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A REMARK ON CHEN'S THEOREM WITH SMALL PRIMES

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Abstract. Let N denote a sufficiently large even integer. In this paper it is proved that for $0.941 \leq \theta \leq 1$, the equation

$$N = p + P_2, \qquad p \leq N^{\theta}$$

is solvable, where p is a prime and P_2 is an almost prime with at most two prime factors. The range $0.941 \leq \theta \leq 1$ extended the previous one $0.945 \leq \theta \leq 1$.

1. Introduction

In 1966 J. R. Chen [4] announced his remarkable Theorem (the detailed proof was published in [5]): Let N denote a sufficiently large even integer, then the equation

$$N = p + P_2$$

is solvable, where and in what follows, p, with or without subscripts, denotes a prime and P_2 denotes an almost prime with at most two prime factors, counted according to multiplicity.

Chen's theorem with small primes was first studied in [3]. For $0 < \theta \leq 1$ put $U = N^{\theta}$ and let $S(N, \theta)$ denote the number of solutions of the equation

$$N = p + P_2, \qquad p \leq U.$$

Then it is proved in [3] that for $0.95 \leq \theta \leq 1$, we have

$$S(N, \theta) \geq \frac{0.001 C(N) U}{\log^2 N},$$

where

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$$C(N)=\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p|N\\p>2}}\frac{p-1}{p-2}.$$

In [15] the range $0.95\le\theta\le 1$ was extended to $0.945\le\theta\le 1$.

While working on [15], we tried to extend the range $0.945\le\theta\le 1$ by the sieve process in [2] but failed, since we faced the difficulty which cannot be surmounted that the mean value theorem in [16] is not sufficient to deal with some of the sieve error terms involved. Now by establishing new mean value theorems for products of large primes over short intervals, we can insert the delicate sieve process in [2] into the arguments in [3] to obtain the following sharper result.

Theorem. Let N be a sufficiently large even integer. Then for $0.941\le\theta\le 1$ we have

$$S(N,\theta)\ge\frac{0.001C(N)U}{\log^2 N}.$$

Notations. In this paper x always denotes a sufficiently large real number, and $x^{\frac{13}{13.9}}<y\le x\exp(-10\log^{\frac{1}{5}}x), w=x^{\frac{1}{14}}, K$ is an integer not exceeding x. The letter c denotes a positive constant which will not be the same at different occurrences. The constants in O-terms and $\ll$-symbols are absolute. The letter p is reserved for prime numbers. As usual, $\varphi(n)$ denotes the Euler's function, and $\Omega(n)$ denotes the number of prime factors of n(counted according to multiplicity). We always denote by χ a Dirichlet character $(\bmod q)$, by χ_0 a principal Dirichlet character $(\bmod q)$, and by $\sum\limits_{\chi_q}$ and $\sum\limits_{\chi_q}{}^{*}$ sums with χ running over the characters and primitive characters $(\bmod q)$ respectively. Let

$$\begin{aligned}
\mathscr{M}^{(1)}&=\{p_1p_2p_3p_4|x^{\frac{1}{14}}\le p_1\le p_2\le p_3\le p_4\le x^{\frac{1}{8.8}},(p_1p_2p_3p_4,K)=1\},\\
\mathscr{M}^{(2)}&=\{p_1p_2p_3p_4|x^{\frac{1}{14}}\le p_1\le p_2\le p_3\le x^{\frac{1}{8.8}}\\
&\quad\le p_4\le x^{\frac{\theta}{2}-\frac{3}{14}}p_3^{-1},(p_1p_2p_3p_4,K)=1\},\\
\mathscr{N}^{(j)}&=\{p_1p_2p_3p_4n|p_1p_2p_3p_4\in\mathscr{M}^{(j)},x-y<p_1p_2p_3p_4n\\
&\quad\le x,p|n\Rightarrow p\ge p_2,(n,K)=1\},\\
\mathscr{N}_r^{(j)}&=\{l|l\in N^{(j)},\Omega(l)=r\},\qquad r\ge 5,\\
w(x,y,s)&=\frac{x^s-(x-y)^s}{s},\qquad b=1+\frac{1}{\log x}.
\end{aligned}$$

2. Some Preliminary Lemmas

Let $\mathscr{A}$ denote a finite set of integers, $\mathscr{P}$ denote an infinite set of primes, $\overline{\mathscr{P}}$ denote the set of primes that do not belong to $\mathscr{P}$. For $z\ge 2$, put

$$P(z)=\prod_{p<z,p\in\mathscr{P}}p,\qquad S(\mathscr{A};\mathscr{P},z)=\sum_{a\in\mathscr{A},(a,P(z))=1}1,$$

$$A_d=\{a|a\in\mathscr{A},a\equiv 0(\bmod d)\},\qquad \mathscr{P}(q)=\{p|p\in\mathscr{P},(p,q)=1\}.$$

Lemma 1. ([10]). *If*

$$A_1)\ |\mathscr{A}_d| = \frac{\omega(d)}{d}X + r_d, \qquad \mu(d) \neq 0, \qquad (d, \overline{\mathscr{P}}) = 1;$$

$$A_2)\ \sum_{z_1 \le p < z_2} \frac{\omega(p)}{p} = \log\frac{\log z_2}{\log z_1} + O\left(\frac{1}{\log z_1}\right), \qquad z_2 > z_1 \ge 2,$$

where $\omega(d)$ *is a multiplicative function,* $0 \le \omega(p) < p$, $X > 1$ *is independent of* d, *then*

$$S(\mathscr{A}; \mathscr{P}, z) \ge XV(z)\left\{f(s) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} - R_D,$$

$$S(\mathscr{A}; \mathscr{P}, z) \le XV(z)\left\{F(s) + O\left(\frac{1}{\log^{\frac{1}{3}} D}\right)\right\} + R_D,$$

where

$$s = \frac{\log D}{\log z}, \qquad R_D = \sum_{d<D, d|P(z)} |r_d|,$$

$$V(z) = C(\omega)\frac{e^{-\gamma}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right),$$

$$C(\omega) = \prod_p \left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-1},$$

where γ *denotes the Euler's constant,* $f(s)$ *and* $F(s)$ *are determined by the following differential-difference equation*

$$\begin{cases} F(s) = \frac{2e^{\gamma}}{s}, \qquad f(s) = 0, \qquad 0 < s \le 2, \\ (sF(s))' = f(s-1), \qquad (sf(s))' = F(s-1), \qquad s \ge 2. \end{cases}$$

Lemma 2. ([7]). *We have*

$$\begin{aligned} F(s) &= \frac{2e^{\gamma}}{s}, \qquad 0 < s \le 3; \\ F(s) &= \frac{2e^{\gamma}}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t}dt\right), \qquad 3 \le s \le 5; \\ F(s) &= \frac{2e^{\gamma}}{s}\left(1 + \int_2^{s-1} \frac{\log(t-1)}{t}dt\right. \\ &\quad \left. + \int_2^{s-3} \frac{\log(t-1)}{t}dt \int_{t+2}^{s-1} \frac{1}{u}\log\frac{u-1}{t+1}du\right), \qquad 5 \le s \le 7; \end{aligned}$$

$$f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, \qquad 2 \le s \le 4;$$

$$f(s) = \frac{2e^{\gamma}}{s}\left(\log(s-1) + \int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}du\right),$$
$$4 \le s \le 6;$$

$$f(s) = \frac{2e^{\gamma}}{s}\left(\log(s-1) + \int_3^{s-1}\frac{dt}{t}\int_2^{t-1}\frac{\log(u-1)}{u}du\right.$$
$$\left. + \int_2^{s-4}\frac{\log(t-1)}{t}dt\int_{t+2}^{s-2}\frac{1}{u}\log\frac{u-1}{t+1}\log\frac{s}{u+2}du\right),$$
$$6 \le s \le 8.$$

Lemma 3. *Let*

$$y = x\exp(-10\log^{\frac{1}{5}}x), \qquad z = x^{\frac{1}{u}}, \qquad Q(z) = \prod_{p<z} p.$$

Then for $1 < u \le 100$*, we have*

$$\sum_{\substack{x-y<n\le x\\(n,Q(z))=1}} 1 = w(u)\frac{y}{\log z} + O\left(\frac{y}{\log^2 x}\right),$$

where $w(u)$ *is determined by the following differential-difference equation*

$$\begin{cases} w(u) = \frac{1}{u}, & 1 \le u \le 2, \\ (uw(u))' = w(u-1), & u \ge 2. \end{cases}$$

Moreover, we have

$$\begin{cases} w(u) \le \frac{1}{1.763}, & u \ge 2; \\ w(u) < \frac{1}{1.7803}, & u \ge 4. \end{cases}$$

Proof. The above asymptotic formula follows from the prime number theorem with Vinogradov's error term and the inductive arguments in section A.2 in [8]. For the upper bounds of $w(u)$, see [11]. ∎

Lemma 4. ([13]). *Let*

$$S(s,\chi) = \sum_{n=M+1}^{M+N}\frac{a(n)\chi(n)}{n^s},$$

where χ is a Dirichlet character modulo q. Then

$$\sum_{Q\le q<2Q}\frac{1}{\varphi(q)}\sum_{\chi_q}{}^{*}\int_{T_0}^{T_0+T}|S(it,\chi)|^2dt\ll\left(QT+\frac{N}{Q}\right)\sum_{n=M+1}^{M+N}|a(n)|^2$$

for any $Q\ge 1, T_0$ and $T>0$.

Lemma 5. *Let $|t|\le x, L\gg x^{\Delta}, q\ll \log^A x$, where Δ and A are positive constants. Then for a Dirichlet character $\chi(\mathrm{mod} q)$, we have*

$$\sum_{L_1<l\le L_2}\frac{\Lambda(l)\chi(l)}{l^{\frac12+it}}=E_0\frac{(L_2)^{\frac12-it}-(L_1)^{\frac12-it}}{\frac12-it}+O(L\exp(-c\log^{\frac15}x)),$$

where $L\le L_1\le L_2\le 2L$ and

$$E_0 = \begin{cases} 1, & \text{if } \chi=\chi_0,\\ 0, & \text{otherwise.}\end{cases}$$

The proof of Lemma 5 is similar to that of (4) in [12], so we omit it here.

Lemma 6. *For $j=1,2$ we have*

$$\sum_{l\in\mathcal{N}^{(j)}}1=(1+o(1))\sum_{p_1p_2p_3p_4\in\mathcal{M}^{(j)}}\frac{y}{p_1p_2p_3p_4\log p_2}w\left(\frac{\log\frac{x}{p_1p_2p_3p_4}}{\log p_2}\right)+O\left(\frac{y}{\log^2 x}\right),$$

where $w(t)$ is defined in Lemma 3.

Proof. We are motivated by [9]. We prove Lemma 6 in the case $j=1$ only, the same argument can be applied to the case $j=2$. We first show that the formula

$$\sum_{\substack{x-y<p_1p_2\cdots p_r\le x\\ p_1p_2\cdots p_r\in\mathcal{N}_r^{(j)}}}=yC_r(x)+O(y\exp(-c\log^{\frac15}x)) \tag{2.1}$$

holds, where $C_r(x)$ is independent of y and $5\le r\le 14$.

By a splitting argument we know that the left hand side of (2.1) is the sum of $O(\log^r x)$ sums of the form

$$S_r=\sum_{\substack{x-y<p_1p_2\cdots p_r\le x\\ p_j\in\mathcal{J}_j,1\le j\le r}}1,$$

where

$$\begin{aligned}&\mathcal{J}_j=(N_j,N_j'],\quad N_j<N_j'\le 2N_j,\quad 1\le j\le r,\\ &w\le N_1<N_1'\le N_2<N_2'\le\cdots\le N_r<N_r',\\ &x\ll \textstyle\prod_{j=1}^r N_j\ll x.\end{aligned}\tag{2.2}$$

Set

$$f_j(s)=\sum_{p_j\in\mathcal{J}_j}\frac{1}{p^s},\qquad F_r(s)=\prod_{j=1}^r f_j(s),$$
$$T=\frac{x}{y}\exp(\log^{\frac12}x),\qquad T_0=\exp(\log^{\frac15}x).$$

Then by Perron's formula (Lemma 3.12 in [14]) we get

$$S_r=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F_r(s)w(x,y,s)ds+O(y\exp(-c\log^{\frac15}x)).$$

On moving the line of integration we obtain

$$\begin{aligned}S_r&=\frac{1}{2\pi i}\int_{\frac12-iT}^{\frac12+iT}F_r(s)w(x,y,s)ds\\ &\quad+O\left(\max_{\frac12\le\sigma\le b}x^{1-\sigma}\tfrac{x^\sigma}{T}\right)+O(y\exp(-c\log^{\frac15}x))\\ &=\frac{1}{2\pi i}\left(\int_{\frac12-iT_0}^{\frac12+iT_0}+\int_{\frac12-iT}^{\frac12-iT_0}+\int_{\frac12+iT_0}^{\frac12+iT}\right)F_r(s)w(x,y,s)ds\\ &\quad+O(y\exp(-c\log^{\frac15}x)).\end{aligned}\tag{2.3}$$

Since for $\mathrm{Re}(s)=\frac12$ we have

$$|F(s)|\ll x^{\frac12}$$

and

$$w(x,y,s)=x^{s-1}y+O(|s|x^{-\frac32}y^2)\quad\text{for}\quad |\text{Im(s)}|\le T_0,$$

we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\frac12-iT_0}^{\frac12+iT_0}F_r(s)w(x,y,s)ds\\ &=\frac{y}{2\pi i}\int_{\frac12-iT_0}^{\frac12+iT_0}F_r(s)x^{s-1}ds+O(T_0^2x^{-\frac32}y^2x^{\frac12})\\ &=yE_r(x)+O(y\exp(-c\log^{\frac15}x)),\end{aligned}\tag{2.4}$$

where $E_r(x)$ is independent of y.

Next we show that for $T_0 \le T_1 \le T$,

$$\int_{T_1}^{2T_1} \left|F_r\left(\frac{1}{2}+it\right)\right|\left|w\left(x,y,\frac{1}{2}+it\right)\right| dt = O(y\exp(-c\log^{\frac{1}{5}} x)). \tag{2.5}$$

By Lemma 5, for $|t| \ge T_0$, we have

$$\left|f_1\left(\frac{1}{2}+it\right)\right| \ll N_1^{\frac{1}{2}} \exp(-c\log^{\frac{1}{5}} x). \tag{2.6}$$

By (2.6) and Cauchy's inequality we get

$$\begin{aligned}
&\int_{T_1}^{2T_1} \left|F\left(\frac{1}{2}+it\right)\right|\left|w\left(x,y,\frac{1}{2}+it\right)\right| dt \\
&\ll \min\left(\frac{y}{x^{\frac{1}{2}}},\frac{x^{\frac{1}{2}}}{T_1}\right) N_1^{\frac{1}{2}} \exp(-c\log^{\frac{1}{5}} x)\left(\int_{T_1}^{2T_1}\left|f_2\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}} \\
&\times\left(\int_{T_1}^{2T_1}\left|\prod_{j=3}^{r} f_j\left(\frac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2}} \\
&\ll \min\left(\frac{y}{x^{\frac{1}{2}}},\frac{x^{\frac{1}{2}}}{T_1}\right) N_1^{\frac{1}{2}} \exp(-c\log^{\frac{1}{5}} x)\,(T_1+N_2)^{\frac{1}{2}}\left(T_1+\frac{x}{N_1N_2}\right)^{\frac{1}{2}} \\
&\ll y\exp(-c\log^{\frac{1}{5}} x),
\end{aligned}$$

where the bounds

$$w \le N_1 \ll N_2 \ll x^{\frac{1}{4}}, \qquad \left|w\left(x,y,\frac{1}{2}+it\right)\right| \ll \min\left(\frac{y}{x^{\frac{1}{2}}},\frac{x^{\frac{1}{2}}}{T_1}\right) \tag{2.7}$$

and Lemma 4 with $Q=1$ are applied.

Now (2.1) follows from (2.3)-(2.5) easily.

By (2.1) we get

$$\sum_{l\in\mathscr{N}^{(1)}} 1 = yE_0(x) + O(y\exp(-c\log^{\frac{1}{5}} x)), \tag{2.8}$$

with $E_0(x)$ independent of y. From Lemma 3 with $y = x\exp(-10\log^{\frac{1}{5}} x))$, we have

$$\begin{aligned}
&\sum_{l\in\mathscr{N}^{(1)}} 1 \\
&= \sum_{p_1p_2p_3p_4\in\mathscr{M}^{(1)}}\left(\frac{y}{p_1p_2p_3p_4\log p_2} w\left(\frac{\log\frac{x}{p_1p_2p_3p_4}}{\log p_2}\right) + O\left(\frac{y}{p_1p_2p_3p_4\log^2 x}\right)\right) \\
&= \sum_{p_1p_2p_3p_4\in\mathscr{M}^{(1)}} \frac{y}{p_1p_2p_3p_4\log p_2} w\left(\frac{\log\frac{x}{p_1p_2p_3p_4}}{\log p_2}\right) + O\left(\frac{y}{\log^2 x}\right).
\end{aligned} \tag{2.9}$$

Upon comparing (2.8) and (2.9) we get

$$E_0(x) = (1 + o(1)) \sum_{p_1p_2p_3p_4 \in \mathscr{M}^{(1)}} \frac{1}{p_1p_2p_3p_4 \log p_2} w\left(\frac{\log \frac{x}{p_1p_2p_3p_4}}{\log p_2}\right),$$

and the proof of Lemma 6 is completed.

3. Mean Value Theorems

In the proof of the Theorem we need the following mean value theorems.

Lemma 7. *Let $g(n)$ be an arithmetical function such that*

$$\sum_{n \le x} \frac{g^2(n)}{n} \ll \log^c x$$

for some $c > 0$. Let $r_1(a,h)$ and $r_2(a,h)$ be positive functions such that

$$u \le ar_1(a,h), ar_2(a,h) \le u + h.$$

For $(al, q) = 1$, define

$$\overline{H}(u,h,a,q,l) = \sum_{\substack{r_1(a,h) \le p < r_2(a,h) \\ ap \equiv l(q)}} 1 - \frac{1}{\varphi(q)}\left(Li(r_2(a,h)) - Li(r_1(a,h))\right).$$

Then for any given constant $A > 0$, there exists a constant $B = B(A,c) > 0$ such that for

$$\frac{3}{5} < \theta \le 1, \quad k = x^\theta, \quad 0 \le \beta < \frac{5\theta - 3}{2}, \quad \lambda = \theta - \frac{1}{2}, \quad D = x^\lambda \log^{-B} x,$$

we have

$$\sum_{d \le D} \max_{(l,d)=1} \max_{h \le k} \max_{\frac{x}{2} \le u \le x} \left| \sum_{a \le x^\beta, (a,d)=1} g(a)\overline{H}(u,h,a,d,l) \right| \ll \frac{k}{\log^A x}.$$

Proof. This result can be proved in the same way as the Theorem in [16]. ∎

Lemma 8. *For $j = 1, 2$ and any $A > 0$, there exists a constant $B = B(A) > 0$ such that for $Q = yx^{-\frac{1}{2}} \log^{-B} x$, we have*

$$\sum_{q \le Q} \max_{(a,q)=1} \left| \sum_{\substack{m \in \mathscr{N}^{(j)} \\ m \equiv a (\bmod q)}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{m \in \mathscr{N}^{(j)} \\ (m,q)=1}} 1 \right| \ll \frac{y}{\log^A x}.$$

Proof. We prove Lemma 8 in the case $j = 1$ only, the same argument can be applied to the case $j = 2$. We may assume $A \geq 300$. It suffice to show that for $5 \leq r \leq 14$, the bounds

$$\sum_{q\leq Q}\max_{(a,q)=1}\left|\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{N}_r^{(1)}\\ p_1p_2\cdots p_r\equiv a(\bmod q)}}1-\frac{1}{\varphi(q)}\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{N}_r^{(1)}\\ (p_1p_2\cdots p_r,q)=1}}1\right|\ll\frac{y}{\log^A x}$$

hold. ∎

By the orthogonality of Dirichlet character, for $(a, q) = 1$ we have

$$\begin{aligned}&\left|\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{N}_r^{(1)}\\ p_1p_2\cdots p_r\equiv a(\bmod q)}}1-\frac{1}{\varphi(q)}\sum_{\substack{p_1p_2\cdots p_r\in\mathcal{N}_r^{(1)}\\ (p_1p_2\cdots p_r,q)=1}}1\right|\\ &\leq\frac{1}{\varphi(q)}\sum_{\chi_q\neq\chi_q^0}\left|\sum_{p_1p_2\cdots p_r\in\mathcal{N}_r^{(1)}}\chi_q(p_1p_2\cdots p_r)\right|.\end{aligned}\tag{3.1}$$

It is easy to see that the inner sum in (3.1) is the sum of $O(\log^r x)$ sums of the form

$$S_r(\chi_q)=\sum_{\substack{p_j\in\mathcal{J}_j,(p_j,K)=1\\ 1\leq j\leq r}}\chi_q(p_1p_2\cdots p_r),$$

where the $\mathcal{J}_j$ is defined by (2.2).

Put

$$f_j(s,\chi)=\sum_{p_j\in\mathcal{J}_j,(p_j,K)=1}\frac{\chi(p)}{p^s},\qquad F_r(s,\chi)=\prod_{j=1}^{r}f_j(s,\chi),\qquad T=x^{\frac{1}{2}}.$$

Then by Perron's formula (Lemma 3.12 in [14]) we get

$$S_r(\chi_q)=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}F_r(s,\chi)w(x,y,s)ds+O(x^{\frac{1}{2}}).$$

On moving the line of integration we obtain

$$\begin{aligned}S_r(\chi_q)&=\frac{1}{2\pi i}\int_{\frac12-iT}^{\frac12+iT}F_r(s,\chi)w(x,y,s)ds+O\left(\max_{\frac12\leq\sigma\leq b}x^{1-\sigma}\frac{x^\sigma}{T}\right)+O(x^{\frac12})\\ &=\frac{1}{2\pi i}\int_{\frac12-iT}^{\frac12+iT}F_r(s,\chi)w(x,y,s)ds+O(x^{\frac12}).\end{aligned}\tag{3.2}$$

Let χ_d^* denote the primitive character which induce χ_q, then $1 < d|q$, $\chi_q = \chi_q^0\chi_d^*$ and

$$\sum_{q\le Q}\frac{1}{\varphi(q)}\sum_{\chi_q\ne\chi_q^0}|S_r(\chi_q)| \ll \sum_{1<d\le Q}\sum_{\substack{q\le Q\\ d|q}}\frac{1}{\varphi(q)}\sum_{\chi_d}{}^{*}|S_r(\chi_q)|. \tag{3.3}$$

Now by (3.2)-(3.3) we need only to show that

$$\sum_{1<d\le Q}\left(\sum_{\substack{q\le Q\\ d|q}}\frac{1}{\varphi(q)}\right)\sum_{\chi_d}{}^{*}\int_{\frac12-iT}^{\frac12+iT}|F_r(s,\chi_d)||w(x,y,s)||ds| \ll y\log^{-5A}x. \tag{3.4}$$

By the estimation

$$\sum_{\substack{q\le Q\\ d|q}}\frac{1}{\varphi(q)} \ll \frac{\log Q}{\varphi(d)}$$

and a splitting argument we need only to show that for

$$T_1 \le T, \quad 1 \le D \le Q$$

we have the estimation

$$\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|F_r\left(\frac12+it,\chi_d\right)\right|\left|w\left(x,y,\frac12+it\right)\right|dt \ll y\log^{-6A}x. \tag{3.5}$$

Now we prove (3.5) in two cases.

(1) $D \ll \log^{100A}x$. By Lemma 5 and summation by parts we obtain

$$\begin{aligned}\left|f_1\left(\tfrac12+it\right)\right| &\ll N_1^{\frac12}\exp(-\log^{\frac16}x) + N_1^{-\frac12}\\ &\ll N_1^{\frac12}\exp(-\log^{\frac16}x).\end{aligned} \tag{3.6}$$

From (2.7), (3.6), Cauchy's inequality and Lemma 4 we obtain

$$\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|F_r\left(\frac12+it,\chi_d\right)\right|\left|w\left(x,y,\frac12+it\right)\right|dt$$
$$\ll \min\left(\frac{y}{x^{\frac12}},\frac{x^{\frac12}}{T_1}\right)N_1^{\frac12}\exp(-\log^{\frac16}x)$$
$$\left(\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|f_2\left(\frac12+it,\chi_d\right)\right|^2dt\right)^{\frac12}$$

$$\times\left(\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|\prod_{j=3}^{r} f_j\left(\frac{1}{2}+it,\chi_d\right)\right|^2 dt\right)^{\frac12}$$
$$\ll \min\left(\frac{y}{x^{\frac12}},\frac{x^{\frac12}}{T_1}\right)N_1^{\frac12}\exp(-\log^{\frac16}x)\left(DT_1+\frac{N_2}{D}\right)^{\frac12}\left(DT_1+\frac{x}{N_1N_2D}\right)^{\frac12}$$
$$\ll y\log^{-6A}x.$$

(2) $D\gg \log^{100A}x$. By Cauchy's inequality and Lemma 4 we have

$$\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|F_r\left(\frac12+it,\chi_d\right)\right|\left|w\left(x,y,\frac12+it\right)\right|dt$$
$$\ll \min\left(\frac{y}{x^{\frac12}},\frac{x^{\frac12}}{T_1}\right)\left(\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|\prod_{j=1}^{2} f_j\left(\frac12+it,\chi_d\right)\right|^2 dt\right)^{\frac12}$$
$$\times\left(\sum_{D\le d\le 2D}\frac{1}{\varphi(d)}\sum_{\chi_d}{}^{*}\int_{T_1}^{2T_1}\left|\prod_{j=3}^{r} f_j\left(\frac12+it,\chi_d\right)\right|^2 dt\right)^{\frac12}$$
$$\ll \min\left(\frac{y}{x^{\frac12}},\frac{x^{\frac12}}{T_1}\right)\left(DT_1+\frac{N_1N_2}{D}\right)^{\frac12}\left(DT_1+\frac{x}{N_1N_2D}\right)^{\frac12}$$
$$\ll y\log^{-6A}x.$$

Now (3.5) is proved, and the proof of Lemma 8 is completed.

4. Weighted Sieve Method

Let

$$U=N^{0.941},\qquad \mathscr{A}=\{N-p\,|p\le U\},\qquad \mathscr{P}=\{p|\ (p,N)=1\}. \tag{4.1}$$

Lemma 9. *We have*

$$2S(N,\theta)\ge \frac32 S(\mathscr{A};\mathscr{P},N^{\frac{1}{14}})+\frac12 S(\mathscr{A};\mathscr{P},N^{\frac{1}{8.8}})$$
$$+\frac12\sum_{\substack{N^{\frac{1}{14}}\le p_1<p_2<N^{\frac{1}{8.8}}\\(p_1p_2,N)=1}} S(\mathscr{A}_{p_1p_2};\mathscr{P},N^{\frac{1}{14}})$$
$$+\frac12\sum_{\substack{N^{\frac{1}{14}}\le p_1<N^{\frac{1}{8.8}}\le p_2<U^{\frac12}N^{-\frac{2}{14}}p_1^{-1}\\(p_1p_2,N)=1}} S(\mathscr{A}_{p_1p_2};\mathscr{P},N^{\frac{1}{14}})$$

$$-\frac{1}{2}\sum_{\substack{N^{\frac{1}{14}}\le p<N^{\frac{4.0871}{14}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{14}})-\frac{1}{2}\sum_{\substack{N^{\frac{1}{14}}\le p<U^{\frac{1}{2}}N^{-\frac{3}{14}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{14}})$$
$$-\frac{1}{2}\sum_{\substack{N^{\frac{1}{14}}\le p_1<N^{\frac{1}{3.106}}\le p_2<\left(\frac{N}{p_1}\right)^{\frac{1}{2}}\\(p_1p_2,N)=1}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)$$
$$-\frac{1}{2}\sum_{\substack{N^{\frac{1}{8.8}}\le p_1<N^{\frac{1}{3.73}}\le p_2<\left(\frac{N}{p_1}\right)^{\frac{1}{2}}\\(p_1p_2,N)=1}} S\left(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),\left(\frac{N}{p_1p_2}\right)^{\frac{1}{2}}\right)$$
$$-\frac{1}{2}\sum_{\substack{N^{\frac{4.0871}{14}}\le p<N^{\frac{1}{3.106}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{14}})-\frac{1}{2}\sum_{\substack{U^{\frac{1}{2}}N^{-\frac{3}{14}}\le p<N^{\frac{1}{3.73}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{8.8}})$$
$$-\frac{1}{2}\sum_{\substack{N^{\frac{1}{14}}\le p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}\\(p_1p_2p_3p_4,N)=1}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)$$
$$-\frac{1}{2}\sum_{\substack{N^{\frac{1}{14}}\le p_1<p_2<p_3<N^{\frac{1}{8.8}}\le p_4<U^{\frac{1}{2}}N^{-\frac{3}{14}}p_3^{-1}\\(p_1p_2p_3p_4,N)=1}} S(\mathscr{A}_{p_1p_2p_3p_4};\mathscr{P}(p_1),p_2)$$
$$-\sum_{\substack{N^{\frac{1}{3.106}}\le p_1<p_2<\left(\frac{N}{p_1}\right)^{\frac{1}{2}}\\(p_1p_2,N)=1}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)$$
$$-\sum_{\substack{N^{\frac{1}{3.73}}\le p_1<p_2<\left(\frac{N}{p_1}\right)^{\frac{1}{2}}\\(p_1p_2,N)=1}} S(\mathscr{A}_{p_1p_2};\mathscr{P}(p_1),p_2)+O(N^{\frac{13}{14}})$$
$$=\frac{1}{2}(3S_{11}+S_{12})+\frac{1}{2}(S_{21}+S_{22})-\frac{1}{2}(S_{31}+S_{32})$$
$$-\frac{1}{2}(S_{41}+S_{42})-\frac{1}{2}(S_{51}+S_{52})-\frac{1}{2}(S_{61}+S_{62})-(S_{71}+S_{72})$$
$$=\frac{1}{2}\Sigma_1+\frac{1}{2}\Sigma_2-\frac{1}{2}\Sigma_3-\frac{1}{2}\Sigma_4-\frac{1}{2}\Sigma_5-\frac{1}{2}\Sigma_6-\Sigma_7+O(N^{\frac{13}{14}}).$$

Since the Proof of Lemma 10 is similar to that of Lemma 6 in [2], so we omit it here.

5. Proof of the Theorem

In this section, the sets $\mathscr{A}$ and $\mathscr{P}$ are defined by (4.1) and (4.2) respectively, and $\theta = 0.941$. Then in Lemma 1 we have

$$X=\int_2^U \frac{dt}{\log t}\sim\frac{U}{\theta\log N};\qquad C(\omega)=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2}=2C(N).$$

(1) Evaluation of $\Sigma_1,\Sigma_2,\Sigma_3$.

By Lemma 1, Lemma 2, Bombieri's mean value theorem in [1] and some routine arguments as in [3], we get that

$$\begin{aligned}
S_{11} &\geq (1+o(1))\left(\log(7\theta-1)+\int_2^{7\theta-2}\frac{\log(s-1)}{s}\log\frac{7\theta-1}{s+1}ds\right)\frac{8C(N)U}{\theta^2\log^2 N}\\
&\geq 16.7064\frac{C(N)U}{\log^2 N},\\
S_{12} &\geq (1+o(1))\left(\log(4.4\theta-1)+\int_2^{4.4\theta-2}\frac{\log(s-1)}{s}\log\frac{4.4\theta-1}{s+1}ds\right)\frac{8C(N)U}{\theta^2\log^2 N}\\
&\geq 10.3392\frac{C(N)U}{\log^2 N},\\
\Sigma_1 &= 3S_{11}+S_{12}\geq 60.4580\frac{C(N)U}{\log^2 N},
\end{aligned}\tag{5.1}$$

$$\begin{aligned}
S_{21} &\geq (1+o(1))\frac{8C(N)U}{\theta\log^2 N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{t_1}^{\frac{1}{8.8}}\frac{\log(7\theta-1-14(t_1+t_2))}{t_1t_2(\theta-2(t_1+t_2))}dt_1dt_2,\\
S_{22} &\geq (1+o(1))\frac{8C(N)U}{\theta\log^2 N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{\frac{1}{8.8}}^{\frac{\theta}{2}-\frac{2}{14}-t_1}\frac{\log(7\theta-1-14(t_1+t_2))}{t_1t_2(\theta-2(t_1+t_2))}dt_1dt_2,\\
\Sigma_2 &= S_{21}+S_{22}\\
&\geq (1+o(1))\frac{8C(N)U}{\theta\log^2 N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\int_{t_1}^{\frac{\theta}{2}-\frac{2}{14}-t_1}\frac{\log(7\theta-1-14(t_1+t_2))}{t_1t_2(\theta-2(t_1+t_2))}dt_1dt_2\\
&\geq 5.9160\frac{C(N)U}{\log^2 N}.
\end{aligned}\tag{5.2}$$

$$\begin{aligned}
S_{31} &\leq (1+o(1))\frac{8C(N)U}{\theta^2\log^2 N}\left(\log\frac{4.0871(14\theta-2)}{14\theta-8.1742}\right.\\
&+\int_2^{7\theta-2}\frac{\log(s-1)}{s}\log\frac{(7\theta-1)(7\theta-1-s)}{s+1}ds\\
&+\left.\int_2^{7\theta-4}\frac{\log(s-1)}{s}ds\int_{s+2}^{7\theta-2}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{(7\theta-1-t)(7\theta-1)}{t+1}dt\right)\\
&\leq 24.6357\frac{C(N)U}{\log^2 N},
\end{aligned}\tag{5.3}$$

$$\begin{aligned}
S_{32} \le\ & (1+o(1))\frac{8C(N)U}{\theta^2\log^2 N}\left(\log\frac{(7\theta-3)(7\theta-1)}{3}\right.\\
&+\int_2^{7\theta-2}\frac{\log(s-1)}{s}\log\frac{(7\theta-1)(7\theta-1-s)}{s+1}ds\\
&\left.+\int_2^{7\theta-4}\frac{\log(s-1)}{s}ds\int_{s+2}^{7\theta-2}\frac{1}{t}\log\frac{t-1}{s+1}\log\frac{(7\theta-1-t)(7\theta-1)}{t+1}dt\right)\\
\le\ & 21.8089\frac{C(N)U}{\log^2 N},\\
\Sigma_3 =\ & S_{31}+S_{32}\le 46.4447\frac{C(N)U}{\log^2 N}.
\end{aligned}$$

(2) Evaluation of Σ_4, Σ_7.

By Chen's switching principle, Lemma 1, Lemma 2, Lemma 7 and some routine arguments used in [3], we get

(5.4)
$$\begin{aligned}
S_{41} \le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log N}\sum_{N^{\frac{1}{14}}\le p_1<N^{\frac{1}{3.106}}\le p_2<\left(\frac{N}{p_1}\right)^{\frac12}}\frac{1}{p_1p_2\log\frac{N}{p_1p_2}}\\
\le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log^2 N}\int_{2.106}^{13}\frac{\log\left(2.106-\frac{3.106}{t+1}\right)}{t}dt\\
\le\ & 7.2603\frac{C(N)U}{\log^2 N},\\
S_{42} \le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log N}\sum_{N^{\frac{1}{8.8}}\le p_1<N^{\frac{1}{3.73}}\le p_2<\left(\frac{N}{p_1}\right)^{\frac12}}\frac{1}{p_1p_2\log\frac{N}{p_1p_2}}\\
\le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log^2 N}\int_{2.73}^{7.8}\frac{\log\left(2.73-\frac{3.73}{t+1}\right)}{t}dt\\
\le\ & 6.8019\frac{C(N)U}{\log^2 N},\\
\Sigma_4 =\ & S_{41}+S_{42}\le 14.0622\frac{C(N)U}{\log^2 N},
\end{aligned}$$

(5.5)
$$\begin{aligned}
S_{71} \le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log N}\sum_{N^{\frac{1}{3.106}}\le p_1<p_2<\left(\frac{N}{p_1}\right)^{\frac12}}\frac{1}{p_1p_2\log\frac{N}{p_1p_2}}\\
\le\ & (1+o(1))\frac{8C(N)U}{(2\theta-1)\log^2 N}\int_2^{2.106}\frac{\log(t-1)}{t}dt\\
\le\ & 0.0238\frac{C(N)U}{\log^2 N},
\end{aligned}$$

$$\begin{aligned}
S_{72} &\le (1+o(1))\frac{8C(N)U}{(2\theta-1)\log N}\sum_{N^{\frac{1}{3.73}}\le p_1<p_2<\left(\frac{N}{p_1}\right)^{\frac12}}\frac{1}{p_1p_2\log\frac{N}{p_1p_2}}\\
&\le (1+o(1))\frac{8C(N)U}{(2\theta-1)\log^2 N}\int_2^{2.73}\frac{\log(t-1)}{t}dt\\
&\le 0.8040\frac{C(N)U}{\log^2 N},\\
&\le 21.8089\frac{C(N)U}{\log^2 N},\\
\Sigma_7 &= S_{71}+S_{72}\le 0.8278\frac{C(N)U}{\log^2 N}.
\end{aligned}$$

(3) Evaluation of Σ_6.

We have

$$S_{61} = \sum_{\substack{N^{\frac{1}{14}}\le p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}\\ (p_1p_2p_3p_4,N)=1}}\ \sum_{\substack{p=N-p_1p_2p_3p_4n\\ N-U\le p_1<p_2<p_3<p_4n\le N\\ (n,p_1^{-1}NP(p_2))=1}} 1 = S'_{61}. \tag{5.6}$$

Let

$$\begin{aligned}
\mathscr{E} &= \Big\{e\,\Big|\,e=p_1p_2p_3p_4n, N^{\frac{1}{14}}\le p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}, (p_1p_2p_3p_4,N)=1,\\
&\qquad N-U\le p_1p_2p_3p_4n\le N, (n,p_1^{-1}NP(p_2))=1\Big\},\\
\mathscr{L} &= \{l\,|\,l=N-e, e\in\mathscr{E}\}.
\end{aligned}$$

Then S'_{61} does not exceed the number of primes in $\mathscr{L}$, hence

$$S_{61}\le S(\mathscr{L},D^{\frac12})+O(D^{\frac12}),\quad D\le N^{\frac12}. \tag{5.7}$$

By Lemma 1 we get

$$S(\mathscr{L},D^{\frac12})\le(1+o(1))\frac{8C(N)|\mathscr{L}|}{(2\theta-1)\log N}+R_1+R_2, \tag{5.8}$$

where

$$D = N^{\theta-\frac12}\log^{-B}N \qquad (B=B(10)\text{ in Lemma 8}),$$

$$R_1 = \sum_{d\le D,(d,N)=1}\left|\sum_{\substack{e\in\mathscr{E}\\(e,d)=1\\e\equiv N(d)}}1-\frac{1}{\varphi(d)}\sum_{\substack{e\in\mathscr{E}\\(e,d)=1}}1\right|,$$

$$R_2 = \sum_{d\le D,(d,N)=1}\frac{1}{\varphi(d)}\sum_{\substack{e\in\mathscr{E}\\(e,d)>1}}1.$$

It is easy to show that

$$
\begin{aligned}
R_2 &\ll \sum_{d\le D}\frac{1}{\varphi(d)}\sum_{m|d,m\ge N^{\frac{1}{14}}}\sum_{\substack{e\in\mathscr{E}\\(e,d)=m}}1\\
&\ll U\sum_{d\le D}\frac{1}{\varphi(d)}\sum_{m|d,m\ge N^{\frac{1}{14}}}\frac{1}{m}\\
&\ll U\sum_{N^{\frac{1}{14}}\le m\le D}\frac{1}{m\varphi(m)}\sum_{d\le\frac{D}{m}}\frac{1}{\varphi(d)}\\
&\ll UN^{-\frac{1}{14}}\log^2 N.
\end{aligned}
\tag{5.9}
$$

By Lemma 8 we get

$$
R_1 \ll \frac{U}{\log^{10} N}. \tag{5.10}
$$

Now by Lemma 6 and Lemma 3, we have

$$
\begin{aligned}
|\mathscr{L}| &= \sum_{\substack{N^{\frac{1}{14}}\le p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}\\(p_1p_2p_3p_4,N)=1}}\ \sum_{\substack{N-U\le np_1p_2p_3p_4\le N\\(n,p_1^{-1}NP(p_2))=1}}1\\
&< (1+o(1))\frac{1}{1.7803}\sum_{N^{\frac{1}{14}}\le p_1<p_2<p_3<p_4<N^{\frac{1}{8.8}}}\frac{U}{p_1p_2p_3p_4\log p_2}\\
&= (1+o(1))\frac{U}{1.7803\log N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.8t_2}dt_2.
\end{aligned}
\tag{5.11}
$$

By (5.6)-(5.11) we get

$$
\begin{aligned}
&S_{61}\\
&\le (1+o(1))\frac{8C(N)U}{1.7803(2\theta-1)\log^2 N}\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\frac{1}{8.8t_2}dt_2.
\end{aligned}
\tag{5.12}
$$

By a similar method we get

$$
\begin{aligned}
S_{62} \le\ & (1+o(1))\int_{\frac{1}{14}}^{\frac{1}{8.8}}\frac{dt_1}{t_1}\int_{t_1}^{\frac{1}{8.8}}\frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right)\log\left(8.8\left(\frac{\theta}{2}-\frac{2}{14}-t_2\right)\right)dt_2\\
&\times\frac{8C(N)U}{1.7803(2\theta-1)\log^2 N}.
\end{aligned}
\tag{5.13}
$$

By (5.12) and (5.13) we obtain

$$
\begin{aligned}
\Sigma_6 &= S_{61}+S_{62} \\
&\le (1+o(1)) \int_{\frac{1}{14}}^{\frac{1}{8.8}} \frac{dt_1}{t_1} \int_{t_1}^{\frac{1}{8.8}} \frac{1}{t_2}\left(\frac{1}{t_1}-\frac{1}{t_2}\right) \log\left(\frac{\frac{\theta}{2}-\frac{2}{14}}{t_2}-1\right) dt_2 \\
&\times \frac{8C(N)U}{1.7803(2\theta-1)\log^2 N} \\
&\le 0.7691\frac{C(N)U}{\log^2 N}.
\end{aligned} \tag{5.14}
$$

(4) Evaluation of Σ_5.

Let $D = N^{\theta-\frac{1}{2}}\log^{-B} N$ with $B = B(10) > 0$ in Bombieri's theorem [1], and $\underline{p} = \frac{D}{p}$.

Lemma 10. ([2, 6, 17]). *For $N^{\frac{1}{4.5}} < D_1 < D_2 < N^{\frac{1}{3}}$ We have*

$$
\begin{aligned}
&\sum_{\substack{D_1\le p<D_2\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},\underline{p}^{\frac{1}{2.5}}) \le \sum_{\substack{D_1\le p<D_2\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},\underline{p}^{\frac{1}{3.675}}) \\
&-\frac{1}{2}\sum_{\substack{D_1\le p<D_2\\(p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}}\le p_1<\underline{p}^{\frac{1}{2.5}}\\(p_1p_2,N)=1}} S(\mathscr{A}_{pp_1};\mathscr{P},\underline{p}^{\frac{1}{3.675}}) \\
&+\frac{1}{2}\sum_{\substack{D_1\le p<D_2\\(p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}}\le p_1<p_2<p_3<\underline{p}^{\frac{1}{2.5}}\\(p_1p_2p_3,N)=1}} S(\mathscr{A}_{pp_1p_2p_3};\mathscr{P}(p_1),p_2)+O(UN^{-\frac{1}{20}}).
\end{aligned}
$$

For $p \ge N^{\frac{4.0871}{14}}$, we have

$$
\underline{p}^{\frac{1}{2.5}} \le N^{\frac{1}{14}}, \qquad S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{14}}) \le S(\mathscr{A}_p;\mathscr{P},\underline{p}^{\frac{1}{2.5}}).
$$

By Lemma 10 we have

$$
\begin{aligned}
S_{51} = &\sum_{\substack{N^{\frac{4.0871}{14}}\le p<N^{\frac{1}{3.106}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},N^{\frac{1}{14}}) \le \sum_{\substack{N^{\frac{4.0871}{14}}\le p<N^{\frac{1}{3.106}}\\(p,N)=1}} S(\mathscr{A}_p;\mathscr{P},\underline{p}^{\frac{1}{2.5}}) \\
&\le \Gamma_1 - \frac{1}{2}\Gamma_2 + \frac{1}{2}\Gamma_3 + O(UN^{-\frac{1}{20}}).
\end{aligned} \tag{5.15}
$$

By Lemma 1, 2, Bombieri's theorem [1] and some routine argument we get

$$\begin{aligned}
\Gamma_1 &= \sum_{\substack{N^{\frac{4.0871}{14}} \le p < N^{\frac{1}{3.106}} \\ (p,N)=1}} S(\mathscr{A}_p; \mathscr{P}, \underline{p}^{\frac{1}{3.675}}) \\
&\le (1+o(1)) \left(\int_{\frac{4.0871}{14}}^{\frac{1}{3.106}} \frac{dt}{t(\theta-2t)} \right) \left(1 + \int_2^{2.675} \frac{\log(t-1)}{t} dt \right) \\
&\times \frac{8C(N)U}{\theta \log^2 N},
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
\Gamma_2 &= \sum_{\substack{N^{\frac{4.0871}{14}} \le p < N^{\frac{1}{3.106}} \\ (p,N)=1}} \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \le p_1 < \underline{p}^{\frac{1}{2.5}} \\ (p_1,N)=1}} S(\mathscr{A}_{pp_1}; \mathscr{P}, \underline{p}^{\frac{1}{3.675}}) \\
&\ge (1+o(1)) \left(\int_{\frac{4.0871}{14}}^{\frac{1}{3.106}} \frac{dt}{t(\theta-2t)} \right) \left(\int_{1.5}^{2.675} \frac{\log(2.675 - \frac{3.675}{t+1})}{t} dt \right) \\
&\times \frac{8C(N)U}{\theta \log^2 N}.
\end{aligned} \tag{5.17}$$

By an argument similar to the evaluation of S_{61} we obtain

$$\begin{aligned}
\Gamma_3 &\le (1+o(1)) \frac{8C(N)}{(2\theta-1)\log N} \sum_{\substack{N^{\frac{4.0871}{14}} \le p < N^{\frac{1}{3.106}} \\ (p,N)=1}} \\
&\times \sum_{\substack{\underline{p}^{\frac{1}{3.675}} \le p_1 < p_2 < p_3 < \underline{p}^{\frac{1}{2.5}} \\ (p_1p_2p_3,N)=1}} \sum_{\substack{N-U \le npp_1p_2p_3 \le N \\ (n, p_1^{-1} NP(p_2))=1}} 1 \\
&\le (1+o(1)) \frac{8C(N)U}{1.763(2\theta-1)\log N} \\
&\times \sum_{\substack{N^{\frac{4.0871}{14}} \le p < N^{\frac{1}{3.106}} \\ (p,N)=1}} \frac{1}{p \log \underline{p}} \int_{\frac{1}{3.675}}^{\frac{1}{2.5}} \int_{t_1}^{\frac{1}{2.5}} \int_{t_2}^{\frac{1}{2.5}} \frac{dt_1 dt_2 dt_3}{t_1 t_2^2 t_3} \\
&\le (1+o(1)) \left(\int_{\frac{4.0871}{14}}^{\frac{1}{3.106}} \frac{dt}{t(\theta-2t)} \right) \left(6.175 \log \frac{3.675}{2.5} - 2.35 \right) \\
&\times \frac{16C(N)U}{1.763(2\theta-1)\log^2 N}.
\end{aligned} \tag{5.18}$$

By (5.15)-(5.18) we get

$$
\begin{aligned}
S_{51} &\le (1+o(1))(1-\Delta)\left(\int_{\frac{4.0871}{14}}^{\frac{1}{3.106}} \frac{dt}{t(\theta-2t)}\right)\frac{8C(N)U}{\theta\log^2 N} \\
&\le 2.5212\frac{C(N)U}{\log^2 N},
\end{aligned}
\tag{5.19}
$$

where and below

$$
\begin{aligned}
\Delta &= \frac{1}{2}\int_{1.5}^{2.675}\frac{\log(2.675-\frac{3.675}{t+1})}{t}dt - \int_{2}^{2.675}\frac{\log(t-1)}{t}dt \\
&- \frac{\theta}{1.763(2\theta-1)}\left(6.175\log\frac{3.675}{2.5}-2.35\right) \\
&\ge 0.01032.
\end{aligned}
$$

In the same way we have

$$
\begin{aligned}
S_{52} &\le (1+o(1))(1-\Delta)\left(\int_{\frac{\theta}{2}-\frac{3}{14}}^{\frac{1}{3.73}} \frac{dt}{t(\theta-2t)}\right)\frac{8C(N)U}{\theta\log^2 N} \\
&\le 0.9155\frac{C(N)U}{\log^2 N}.
\end{aligned}
\tag{5.20}
$$

Finally, by (5.19) and (5.20) we find that

$$
\Sigma_5 = S_{51}+S_{52} \le 3.4367\frac{C(N)U}{\log^2 N}. \tag{5.21}
$$

6) Proof of the Theorem

By (5.1)-(5.4), (5.14) and (5.21) we get

$$
\Sigma_1+\Sigma_2 \ge 66.3740\frac{C(N)U}{\log^2 N}, \tag{5.22}
$$

$$
\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6 \le 64.7127\frac{C(N)U}{\log^2 N}. \tag{5.23}
$$

By (5.5), (5.22), (5.23) and Lemma 10 we obtain

$$
\begin{aligned}
2S(N,U) &\ge \frac{1}{2}(\Sigma_1+\Sigma_2) - \frac{1}{2}(\Sigma_3+\Sigma_4+\Sigma_5+\Sigma_6) - \Sigma_7 \\
&\ge \left(\frac{66.3740-64.7124}{2} - 0.8278\right)\frac{C(N)U}{\log^2 N} \\
&> \frac{0.002C(N)U}{\log^2 N}, \\
S(N,U) &> \frac{0.001C(N)U}{\log^2 N}.
\end{aligned}
$$

The Theorem is proved.

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