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ZEROS OF A QUASI-MODULAR FORM OF WEIGHT 2 FOR $\Gamma_0^+(N)$

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Abstract. Basraoui and Sebbar showed that the Eisenstein series E_2 has infinitely many $\mathrm{SL}_2(\mathbb{Z})$-inequivalent zeros in the upper half-plane $\mathbb{H}$, yet none in the standard fundamental domain $\mathfrak{F}$. They also found infinitely many such regions containing a zero of E_2 and infinitely many regions which do not have any zeros of E_2. In this paper we study the zeros of the quasi-modular form $E_2(z) + NE_2(Nz)$ of weight 2 for $\Gamma_0^+(N)$.

1. Introduction and Preliminaries

It is well known by the the Valence formula [12, Section 1.3, Proposition 2] that every nonzero modular form has finitely many $\mathrm{SL}_2(\mathbb{Z})$-inequivalent zeros in the upper half-plane $\mathbb{H}$. Several authors investigated the zeros of special modular forms for $\mathrm{SL}_2(\mathbb{Z})$ (for example, see [3, 4, 5, 9]). It has been proved that for an even integral weight k the Eisenstein series E_k for $\mathrm{SL}_2(\mathbb{Z})$, the zeros of E_k in the fundamental domain of the modular group $\mathrm{SL}_2(\mathbb{Z})$ lie in the arc of the unit circle for $4 \le k \le 26$ by Wohlfahrt [11] and for every $k > 2$, by Rankin and Swinnerton-Dyer [8] later. Rankin [7] generalized this result to a certain class of Poincaré series for $\mathrm{SL}_2(\mathbb{Z})$.

For higher level cases, let $\Gamma_0^+(N)$ denote the group generated by the Hecke congruence group $\Gamma_0(N)$ and the Fricke involution $w_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Shigezumi [6] investigated the zeros of the Eisenstein series for $\Gamma_0^+(2)$ and $\Gamma_0^+(3)$. Recently Basraoui and Sebbar [1] investigated some properties of zeros of the Eisenstein series E_2 for $\mathrm{SL}_2(\mathbb{Z})$ which is a quasi-modular form. They showed that there are infinitely many

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inequivalent zeros of E_2 in the half strip $\mathfrak{S} := \{\tau \in \mathbb{H} \mid -1/2 < \mathrm{Re}(\tau) \leq 1/2\}$ and proved that the fundamental domain $\mathfrak{F}$ for $\mathrm{SL}_2(\mathbb{Z})$ and infinitely many of its conjugates in $\mathfrak{S}$ contain no zeros of E_2, while there are infinitely many conjugates of $\mathfrak{F}$ in $\mathfrak{S}$ which contain zeros of E_2. This is a different phenomenon from the cases for modular forms.

In this paper, by applying the arguments in [1] we study the zeros of the quasi-modular form $E_2(z) + NE_2(Nz)$ of weight 2 for $\Gamma_0^+(N)$, whose definition is given in Definition 1.1. In particular, we show how to take care of the parts related with the Fricke involution while the proofs in [1] deal with $\mathrm{SL}_2(\mathbb{Z})$.

Throughout this paper, we let $z = x + iy$ with $x,\ y > 0 \in \mathbb{R}$ and denote $\Gamma_0(N)$ or $\Gamma_0^+(N)$ by Γ.

Definition 1.1. [12, page 58] For a positive even integer k, *an almost holomorphic modular form* of weight k and depth $\leq M$ for Γ is a holomorphic function $F(z)$ on $\mathbb{H}$ such that

$$F\left(\frac{az+b}{cz+d}\right) = (\det\gamma)^{-k/2}\,(cz+d)^k F(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and the growth condition that it has the form

$$F(z) = \sum_{m=0}^{M} f_m(z)(-4\pi y)^{-m},\ \text{(where } f_0(z), \ldots, f_M(z) \text{ are holomorphic on } \mathbb{H})$$

for some nonnegative integer M (which is necessarily at most $k/2$).

The constant term, $f_0(z)$ of such a F is called *a quasi-modular form* of weight k for Γ. We let $\widetilde{M_k}(\Gamma)$ be the $\mathbb{C}$-linear space of quasi-modular forms of weight k for Γ. Then the space $\widetilde{M_*}(\Gamma) = \bigoplus \widetilde{M_k}(\Gamma)$ is a graded ring. Note that as mentioned in [12, page 58], a direct definition of a quasi-modular form of weight k and depth $\leq M$ on Γ can be given as a holomorphic function f on $\mathbb{H}$ such that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, the function $(\det\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ is a polynomial of degree $\leq M$ in $\frac{c}{cz+d}$.

Indeed, if we choose a holomorphic function ϕ on $\mathbb{H}$ such that the function $\phi^*(z) := \phi(z) - 1/(4\pi y)$ satisfies the following,

$$\phi^*(\gamma z) = (\det\gamma)^{-1}(cz+d)^2\phi^*(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{1}$$

where $z = x + iy$, then clearly ϕ is a quasi-modular form of weight 2 for Γ. We can show that every quasi-modular form of weight k for Γ is presented as a polynomial of a quasi-modular form ϕ of weight 2 with coefficients of modular forms as follows:

Proposition 1.2. [12, page 59] *For a positive even integer k and an integer r such that $0 \leq r \leq k/2$, let $M_{k-2r}(\Gamma)$ be the space of modular forms of weight $k-2r$ for Γ where Γ is $\Gamma_0(N)$ or $\Gamma_0^+(N)$. A quasi-modular form of weight k for Γ is an element in the ring $\bigoplus_{r=0}^{k/2} M_{k-2r}(\Gamma) \cdot \phi^r$, where ϕ is a holomorphic function on $\mathbb{H}$ satisfying the condition* (1)*.*

We recall that the Eisenstein series $E_2(z)$ is written as

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \text{ where } \sigma_1(n) = \sum_{1 \leq d | n} d.$$

Then this is a quasi-modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$ and it satisfies that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) - \frac{6i}{\pi} c(cz+d). \tag{2}$$

(This is by normalization of [12, Section 2.3, Eq. (17) and (19)].)

For convenience, we define the slash operator $f \mapsto f|_2\gamma$ by

$$(f|_2\gamma)(z) = (\det\gamma)(cz+d)^{-2} f\left(\frac{az+b}{cz+d}\right), \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

and so we have the definition,

$$(f(g)|_2\gamma)(z) = (\det\gamma)(cz+d)^{-2} f((g(\gamma z))), \text{ for a function } g : \mathbb{H} \to \mathbb{H}.$$

We now prove that $E_2(z) + NE_2(Nz)$ is a quasi-modular form of weight 2 for $\Gamma_0^+(N)$ and calculate some special values of $E_2(z) + NE_2(Nz)$ which will be needed later.

Proposition 1.3.

(1) $E_2(z) + NE_2(Nz)$ *is a quasi-modular form of weight* 2 *on* $\Gamma_0^+(N)$.

(2) $E_2(z) - NE_2(Nz)$ *is a modular form of weight* 2 *on* $\Gamma_0(N)$.

Proof. We let

$$E_2^*(z) := E_2(z) - \frac{3}{\pi y}.$$

Then E_2^* is invariant under the slash operator $|_2$ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(1) Let $E(z) = E_2(z) + NE_2(Nz)$. Then

$$E(z) = E_2^*(z) + \frac{3}{\pi y} + N\left(E_2^*(Nz) + \frac{3}{\pi N y}\right) = E_2^*(z) + NE_2^*(Nz) + \frac{6}{\pi y}. \tag{3}$$

Hence

$$E(z) - \frac{6}{\pi y} = E_2^*(z) + NE_2^*(Nz). \tag{4}$$

Let $g(z) = Nz$. Considering $E_2^*(Nz) = E_2^*(g(z))$, we have that for any $\gamma = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$,

$$\begin{aligned}(E_2^*(g)|_2\, \gamma)(z) &= E_2^*(N\gamma z)(cNz + d)^{-2} \\ &= E_2^*\left(\frac{a(Nz) + bN}{c(Nz) + d}\right)(cNz + d)^{-2} \\ &= (E_2^*|_2\gamma')(Nz) = E_2^*(Nz) = E_2^*(g(z)),\end{aligned} \tag{5}$$

where $\gamma' = \begin{pmatrix} a & bN \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. (Note that the last equality follows from the fact that E_2^* is invariant under the slash operator $|_2$.)

Hence this implies that for all $\gamma \in \Gamma_0(N)$,

$$((E_2^* + NE_2^*(g))|_2\gamma)(z) = E_2^*(z) + NE_2^*(Nz).$$

Now for $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, we have that

$$\begin{aligned}((E_2^* + NE_2^*(g))|_2 w_N)(z) &= (\sqrt{N}z)^{-2}\left(E_2^*\left(\frac{-1}{Nz}\right) + NE_2^*\left(\frac{-1}{z}\right)\right) \\ &= N^{-1}z^{-2}E_2^*\left(\frac{-1}{Nz}\right) + z^{-2}E_2^*\left(\frac{-1}{z}\right) \\ &= N(Nz)^{-2}E_2^*\left(\frac{-1}{Nz}\right) + z^{-2}E_2^*\left(\frac{-1}{z}\right) \\ &= E_2^*(z) + NE_2^*(Nz).\end{aligned} \tag{6}$$

Note that the last inequality follows from the modularity under $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Hence we have shown that for $g(z) = Nz$, $((E_2^* + NE_2^*(g))|_2\gamma)(z) = (E_2^*(z) + NE_2^*(Nz))$, for all $\gamma \in \Gamma_0^+(N)$. This fact together with two conditions (1) and (4) implies that $E(z)$ is a quasi-modular form of weight 2 on $\Gamma_0^+(N)$.

(2) Let $g(z) = Nz$. For all $\gamma \in \Gamma_0(N)$, we have

$$
\begin{aligned}
((E_2 - NE_2(g))|_2\gamma)(z) &= ((E_2^* - NE_2^*(g)|_2\gamma)(z) \\
&= E_2^*(z) - NE_2^*(Nz) \\
&= E_2(z) - NE_2(Nz).
\end{aligned} \tag{7}
$$

Also, we note from (2) that for each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$E_2\left(\frac{az+b}{cz+d}\right)(cz+d)^{-2} = E_2(z) - \frac{6i}{\pi}\frac{c}{(cz+d)}.$$

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and let $s := \gamma\infty = \frac{a}{c}$. Then $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\gamma = \gamma' U$ for some $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $U = \begin{pmatrix} x & y \\ 0 & w_s \end{pmatrix} \in M_2(\mathbb{Z})$. So $N = xw_s$, $c = c'x$ and $d = c'y + d'w_s$. Hence $N/w_s = c/c'$. Therefore, we have

$$
\begin{aligned}
E_2(N\gamma z) &= E_2(\gamma' Uz) \\
&= (c'Uz + d')^2 E_2(Uz) - \frac{6c'i}{\pi}(c'Uz + d') \\
&= \frac{(cz+d)^2 E_2(Uz)}{w_s^2} - \frac{6c'i(cz+d)}{\pi w_s}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
E_2(N\gamma z)(cz+d)^{-2} &= \frac{E_2(Uz)}{w_s^2} - \frac{6c'i}{\pi w_s}\frac{1}{(cz+d)} \\
&= \frac{E_2(Uz)}{w_s^2} - \frac{6ci}{N\pi}\frac{1}{(cz+d)}.
\end{aligned} \tag{8}
$$

So

$$
\begin{aligned}
((E_2 - NE_2(g))|_2\gamma)(z) &= (E_2(\gamma z) - NE_2(N\gamma z))(cz+d)^{-2} \\
&= E_2(z) - \frac{6ci}{\pi}\frac{1}{(cz+d)} - \frac{N}{w_s^2}E_2(Uz) + \frac{6ci}{\pi}\frac{1}{(cz+d)} \qquad (9) \\
&= E_2(z) - \frac{N}{w_s^2}E_2(Uz)
\end{aligned}
$$

and this implies that $E_2(z) - NE_2(Nz)$ is holomorphic at the cusp s. Consequently $E_2(z) - NE_2(Nz)$ is a modular form of weight 2 on $\Gamma_0(N)$. ∎

Throughout this paper, as in the proof of Proposition 1.3 we let

$$E(z) := E_2(z) + NE_2(Nz)$$

for $z \in \mathbb{H}$. Then for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we can easily show by (2) that

$$E\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^2E(z) - \frac{12i}{\pi}c(cz+d). \qquad (10)$$

Note that $\rho_2 := e^{i(3\pi/4)}/\sqrt{2}$ is an elliptic point of nonzero modular functions of weight k for $\Gamma_0^+(2)$ by [6, Proposition 3.1] and $\rho_3 := e^{i(5\pi/6)}/\sqrt{3}$ is an elliptic point for $\Gamma_0^+(3)$ by [6, Proposition 4.3].

Lemma 1.4.

(a) $E(\rho_2) = \dfrac{12}{\pi}$ *for* $N = 2$.

(b) $E(\rho_3) = \dfrac{12\sqrt{3}}{\pi}$ *for* $N = 3$.

Proof. Note that for $\tau \in \mathbb{H}$,

$$
\begin{aligned}
E\Big(-\frac{1}{N\tau}\Big) &= E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right) \\
&= E_2\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right) + NE_2\left(N\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(N\tau)\right) \qquad (11) \\
&= (N\tau)^2E_2(N\tau) + \frac{6}{\pi i}(N\tau) + NE_2\left(-\frac{1}{\tau}\right) \text{ by (2)} \\
&= \tau^2NE(\tau) + \frac{12N}{\pi i}\tau.
\end{aligned}
$$

(a) By (11), for $\tau = \rho_2 = e^{i(3\pi/4)}/\sqrt{2}$ with $N = 2$,

(12) $$E(-\frac{1}{2\rho_2}) = -iE(\rho_2) + \frac{12}{\pi i}(-1+i).$$

Now since $\alpha_2 w_2 \rho_2 = \rho_2$ for $\alpha_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \in \Gamma_0(2)$ and $w_2 = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$, we get from (10) and (12):

$$\begin{aligned}
E(\rho_2) &= E((\alpha_2 w_2)\rho_2) = E(\alpha_2(w_2\rho_2)) \\
&= (-2w_2\rho_2+1)^2 E(w_2\rho_2) + \frac{-24}{\pi i}(-2w_2\rho_2+1) \\
&= \left(\frac{1}{\rho_2}+1\right)^2 E\left(-\frac{1}{2\rho_2}\right) - \frac{24}{\pi i}\left(\frac{1}{\rho_2}+1\right) \\
&= iE(\rho_2) + \frac{12}{\pi i}(1+i)
\end{aligned}$$

by (12).

Hence we solve $E(\rho_2) = iE(\rho_2) + \frac{12}{\pi i}(1+i)$ for $E(\rho_2)$ and we get

$$E(\rho_2) = \frac{12}{\pi}.$$

(b) Similarly, with $\rho_3 = e^{i(5\pi/6)}/\sqrt{3}$ and $N = 3$, we have from (11) that

(13) $$E\Big(-\frac{1}{3\rho_3}\Big) = \Big(\frac{1-\sqrt{3}i}{2}\Big)E(\rho_3) + \frac{6}{\pi i}(-3+\sqrt{3}i).$$

And since $\alpha_3 w_3 \rho_3 = \rho_3$ for $\alpha_3 = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$ and $w_3 = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}$, we have that

$$\begin{aligned}
E(\rho_3) &= E((\alpha_3 w_3)\rho_3) = E(\alpha_3(w_3\rho_3)) \\
&= (-3w_3\rho_3+1)^2 E(w_3\rho_3) + \frac{-36}{\pi i}(-3w_3\rho_3+1) \\
&= \left(\frac{1}{\rho_3}+1\right)^2 E\left(-\frac{1}{3\rho_3}\right) - \frac{36}{\pi i}\left(\frac{1}{\rho_3}+1\right) \\
&= \left(\frac{1+\sqrt{3}i}{2}\right) E(\rho_3) + \frac{6}{\pi i}(3+\sqrt{3}i)
\end{aligned}$$

by (13).

So we solve $E(\rho_3) = (\frac{1+\sqrt{3}i}{2})E(\rho_3) + \frac{6}{\pi i}(3+\sqrt{3}i)$ for $E(\rho_3)$ and get

$$E(\rho_3) = \frac{12\sqrt{3}}{\pi}.$$ ■

2. Zeros of E for $\Gamma_0^+(N)$

In this section we study the zeros of E for $\Gamma_0^+(N)$, where $E(z) = E_2(z) + NE_2(Nz)$.

Proposition 2.1. *For a positive integer N, the quasi-modular form E for $\Gamma_0^+(N)$ has a unique zero τ_0 on the imaginary axis. And for $N = 2, 3$, E for $\Gamma_0^+(N)$ has a zero τ_1 on the axis $\mathrm{Re}(z) = \frac{1}{2}$.*

Proof. This uses the proof of [1, Proposition 3.1] for E_2.

For $\tau = iy$, since $E_2(\tau)$ is real and increasing on $(0, \infty)$ by definition of E_2, $E(\tau)$ is also real and increasing on $(0, \infty)$.

Also since $\lim\limits_{y\to 0} E_2(iy) = -\infty$ and $\lim\limits_{y\to\infty} E_2(iy) = 1$,

$$\lim_{y\to 0} E(iy) = -\infty \quad \text{and} \quad \lim_{y\to\infty} E(iy) = 1 + N > 1. \tag{14}$$

Since $E(iy)$ is continuous and increasing, this implies that E has a unique zero, say τ_0 on the purely imaginary axis.

Note that $E_2(\tau)$ is real for $\tau = \frac{1}{2} + iy$, $y > 0$, and $\lim\limits_{y\to 0} E_2(\frac{1}{2} + iy) = -\infty$. If N is even, then

$$\lim_{y\to 0} E\Big(\frac{1}{2} + iy\Big) = \lim_{y\to 0} \Big(E_2\Big(\frac{1}{2} + iy\Big) + NE_2(Niy)\Big) = -\infty,$$

and if N is odd, then

$$\lim_{y\to 0} E\Big(\frac{1}{2} + iy\Big) = \lim_{y\to 0} \Big(E_2\Big(\frac{1}{2} + iy\Big) + NE_2\Big(\frac{1}{2} + Niy\Big)\Big) = -\infty.$$

If $N = 2$, by Lemma 1.4 (1), $E(\rho_2) = E(\rho_2 + 1) = \frac{12}{\pi} > 0$, hence we conclude that there exists a zero τ_1 of real part $1/2$ and whose imaginary part is less than $1/2$.

If $N = 3$, by Lemma 1.4 (2), $E(\rho_3) = E(\rho_3 + 1) = \frac{12\sqrt{3}}{\pi} > 0$, hence we conclude that there exists a zero τ_1 of real part $1/2$ and whose imaginary part is less than $1/(2\sqrt{3})$. ∎

Proposition 2.2. *For each integer $N \geq 2$, two zeros of E are $\Gamma_0^+(N)$-equivalent if and only if one is a translation of the other by an integer.*

Proof. Suppose that z_1 and z_2 are any two zeros of E in $\mathbb{H}$ that are equivalent modulo $\Gamma_0^+(N)$, i.e. $z_1 = \alpha z_2$ for some $\alpha \in \Gamma_0^+(N)$.

If $\alpha \in \Gamma_0(N)$, α must be a translation as in the proof of [1, Proposition 3.3].

If $\alpha = \gamma w_N$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $w_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, then we have from (10) and (11) that

$$0 = E(z_1) = E(\gamma(w_N \cdot z_2)) = (cw_N z_2 + d)^2 E(w_N z_2) + \frac{12c}{\pi i}(cw_N z_2 + d)$$

and

$$E(w_N z_2) = \frac{12N}{\pi i} z_2 + N z_2^2 E(z_2) = \frac{12N}{\pi i} z_2.$$

Hence $0 = (cw_N z_2 + d)^2 \frac{12N}{\pi i} z_2 + \frac{12C}{\pi i}(cw_N z_2 + d)$ implies that $cw_N z_2 + d = 0$ or $(cw_N z_2 + d)N z_2 + c = 0$. Note that $w_N \cdot z_2 \in \mathbb{H}$ implies that $cw_N \cdot z_2 + d \neq 0$, since $\gamma \in \Gamma_0(N)$. So $0 = (cw_N z_2 + d)N z_2 + c = (-\frac{c}{N z_2} + d)N z_2 + c = dN z_2$. Then $d = 0$ and $-bc = 1$, so $c = \pm 1$, and then $\gamma \notin \Gamma_0(N)$, which is a contradiction.

The invariance of E under translation proves the converse. ■

Corollary 2.3. *For each integer $N \geq 2$, no two distinct zeros of E for $\Gamma_0^+(N)$ in the half-strip $\mathfrak{S} = \{\tau \in \mathbb{H} : -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}\}$ are equivalent modulo $\Gamma_0^+(N)$.*

Theorem 2.4. *For each integer $N \geq 2$, the quasi-modular form E for $\Gamma_0^+(N)$ has infinitely many $\Gamma_0^+(N)$-inequivalent zeros in the half-strip $\mathfrak{S}$.*

Proof. By [10, Proposition 5.3] with $f = NE_2(Nz) - E_2$ and $\phi_0 = 2E_2$ for $E = f + \phi_0$, E has infinitely many zeros that are inequivalent relative to $\Gamma_0(N)$, so to $\Gamma_0^+(N)$. Hence since it is invariant under translation, the theorem holds. ■

Next, we are interested in Δ_N^+ for $N = 2,\ 3$ defined as in [2, Eq. (10)]:

$$\Delta_N^+ = (\eta(z)\eta(Nz))^\delta, \text{ where } \delta = \begin{cases} 8, & \text{if } N = 2 \\ 12, & \text{if } N = 3. \end{cases} \tag{15}$$

Corollary 2.5. *Δ_N^+ has infinitely many critical points for $N = 2,\ 3$.*

Proof. Note that for $f \in M_k(\Gamma_0^+(N))$,

$$\partial_k f = \theta f - \frac{kE}{24} f \in M_{k+2}(\Gamma_0^+(N)).$$

By (15), $\Delta_2^+ = (\eta(z)\eta(2z))^8$ and $\Delta_2^+ = q + \mathcal{O}(q^2) \in S_8(\Gamma_0^+(2))$. Hence,

$$\begin{aligned} \partial_8 \Delta_2^+ &= \theta \Delta_2^+ - \frac{8E}{24}\Delta_2^+ \\ &= \mathcal{O}(q) - \frac{8E}{24}\mathcal{O}(q) \\ &= \mathcal{O}(q) \in S_{10}(\Gamma_0^+(2)). \end{aligned}$$

Since $\dim(S_{10}(\Gamma_0^+(2))) = \left[\frac{10}{8}\right] - 1 = 0$, we have that $\partial_8 \Delta_2^+ = 0$, so $\theta \Delta_2^+ = \frac{8E}{24}\Delta_2^+$ and $E = 3\left(\frac{\theta \Delta_2^+}{\Delta_2^+}\right)$. Therefore our assertion for $N = 2$ follows from Theorem 2.4.

Again, by (15), $\Delta_3^+ = (\eta(z)\eta(3z))^{12}$ and $\Delta_3^+ = q^2 + \mathcal{O}(q^3) \in S_{12}(\Gamma_0^+(3))$. Hence,

$$\begin{aligned}\partial_{12}\Delta_3^+ &= \theta\Delta_3^+ - \frac{12E}{24}\Delta_3^+\\ &= \mathcal{O}(q^2) - \frac{12E}{24}\mathcal{O}(q^2)\\ &= \mathcal{O}(q^2) \in S_{14}(\Gamma_0^+(3)).\end{aligned}$$

Since $\dim(S_{14}(\Gamma_0^+(3))) = \left[\frac{14}{6}\right] - 1 = 1$, there is no a nonzero modular form with a Fourier expansion at ∞ starting q^n for $n > 1$, which implies that $\partial_{12}\Delta_3^+ = 0$. So $\theta\Delta_3^+ = \frac{12E}{24}\Delta_3^+$ and $E = 2\Big(\frac{\theta\Delta_3^+}{\Delta_3^+}\Big)$. Therefore our assertion for $N = 3$ follows from Theorem 2.4. ∎

3. Distribution of the Zeros of E for $\Gamma_0^+(2)$

Note that a fundamental domain for $\Gamma_0^+(2)$ is given by

$$\mathfrak{F}^+(2) := \{|z| \geq 1/\sqrt{2}, -1/2 \leq \text{Re}(z) \leq 0\} \cup \{|z| > 1/\sqrt{2}, 0 \leq \text{Re}(z) < 1/2\}.$$

(Refer to [6, p. 694].)

We consider fundamental regions within the half-strip that contains zeros of E and fundamental regions that do not contain any zeros of E.

Theorem 3.1. *There exists a positive integer c_0 such that for all odd integers c with $|c| \geq c_0$, there exists a fundamental domain with a vertex at $\frac{c-1}{2c}$ containing a zero of E. Therefore, there exist infinitely many fundamental domains within the half-strip that contains zeros of E.*

Proof. By generalizing the idea of the proof of [1, Theorem 4.1], let τ_0 be the unique zero of E on the imaginary axis and let $\alpha = \begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \Gamma_0(2)$, where $t \neq 0$. Then,

$$E(\tau_0) = 0 = E(\alpha^{-1}(\alpha\tau_0)) = (-v\alpha\tau_0 + t)^2 E(\alpha\tau_0) - \frac{12i}{\pi}(-v)(-v\alpha\tau_0 + t).$$

This is true if and only if

$$\frac{E(\alpha\tau_0)}{\alpha\tau_0 E(\alpha\tau_0) + \frac{12}{\pi i}} = \frac{v}{t}. \tag{16}$$

Note that $\tau_0 \in w_2\mathfrak{F}^+(2)$. In fact, from (11) we have that

$$E\left(-\frac{1}{2\cdot\frac{i}{\sqrt{2}}}\right) = \left(\frac{i}{\sqrt{2}}\right)^2 2E\left(\frac{i}{\sqrt{2}}\right) + \frac{24}{\pi i}\cdot\frac{i}{\sqrt{2}},$$

which implies that

$$E\left(\frac{i}{\sqrt{2}}\right) = \frac{6\sqrt{2}}{\pi} > 0.$$

Since E is strictly increasing on $(0,\infty)$ along the imaginary axis, $\tau_0 = iy$ is below $\frac{i}{\sqrt{2}}$, therefore $0 < y < \frac{1}{\sqrt{2}}$. Note that

$$\tau_0 \in w_2\mathfrak{F}^+(2) \Leftrightarrow w_2\tau_0 = \frac{1}{-2iy} = \frac{i}{2y} \in \mathfrak{F}^+(2) \Leftrightarrow \mathrm{Im}(w_2\tau_0) = \frac{1}{2y} > \frac{1}{\sqrt{2}} \Leftrightarrow 0 < y < \frac{1}{\sqrt{2}}.$$

Hence, when

$$f(z) = \frac{E(z)}{zE(z) + \frac{12}{\pi i}}$$

and $\alpha = S_{-2} := \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, this implies that f maps a neighborhood D_0 of $S_{-2}\tau_0$, which can be chosen to be in the interior of $S_{-2}w_2\mathfrak{F}^+(2)$ onto a neighborhood U_0 of -2.

There exists a positive integer c_0 such that for all integers c such that $|c| \geq c_0$, $-2 - \frac{2}{c} \in U_0$. For each odd integer $|c| \geq c_0$, let $z_c \in D_0$ such that $f(z_c) = -2 - \frac{2}{c}$. Therefore, if $\gamma_c = \begin{pmatrix} c & \frac{c-1}{2} \\ 2c+2 & c \end{pmatrix} \in \Gamma_0(2) \subset \Gamma_0^+(2)$, then since

$$\frac{E(\gamma_c^{-1}(\gamma_c z_c))}{\gamma_c^{-1}(\gamma_c z_c)E(\gamma_c^{-1}(\gamma_c z_c)) + \frac{12}{\pi i}} = \frac{E(z_c)}{z_c E(z_c) + \frac{12}{\pi i}},$$

recalling (16), $\gamma_c z_c$ is a zero of E belonging to $\gamma_c S_{-2} w_2 \mathfrak{F}^+(2)$. For all odd integers c such that $|c| \geq c_0$,

$$\gamma_c S_{-2} w_2 = \begin{pmatrix} c-1 & -1 \\ 2c & -2 \end{pmatrix} \in \Gamma_0^+(2),$$

and $\gamma_c S_{-2} w_2(\infty) = \frac{c-1}{2c}$. Hence $\gamma_c S_{-2} w_2\ \mathfrak{F}^+(2)$ is the fundamental domain which has a vertex at the cusp $\frac{c-1}{2c}$. ∎

Proposition 3.2. *The Eisenstein series E for $\Gamma_0^+(2)$ has no zeros in the fundamental domain $\mathfrak{F}^+(2)$ for $\Gamma_0^+(2)$.*

Proof. Let $\tau_0 = iy_0$ be the unique zero of E on the imaginary axis. Then, by (11), we have that

$$E\left(-\frac{1}{2 \cdot iy_0}\right) = (iy_0)^2 2 \cdot E(iy_0) + \frac{24}{\pi i} \cdot iy_0 = \frac{24}{\pi} y_0 < 3.$$

The last inequality follows from the following : Since $\lim_{y\to\infty} E(iy) = 3$ by (14), and E is strictly increasing on $(0,\infty)$ along the imaginary axis, we have that $E\left(-\frac{1}{2\cdot iy_0}\right) = \frac{24}{\pi}y_0 < 3$.

This inequality implies that $y_0 < \frac{\pi}{8}$. If $\tau = x+iy \in \mathfrak{F}^+(2)$ is a zero of E, then $y = \text{Im}(\tau) > \frac{1}{2} > \frac{\pi}{8} > y_0$. Hence we have

$$\begin{aligned}
\frac{1}{24}|3-E(\tau)| &\le \frac{1}{24}(|1-E_2(\tau)| + 2|1-E_2(2\tau)|)\\
&= \left|\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in\tau}\right| + 2\left|\sum_{n=1}^{\infty}\sigma_1(n)e^{4\pi in\tau}\right|\\
&\le \sum_{n=1}^{\infty}\sigma_1(n)e^{-2\pi ny} + 2\left(\sum_{n=1}^{\infty}\sigma_1(n)e^{-4\pi ny}\right)\\
&< \sum_{n=1}^{\infty}\sigma_1(n)e^{-2\pi ny_0} + 2\left(\sum_{n=1}^{\infty}\sigma_1(n)e^{-4\pi ny_0}\right)\\
&= \frac{1}{24}(3-E(\tau_0)) = \frac{1}{8}.
\end{aligned}$$

Hence $|3-E(\tau)| < 3$, hence τ cannot be a zero of E if $\tau \in \mathfrak{F}^+(2)$. ■

Now we will find more fundamental domains which do not contain any zeros of E.

Lemma 3.3. *For an odd positive integer c, let $S_c^+ = \begin{pmatrix} c-1 & 1-2c \\ 2c & -4c-2\end{pmatrix} \in \Gamma_0(2)w_2$. Then the fundamental domain $S_c^+\,\mathfrak{F}^+(2)$ is the region with the edge joining $\frac{c-1}{2c}$ and $S_c^+(\rho_2)$ which is an arc of the circle $C_1(c)$ centered at $c_1(c) = \frac{5c^2-3c-1}{2c(5c+2)}$ with radius $r_1(c) = \frac{1}{2c(5c+2)}$, and the edge joining $\frac{c-1}{2c}$ and $S_c^+(\rho_2+1)$ which is an arc of the circle $C_2(c)$ centered at $c_2(c) = \frac{3c^2-c-1}{2c(3c+2)}$ with radius $r_2(c) = \frac{1}{2c(3c+2)}$.*

Proof. Note that $S_c^+(\infty) = \frac{1}{2} - \frac{1}{2c}$,

$$S_c^+(\rho_2) = \frac{13c^2-3c-3}{2(13c^2+10c+2)} + \frac{i}{2(13c^2+10c+2)},$$

and

$$S_c^+(\rho_2+1) = \frac{5c^2+c-1}{2(5c^2+6c+2)} + \frac{i}{2(5c^2+6c+2)}.$$

Hence, from the equation of the circle centered at $c_1(c) \in \mathbb{R}$ with radius $r_1(c) := |c_1(c) - \frac{c-1}{2c}|$ passing through $S_c^+(\rho_2)$, we find that

$$c_1(c) = \frac{5c^2 - 3c - 1}{2c(5c+2)} \text{ and } r_1(c) = \frac{1}{2c(5c+2)},$$

and similarly from the equation of the circle centered at $c_2(c) \in \mathbb{R}$ with radius $r_2(c) := |c_2(c) - \frac{c-1}{2c}|$ passing through $S_c^+(\rho_2 + 1)$, we get that

$$c_2(c) = \frac{3c^2 - c - 1}{2c(3c+2)} \text{ and } r_2(c) = \frac{1}{2c(3c+2)}.$$

If we describe the fundamental domain $S_c^+\ \mathfrak{F}^+(2)$ more closely for better understanding, its vertices are

$$\frac{c-1}{2c}, S_c^+(\rho_2), \text{ and } S_c^+(\rho_2 + 1).$$

Also since c is positive, we have that

$$\frac{c-1}{2c} < c_1(c) < c_2(c) < \mathrm{Re}(S_c^+(\rho_2)) < \mathrm{Re}(S_c^+(\rho_2 + 1))$$

and

$$\mathrm{Im}(S_c^+(\rho_2)) < \mathrm{Im}(S_c^+(\rho_2 + 1)) < r_1(c) < r_2(c).$$

Thus we have the following Figure 1.

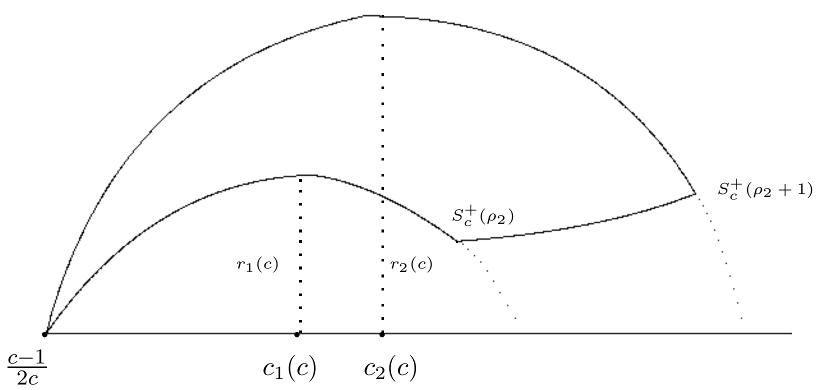

Figure 1. The fundamental domain $S_c^+\ \mathfrak{F}^+(2)$. ■

Theorem 3.4. *For each integer $m \leq -4$ and each odd integer $c \geq 3$, let*

$$S_c^+(m) = \begin{pmatrix} c-1 & m(c-1)-1 \\ 2c & 2(cm-1) \end{pmatrix} \in \Gamma_0(2)w_2.$$

Then E has no zeros in $S_c^+(m)\,\mathfrak{F}^+(2)$.

In particular, there are infinitely many fundamental domains for $\Gamma_0^+(2)$ which contain no zeros of E.

Proof. Suppose there is a zero z_0 of E in the fundamental domain $S_c^+(m)\,\mathfrak{F}^+(2)$. Then, $S_c^+(m)\,\mathfrak{F}^+(2)$ has a vertex at $\frac{c-1}{2c}$, as does $S_c^+\,\mathfrak{F}^+(2)$ given in Lemma 3.3. For convenience, we let

$$b = m(c-1)-1 \text{ and } d = 2(cm-1), \text{ so the given } S_c^+(m) = \begin{pmatrix} c-1 & b \\ 2c & d \end{pmatrix}.$$

Then, since we assume that $m \leq -4$ and $c \geq 3$, we have that $(b,d) \neq (1-2c, -4c-2)$. So $S_c^+\,\mathfrak{F}^+(2) \cap S_c^+(m)\,\mathfrak{F}^+(2)$ is an empty set. Hence, $S_c^+(m)\,\mathfrak{F}^+(2)$ is either within the circle $C_1(c)$ or outside the circle $C_2(c)$ on $\mathbb{H}$ given in Lemma 3.3 with referring Figure 1.

Note that

$$S_c^+(m)(\rho_2) = \frac{(2c^2m^2 - 2c^2m - 2cm^2 + c^2 - 2cm + c + 2m + 1) + i}{4(cm-1)(c(m-1)-1) + 2c^2},$$

and

$$S_c^+(m)(\rho_2+1) = \frac{(2c^2m^2 + 2c^2m - 2cm^2 + c^2 - 6cm - 3c + 2m + 3) + i}{4(cm-1)(c(m+1)-1) + 2c^2},$$

hence, since $m \leq -4$ and $c \geq 3$, we can easily show by computation using MAPLE 16 that

$$\begin{aligned}
&\mathrm{Im}(S_c^+(\rho_2)) - \mathrm{Im}(S_c^+(m)(\rho_2+1)) \\
&= \frac{c(m+3)(c(m-2)-2)}{(13c^2+10c+2)(2(cm-1)(c(m+1)-1)+c^2)} > 0, \\
&\mathrm{Im}(S_c^+(m)(\rho_2)) - \mathrm{Im}(S_c^+(m)(\rho_2+1)) \\
&= \frac{2c(cm-1)}{(2(cm-1)(c(m-1)-1)+c^2)(2(cm-1)(c(m+1)-1)+c^2)} < 0, \\
&\mathrm{Re}(S_c^+(\rho_2)) - \mathrm{Re}(S_c^+(m)(\rho_2+1)) \\
&= \frac{(m+3)(c^2(5m+3)+2c(m-2)-2)}{(13c^2+10c+2)(2(cm-1)(c(m+1)-1)+c^2)} > 0, \\
&\mathrm{Re}(S_c^+(m)(\rho_2+1)) - \mathrm{Re}(S_c^+(m)(\rho_2)) \\
&= \frac{2(cm-1)^2 - c^2}{(2(cm-1)(c(m-1)-1)+c^2)(2(cm-1)(c(m+1)-1)+c^2)} > 0, \\
&\mathrm{Re}(S_c^+(m)(\rho_2)) - \frac{c-1}{2c} = \frac{-(c(2m-1)-2)}{2c(2(cm-1)(c(m-1)-1)+c^2)} > 0,
\end{aligned}$$

so

$$\mathrm{Im}(S_c^+(m)(\rho_2)) < \mathrm{Im}(S_c^+(m)(\rho_2+1)) < \mathrm{Im}(S_c^+(\rho_2)) < \mathrm{Im}(S_c^+(\rho_2+1)),$$

and

$$\frac{c-1}{2c} < \mathrm{Re}(S_c^+(m)(\rho_2)) < \mathrm{Re}(S_c^+(m)(\rho_2+1)) < \mathrm{Re}(S_c^+(\rho_2)) < \mathrm{Re}(S_c^+(\rho_2+1)),$$

which implies that $S_c^+(m)\ \mathfrak{F}^+(2)$ is within the circle $C_1(c)$ on $\mathbb{H}$ with vertices $\frac{c-1}{2c}$, $S_c^+(m)(\rho_2)$ and $S_c^+(m)(\rho_2+1)$ as shown in Figure 2.

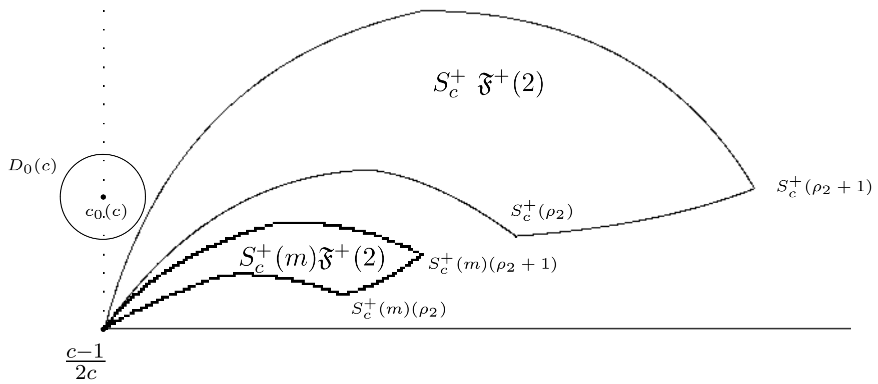

Figure 2. The fundamental domains $S_c^+\ \mathfrak{F}^+(2)$ and $S_c^+(m)\ \mathfrak{F}^+(2)$.

By showing, that a given zero z_0 is outside $C_2(c)$ (hence outside $C_1(c)$ and $S_c^+(m)\ \mathfrak{F}^+(2)$), we will get a contradiction.

Note that $S_c^+(m) = \begin{pmatrix} -b & \frac{c-1}{2} \\ -d & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$ and $-bc + d \cdot \frac{c-1}{2} = 1$. So

$$2(S_c^+(m))^{-1} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} c & -\frac{c-1}{2} \\ d & -b \end{pmatrix}.$$

Note that $(S_c^+(m))^{-1} z_0 = 2(S_c^+(m))^{-1} z_0$. Let $z_1 := \begin{pmatrix} c & -\frac{c-1}{2} \\ d & -b \end{pmatrix} z_0$.

Then,

$$
\begin{aligned}
E((S_c^+(m))^{-1}z_0) &= E(2(S_c^+(m))^{-1}z_0)\\
&= z_1^2\cdot 2\cdot E(z_1)+\frac{24}{\pi i}z_1 \text{ by (11)}\\
&= z_1^2\cdot 2\cdot\Big((dz_0-b)^2E(z_0)+\frac{12}{\pi i}d(dz_0-b)\Big)+\frac{24}{\pi i}z_1 \text{ by (10)}\\
&= \Big(\frac{cz_0-\frac{c-1}{2}}{dz_0-b}\Big)^2\cdot 2\cdot\frac{12}{\pi i}d(dz_0-b)+\frac{24}{\pi i}\Big(\frac{cz_0-\frac{c-1}{2}}{dz_0-b}\Big)\\
&\quad \text{(by the fact that } -bc+d\cdot\frac{c-1}{2}=1)\\
&= \frac{24}{\pi i}c\Big(cz_0-\frac{c-1}{2}\Big).
\end{aligned}
$$

So we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in((S_c^+(m))^{-1}z_0)}+2\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in2((S_c^+(m))^{-1}z_0)}\\
&=\frac{3}{24}-\frac{1}{24}E(2(S_c^+(m))^{-1}z_0)\\
&=\frac{1}{8}-\frac{1}{24}\Big(\frac{24}{\pi i}c\Big(cz_0-\frac{c-1}{2}\Big)\Big)\\
&=-\frac{c^2}{\pi i}\Big(z_0-\Big(\frac{c-1}{2c}+\frac{\pi i}{8c^2}\Big)\Big).
\end{aligned}
$$

Since $(S_c^+(m))^{-1}z_0\in\mathfrak{F}^+(2)$, Im$((S_c^+(m))^{-1}z_0)\geq\frac{1}{2}$. Hence

$$
\begin{aligned}
&\left|\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in((S_c^+(m))^{-1}z_0)}+2\sum_{n=1}^{\infty}\sigma_1(n)e^{2\pi in2((S_c^+(m))^{-1}z_0)}\right|\\
&\leq\sum_{n=1}^{\infty}\sigma_1(n)e^{-n\pi}+2\sum_{n=1}^{\infty}\sigma_1(n)e^{-2n\pi}:=M.
\end{aligned}
$$

Therefore, we have that

$$\Big|z_0-\Big(\frac{c-1}{2c}+\frac{\pi i}{8c^2}\Big)\Big|\leq M\frac{\pi}{c^2}.$$

Let $D_0(c)$ be the disk centered at $c_0(c)=\frac{c-1}{2c}+\frac{\pi i}{8c^2}$ with radius $r_0(c)=M\frac{\pi}{c^2}$. Refer to Figure 2. Then z_0 belongs to $D_0(c)$. In order to show that $D_0(c)$ lies outside the circle $C_2(c)$, we show that $|c_0(c)-c_2(c)|>r_2(c)+r_0(c)$.

Since the cusp $\frac{1}{2}-\frac{1}{2c}$ and $c_0(c)$ are on the same vertical axis,

$$|c_2(c)-c_0(c)|^2=r_2(c)^2+\left(\frac{\pi}{8c^2}\right)^2.$$

So it is enough to show that

$$r_0(c)^2 + 2r_0(c)r_2(c) < \left(\frac{\pi}{8c^2}\right)^2,$$

which is equivalent to

$$64\left(M^2 + M\frac{c}{(3c+2)\pi}\right) < 1.$$

By modifying the proof of [1, Lemma 4.3], we set

$$q = e^{-\pi} \approx 0.04321391825.$$

Then

$$\begin{aligned}
0 < M &= \sum_{n\geq 1}\sigma_1(n)q^n + 2\sum_{n\geq 1}\sigma_1(n)q^{2n} \\
&= \sum_{n\geq 1}\frac{nq^n}{1-q^n} + 2\sum_{n\geq 1}\frac{nq^{2n}}{1-q^{2n}} \text{ (as in the proof of [1, Lemma 4.3])} \\
&\leq \frac{1}{1-q}\sum_{n\geq 1}nq^n + \frac{1}{1-q^2}\sum_{n\geq 1}2nq^{2n} \\
&\leq \frac{q}{(1-q)^3} + 2\frac{q^2}{(1-q^2)^3} \\
&\approx 0.05309361050.
\end{aligned}$$

Since $\frac{c}{(3c+2)} \leq \frac{1}{3}$ for all $c \geq 3 > 1$, we have that

$$64\left(M^2 + M\frac{c}{(3c+2)\pi}\right) \leq 64\left(M^2 + M\frac{1}{3\pi}\right) \approx 0.5409496650 < 1.$$

Hence we have shown that $D_0(c)$ is outside the circle $C_2(c)$. This completes the proof. ∎

Remark 3.5. We note that Theorem 3.4 gives a more general and explicit description of regions comparing from the results in [1]. In particular, we show how to take care of the parts related with the Fricke involution while the proofs in [1] deal with $\mathrm{SL}_2(\mathbb{Z})$.

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