] \subseteq [\alpha,\beta]$ and all sign $>$ ($\le$) is replaced by symbol $\ge$ ($<$).

Corollary 3.4. *Let $b = \infty$. Equation* (3.1) *is oscillatory at $t = \infty$ if any one of the conditions holds:*

(i) *there exist the sequences of numbers* $\alpha_k,\ c_k,\ d_k,\ k \geq 1$ *such that* $a < \alpha_k < c_k < d_k < c_{k+1} < b$, $\alpha_k \to b$ *as* $k \to \infty$ *and*

$$\int_{c_k}^{d_k} v(t)dt > \varphi^-_{v^-}(\alpha_k, c_k) + \varphi^+_{v^-}(d_k, \infty)$$

for all $k \geq 1$*;*

(ii) *for some* $h > 0$

$$\limsup_{c\to\infty} \frac{\int_c^{c+h} v(t)dt}{\varphi^-(c-h, c) + \varphi^+(c+h, b)} > 1.$$

Remark 10. Under the conditions of assertion (i) of Corollary 3.4 if $v^-(t) = 0, \forall t \in (c_k, d_k)$, $\forall k \geq 1$ and

$$\int_{c_k}^{d_k} v^+(t)dt > \varphi^-_{v^-}(\alpha_k, c_k) + \varphi^+_{v^-}(d_k, \infty)$$

for all sufficiently large k then the equation (3.1) is oscillatory at $t = \infty$.

In the case of $v \geq 0$ from Theorems 3.5 and 3.6 we get the following results:

Theorem 3.7. *Let* $a \leq \alpha < \beta \leq b$ *and* $v \geq 0$*. If* $B_v(\alpha, \beta) > 1$ *then the equation* (3.1) *is conjugate on the interval* I_0*, and if* $\lim_{\alpha\to b} B_v(\alpha, b) > 1$ *the equation* (3.1) *is oscillatory at* $t = b$*. If* $\widetilde{\gamma}_p B_v(\alpha, \beta) \leq 1$*, then the equation* (3.1) *is disconjugate on the interval* I_0 *and if there is a point* $c \in I$ *such that* $\widetilde{\gamma}_p B_{v^+}(c, b) \leq 1$ *then the equation* (3.1) *is nonoscillatory at* $t = b$*.*

The general results for half-linear equations in Theorems 3.5, 3.6 and 3.7 and Corollaries 3.2 and 3.3 also hold for the linear equation when we assert $p = 2$.

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