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MULTIPLE SOLUTIONS FOR THE NONHOMOGENEOUS FOURTH ORDER ELLIPTIC EQUATIONS OF KIRCHHOFF-TYPE

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Abstract. This paper considers the following nonhomogeneous fourth order elliptic equations of Kirchhoff type:

$$\begin{cases} \triangle^2 u - (a + b\displaystyle\int_{\mathrm{R}^N} |\nabla u|^2 dx)\triangle u + V(x)u = f(x,u) + h(x), \text{ in } \mathrm{R}^N, \\ u \in H^2(\mathrm{R}^N), \end{cases}$$

where constants $a > 0,\ b \geq 0$. Under certain assumptions on $V(x),\ f(x,u)$ and $h(x)$, we show the existence and multiplicity of solutions by the Ekeland's variational principle and the Mountain Pass Theorem in the critical theory.

1. Introduction and Preliminaries

Consider the following nonhomogeneous fourth order elliptic equations of Kirchhoff type:

$$\begin{aligned} &\triangle^2 u - (a + b\int_{\mathrm{R}^N} |\nabla u|^2 dx)\triangle u + V(x)u = f(x,u) + h(x),\ x \in \mathrm{R}^N, \\ &u \in H^2(\mathrm{R}^N), \end{aligned} \tag{1.1}$$

where constants $a > 0,\ b \geq 0$. We assume that the functions $V(x),\ f(x,u)$ and its primitive $F(x,u) := \int_0^u f(x,s)ds$ satisfy the following hypotheses:

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(V) $V(x) \in C(\mathrm{R}^N, \mathrm{R})$ satisfies $\inf\limits_{x \in \mathrm{R}^N} V(x) \geq a_1 > 0$, where a_1 is a constant. Moreover, for any $M > 0$, meas$\{x \in \mathrm{R}^N : V(x) \leq M\} < \infty$, where meas$(.)$ denotes the Lebesgue measure in R^N.

(f_1) $f(x,u) \in C(\mathrm{R}^N \times \mathrm{R}, \mathrm{R})$ and there exist $2 < p < 2^* = \frac{2N}{N-2}$ and $a_2 > 0$ such that

$$|f(x,u)| \leq a_2(1 + |u|^{p-1}).$$

(f_2)
$$\lim_{u \to 0} \frac{f(x,u)}{u} = 0, \ \forall x \in \mathrm{R}^N.$$

(f_3) There exist $\mu > 4$ and $r > 0$ such that

$$\mu F(x,u) \leq u f(x,u), \ \forall x \in \mathrm{R}^N, \ |u| \geq r.$$

(f_4)
$$\inf_{x \in \mathrm{R}^N, |u| = r} F(x,u) > 0.$$

Let $H := H^2(\mathrm{R}^N)$ with the inner product and the norm

$$\langle u, v \rangle_H = \int_{\mathrm{R}^N} (\triangle u \triangle v + \nabla u \nabla v + uv) dx, \ \|u\|_H = \langle u, u \rangle_H^{\frac{1}{2}}.$$

Define our working space

$$E = \{u \in H : \int_{\mathrm{R}^N} (|\triangle u|^2 + |\nabla u|^2 + V(x) u^2) dx < +\infty\}$$

with the inner product and norm

$$\langle u, v \rangle = \int_{\mathrm{R}^N} (\triangle u \triangle v + a \nabla u \nabla v + V(x) uv) dx, \ \|u\| = \langle u, u \rangle^{\frac{1}{2}},$$

where $\|\cdot\|$ is an equivalent to the norm $\|\cdot\|_H$.

It is clear that system (1.1) is the Euler-Lagrange equations of the functional $I : E \to \mathrm{R}$ defined by

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathrm{R}^N} |\nabla u|^2 dx\right)^2 - \int_{\mathrm{R}^N} F(x,u) dx - \int_{\mathrm{R}^N} h(x) u dx. \tag{1.2}$$

Obviously, I is a well-defined C^1 functional and satisfies

$$\begin{aligned} \langle I'(u), v \rangle = & \int_{\mathrm{R}^N} (\triangle u \triangle v + a \nabla u \nabla v + V(x) uv) dx \\ & + b \int_{\mathrm{R}^N} |\nabla u|^2 dx \int_{\mathrm{R}^N} \nabla u \nabla v dx \\ & - \int_{\mathrm{R}^N} f(x,u) v dx - \int_{\mathrm{R}^N} h(x) v dx, \ \forall \ u, \ v \in E. \end{aligned} \tag{1.3}$$

Let $V(x) = 0$, $h(x) = 0$, replace R^N by a bounded smooth domain $\Omega \subset \mathrm{R}^N$, and set $u = \nabla u = 0$ on Ω, then problem (1.1) reduces to the following homogeneous equations:

$$\begin{aligned} &\triangle^2 u - (a + b\int_\Omega |\nabla u|^2 dx)\triangle u = f(x,u),\ x \in \Omega, \\ &u = 0, \nabla u = 0 \text{ on } \Omega, \end{aligned} \tag{1.4}$$

which is related to the following stationary analogue of the equation of Kirchhoff type:

$$u_{tt} + \triangle^2 u - (a + b\int_\Omega |\nabla u|^2 dx)\triangle u = f(x,u),\ \text{in } \Omega, \tag{1.5}$$

where $\triangle^2$ is the biharmonic operator. In one and two dimensions, (1.5) is used to describe some phenomena appeared in different physical, engineering and other sciences because it is regarded as a good approximation for describing nonlinear vibrations of beams or plates (see [2-3]). Using the mountain pass techniques and the truncation method, wang *et al.* [4] obtained the existence of nontrivial solutions of the following elliptic equations:

$$\begin{cases} \triangle^2 u - \lambda(a + b\int_\Omega |\nabla u|^2 dx)\triangle u = f(x,u),\ x \in \Omega, \\ u = 0, \nabla u = 0 \text{ on } \Omega. \end{cases}$$

More recently, there are several papers having studied (1.1) with $h(x) = 0$, see for example [5-6].

In (1.1), let $a = 0,\ V(x) = 0$ and $h(x) = 0$, then problem (1.1) can be rewritten as the following fourth order equation of Kirchhoff type:

$$\begin{aligned} &\triangle^2 u - b(\int_\Omega |\nabla u|^2 dx)\triangle u = f(x,u) \text{ in } \Omega, \\ &u = \nabla u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.6}$$

By the variational methods, T. F. Ma and F. Wang *etc.* studied (1.6) and obtained the existence and multiplicity of solutions, see [7-9].

If $a = 1,\ b = 0$ and $h(x) = 0$, then (1.1) reduces to the following equations:

$$\begin{aligned} &\triangle^2 u - \triangle u + V(x)u = f(x,u),\ x \in \mathrm{R}^N, \\ &u \in H^2(\mathrm{R}^N). \end{aligned} \tag{1.7}$$

In recent years, there are many results for (1.7), see for instance [10-12]. The solvability of (1.1) without $\triangle^2$ has also been well studied by various authors (see [13-14] and the references therein).

Obviously, the problem (1.1) is nonlocal because of the presence of the term $\int_{\mathrm{R}^N} |\nabla u|^2 dx$ which provokes some mathematical difficulties. This phenomenon makes the study of such a class of problems particularly interesting. To my best knowledge, there are no any work on the existence and multiplicity solutions for the nonhomogeneous fourth order elliptic equation of Kirchhoff type. The object of this paper is to establish the first results in this case. Our tools is the Mountain Pass Theorem [15] and the Ekeland's variational principle [16] in the critical theory. Throughout this paper, C_i denotes various positive constants.

2. Main Results

In order to deduce our results, we need the following lemmas. Motivated by Lemma 3.4 in [1], we can first prove the following Lemma 2.1 in the same way. Here we omit it.

Lemma 2.1. *Under the assumption* (V), *the embedding* $E \hookrightarrow L^s(R^N)$ *is compact for any* $s \in [2, 2^*)$. *Then, for each* $s \in [2, 2^*)$, *there exists* $\eta_s > 0$ *such that* $\|u\|_{L^s} \leq \eta_s \|u\|, \ \forall u \in E$, *where* $\|u\|_{L^s} := (\int_{R^N} |u|^s dx)^{\frac{1}{s}}$, *for any* $s \in [1, \infty)$ *is the norm of the usual Lebesgue space* $L^s(R^N)$.

Lemma 2.2. *Assume* (V) *and* (f_1)-(f_2) *hold. Let* $h \in L^2(R^N)$, *then there exist some constants* $\rho, \ \alpha, \ m_0 > 0$ *such that* $I(u) \geq \alpha > 0$ *with* $\|u\| = \rho$ *for all* $u \in E$ *and* h *satisfying* $\|h\|_{L^2} < m_0$.

Proof. By (f_1) and (f_2), there exists $c(\varepsilon) > 0$ such that

$$|f(x,u)| \leq \varepsilon |u| + c(\varepsilon)|u|^{p-1}, \tag{2.1}$$

and for all $(x,u) \in \mathrm{R}^N \times \mathrm{R}$, one has

$$|F(x,u)| \leq \frac{\varepsilon}{2}|u|^2 + \frac{c(\varepsilon)}{P}|u|^p. \tag{2.2}$$

It follows from (1.2), (2.2), the Hölder inequality and Lemma 2.1 that

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathrm{R}^N} (\frac{\varepsilon}{2}|u|^2 + \frac{c(\varepsilon)}{P}|u|^p) dx - \|h\|_{L^2}\|u\|_{L^2} \\ &= \frac{1}{2}\|u\|^2 - \frac{\varepsilon}{2}\|u\|^2_{L^2} - \frac{c(\varepsilon)}{P}\|u\|^p_{L^p} - \|h\|_{L^2}\|u\|_{L^2} \\ &\geq \|u\|[(\frac{1}{2} - \frac{\varepsilon\eta_2^2}{2})\|u\| - C_1\|u\|^{p-1} - \eta_2\|h\|_{L^2}]. \end{aligned} \tag{2.3}$$

Taking $\varepsilon = \dfrac{1}{2\eta_2^2}$ and setting $g(t) = \dfrac{1}{4}t - C_1 t^{p-1}$ for $t \geq 0$. By direct calculations, we see that $\max\limits_{t\geq 0} g(t) = g(\rho) > 0$, where $\rho = [\dfrac{1}{4C_1(p-1)}]^{\frac{1}{p-2}} > 0$. Then it follows from (2.3) that, if $\|h\|_{L^2} < m_0 := \dfrac{g(\rho)}{2\eta_2} > 0$, there exists $\alpha > 0$ such that $I(u)|_{\|u\|=\rho} \geq \alpha > 0$.

Lemma 2.3. *Assume that* (V), $h(x) \in L^2(R^N)$, $h \geq (\not\equiv)0$ *and* (f_1)-(f_4) *hold, then there exists a function* $v \in E$ *with* $\|v\| > \rho$ *such that* $I(v) < 0$, *where* ρ *is given by Lemma 2.2.*

Proof. For any $x \in \mathrm{R}^N$, $|z| \geq r$, set

$$\tau(t) = F(x, t^{-1}z)t^{\mu}, \forall t \in [1, \frac{|z|}{r}].$$

By (f_3), one has

$$\tau'(t) = t^{\mu-1}[\mu F(x, t^{-1}z) - t^{-1}zf(x, t^{-1}z)] \leq 0.$$

Hence, $\tau(1) \geq \tau(\frac{|z|}{r})$, that is

$$F(x,z) \geq F(x, \frac{r}{|z|}z)\frac{|z|^{\mu}}{r^{\mu}} \geq \inf_{x\in\mathrm{R}^N, \|u\|=r} F(x,u)\frac{|z|^{\mu}}{r^{\mu}} \geq C_2|z|^{\mu} \tag{2.4}$$

for any $x \in \mathrm{R}^N$, $|z| \geq r$. By (f_2), there exists $\delta \leq r$ such that

$$|\frac{f(x,z)z}{z^2}| = |\frac{f(x,z)}{z}| \leq 1,$$

for all $x \in \mathrm{R}^N$, $0 < |z| < \delta$. It follows from (f_1) that there exists a positive constant M_1 such that

$$|\frac{f(x,z)z}{z^2}| \leq \frac{a_2(1+|z|^{p-1})|z|}{z^2} \leq M_1,$$

Thus, one has

$$f(x,z)z \geq -(M_1+1)|z|^2$$

for all $x \in \mathrm{R}^N$, $0 < |z| < \delta$. Using the definition of $F(x,z)$, we have

$$F(x,z) \geq -\frac{1}{2}(M_1+1)|z|^2 \tag{2.5}$$

for all $x \in \mathrm{R}^N$, $0 < |z| < \delta$. Setting $C_3 = \frac{1}{2}(M_1+1) + C_2$, we obtain from (2.4) and (2.5) that

$$F(x,z) \geq C_2|z|^{\mu} - C_3|z|^2 \tag{2.6}$$

for *a.e.* $x \in \mathrm{R}^3$ and all $z \in \mathrm{R}$. Since $E \hookrightarrow L^2(\mathrm{R}^N)$ and $L^2(\mathrm{R}^N)$ is a separable Hilbert space, E has a countable orthogonal basis $\{e_j\}$. Set $E_k = \mathrm{span}\{e_1, e_2, ..., e_k\}$. Then $E = E_k \oplus E_k^{\perp}$ and E_k is finite-dimensional space. Moreover, for any finite dimensional subspace $\tilde{F} \subset E$, there is a positive integral number m such that $\tilde{F} \subset E_m$. Hence, by (2.6) and the assumptions on $h(x)$, we get

$$\begin{aligned} I(u) &\leq \frac{1}{2}\|u\|^2 + \frac{C_4}{4}\|u\|^4 - C_2\|u\|_{L^\mu}^\mu + C_3\|u\|_{L^2}^2 + \int_{\mathrm{R}^N} h(x)|u|dx \\ &\leq \frac{1}{2}\|u\|^2 + \frac{C_4}{4}\|u\|^4 - C_2\gamma^\mu\|u\|^\mu + C_3\eta_2^2\|u\|^2 + \int_{\mathrm{R}^N} h(x)|u|dx \end{aligned}$$

for all $u \in E_m$, where in the last inequality we use the equivalence of all norms on the finite dimensional subspace E_m. Consequently, by $\mu > 4$, there is a point $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$, which completes this lemma.

Lemma 2.4. *Assume* (V) *and* (f_3)-(f_4) *hold. Let* $h \in L^2(R^N)$ *and* $\{u_n\}$ *is a* (PS) *sequence, then* $\{u_n\}$ *is bounded in* E *if* $\|h\|_{L^2} < m_0$.

Proof. Consider a sequence $\{u_n\}$ which satisfies $I(u_n) \to c$ and $\langle I'(u_n), u_n\rangle \to 0$. If $\{u_n\}$ is unbounded in E, we can assume $\|u_n\| \to +\infty$ as $n \to \infty$. Set $\omega_n = \dfrac{u_n}{\|u_n\|}$, then $\|\omega_n\| = 1$ and $\|\omega_n\|_{L^s} \leq \eta_s$ for $s \in [2, 2^*)$. Going if necessary to a subsequence, we may assume that

$$\omega_n \rightharpoonup \omega \text{ in } E,\ \omega_n \to \omega \text{ in } L^s(\mathrm{R}^N)(2 \leq s < 2^*),\ \omega_n \to \omega\ a.e. \text{ on } \mathrm{R}^N. \tag{2.7}$$

Set $\Omega = \{x \in \mathrm{R}^3 : \omega(x) \neq 0\}$. If $\mathrm{meas}(\Omega) > 0$, then $|u_n| \to +\infty$ a.e. $x \in \Omega$ as $n \to \infty$. It follows from (2.6) that

$$f(x, u_n)u_n \geq C_5|u_n|^\mu - C_6|u_n|^2$$

for *a.e.* $x \in \mathrm{R}^3$ and all $u_n \in \mathrm{R}$. Hence

$$\int_{\mathrm{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^\mu}dx \geq C_5\|\omega_n\|_{L^\mu}^\mu - C_7\frac{\|\omega_n\|_{L^2}^2}{\|u_n\|^{\mu-2}}. \tag{2.8}$$

Since $\mu > 4$ and

$$\begin{aligned} \frac{\langle I'(u_n, u_n\rangle}{\|u_n\|^\mu} &= \frac{1}{\|u_n\|^{\mu-4}} + \frac{b(\int_{\mathrm{R}^N} |\nabla u_n|^2dx)^2}{\|u_n\|^\mu} \\ &- \int_{\mathrm{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^\mu}dx - \int_{\mathrm{R}^N} h(x)\frac{u_n}{\|u_n\|^\mu}dx, \end{aligned}$$

one has

$$\lim_{n\to\infty} \int_{\mathrm{R}^N} \frac{f(x, u_n)u_n}{\|u_n\|^\mu}dx = 0. \tag{2.9}$$

Consequently, we obtain from (2.8) and (2.9) that

$$0 = \lim_{n\to\infty} \int_{\mathrm{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|^\mu} dx \geq C_5 \|\omega_n\|^\mu_{L^\mu} > 0,$$

which is a contradiction. Hence, meas$(\Omega) = 0$. Therefore, $\omega(x) = 0$ *a.e.*$x \in \mathrm{R}^N$. It follows from (f_1)-(f_3) that

$$|uf(x,u) - \mu F(x,u)| \leq C_8 u^2, \text{for all } (x,u) \in \mathrm{R}^N \times \mathrm{R}.$$

Thus, for $\|h\|_{L^2} < m_0$,

$$\begin{aligned}
&\frac{1}{\|u_n\|^2}[I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle] \\
&= (\frac{1}{2} - \frac{1}{\mu}) + (\frac{b}{4} - \frac{b}{\mu})(\int_{\mathrm{R}^N} |\nabla u_n|^2 dx)^2/\|u_n\|^2 \\
&+\int_{\mathrm{R}^N} [\frac{1}{\mu} f(x,u_n)u_n - F(x,u_n)]/\|u_n\|^2 dx + (\frac{1}{\mu} - 1)\|h\|_{L^2}\|u_n\|_{L^2}/\|u_n\|^2 \\
&\geq (\frac{1}{2} - \frac{1}{\mu}) - \frac{C_8}{\mu}\int_{\mathrm{R}^N} \omega_n^2 dx + (\frac{1}{\mu} - 1)m_0 \frac{\eta_2}{\|u_n\|}.
\end{aligned} \tag{2.10}$$

Since $\mu > 4$, (2.10) implies $0 \geq \frac{1}{2} - \frac{1}{\mu}$, a contradiction. Hence, $\{u_n\}$ is bounded in E.

Lemma 2.5. *Let (V), (f_1)-(f_2) hold and $\{u_n\}$ is a bounded Palais-Smale sequence of I, then $\{u_n\}$ has a strongly convergent subsequence in E.*

Proof. By (1.3), we have

$$\begin{aligned}
\langle I'(u_n) - I'(u), u_n - u\rangle \geq \|u_n - u\|^2 - b(\int_{\mathrm{R}^N} |\nabla u|^2 dx \\
- \int_{R^N} |\nabla u_n|^2 dx)\int_{\mathrm{R}^N} \nabla u \nabla(u_n - u) dx \\
- \int_{\mathrm{R}^N} [f(x,u_n) - f(x,u)](u_n - u) dx,
\end{aligned}$$

then, one has

$$\begin{aligned}
\|u_n - u\|^2 \leq \langle I'(u_n) - I'(u), u_n - u\rangle + b(\int_{\mathrm{R}^N} |\nabla u|^2 dx \\
- \int_{\mathrm{R}^N} |\nabla u_n|^2 dx)\int_{\mathrm{R}^N} \nabla u \nabla(u_n - u) dx \\
+ \int_{\mathrm{R}^N} [f(x,u_n) - f(x,u)](u_n - u) dx.
\end{aligned} \tag{2.11}$$

Since $\{u_n\}$ is bounded in E, going if necessary to a subsequence, we may assume that

$$u_n \rightharpoonup u \text{ in } E,\ u_n \to u \text{ in } L^s(\mathrm{R}^N)(2 \le s < 2^*),\ u_n \to u \textit{ a.e.} \text{ on } \mathrm{R}^N. \tag{2.12}$$

Then, it follows from (2.1), the boundedness of $\{u_n\}$ and the Hölder inequality that

$$\begin{aligned}
&\int_{\mathrm{R}^N} |f(x,u_n) - f(x,u)||u_n - u|dx \\
&\le \int_{\mathrm{R}^N} (|f(x,u_n)| + |f(x,u)|)|u_n - u|dx \\
&\le \int_{\mathrm{R}^N} \varepsilon(|u_n| + |u|)|u_n - u|dx + c(\varepsilon)\int_{\mathrm{R}^N} (|u_n|^{p-1} + |u|^{p-1})|u_n - u|dx \\
&\le \varepsilon[(\int_{\mathrm{R}^N} |u_n|^2 dx)^{\frac{1}{2}} + (\int_{\mathrm{R}^N} |u|^2 dx)^{\frac{1}{2}}](\int_{\mathrm{R}^N} |u_n - u|^2 dx)^{\frac{1}{2}} \\
&\quad + c(\varepsilon)[(\int_{\mathrm{R}^N} |u_n|^p dx)^{\frac{p-1}{p}} + (\int_{\mathrm{R}^N} |u|^p dx)^{\frac{p-1}{p}}](\int_{\mathrm{R}^N} |u_n - u|^p dx)^{\frac{1}{p}} \\
&\le C_7\|u_n - u\|_{L^2} + C_8\|u_n - u\|_{L^p} \to 0,\ n \to +\infty.
\end{aligned} \tag{2.13}$$

Define the linear functional $g : E \to \mathrm{R}$ by $g(w) = \int_{\mathrm{R}^N} \nabla u \nabla w dx$. Since $g(w) \le \|u\|\|w\|$, we can deduce that g is continuous on E. Using $u_n \rightharpoonup u$ in E, one has

$$\int_{\mathrm{R}^N} \nabla u \nabla(u_n - u)dx \to 0, \text{ as } n \to \infty.$$

Thus, we get from the boundedness of $\{u_n\}$ in E that

$$b(\int_{\mathrm{R}^N} |\nabla u_n|^2 dx - \int_{\mathrm{R}^N} |\nabla u|^2 dx)\int_{\mathrm{R}^N} \nabla u \nabla(u_n - u)dx \to 0, \text{ as } n \to \infty. \tag{2.14}$$

Clearly,

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0, \text{ as } n \to \infty. \tag{2.15}$$

It follows from (2.11), (2.13), (2.14) and (2.15) that $\|u_n - u\| \to 0$. The proof is complete. ■

The following theorems are our main results.

Theorem 2.1. *Assume that* $h(x) \in L^2(R^N)$ *and* $h(x) \ge (\not\equiv)0$. *Let* (V) *and* (f_1)-(f_4) *hold, then there exists a constant* $m_0 > 0$ *such that problem (1.1) possesses at least two nontrivial solutions* $u_0 \in E$ *and* $u_1 \in E$ *satisfying* $I(u_0) < 0 < I(u_1)$ *when* $\|h\|_{L^2} < m_0$.

Proof. We prove Theorem 2.1 by the following two steps.

Step 1. There exists $u_0 \in E$ such that $I(u_0) > 0$ and $I'(u_0) = 0$.

By Lemma 2.2, 2.3 and the Mountain Pass Theorem [15], there exists a sequence $\{u_n\} \subset E$ satisfying $I(u_n) \to c_1 > 0,\ I'(u_n) = 0$. Then it follows from Lemma 2.4 and 2.5 that there exists $u_0 \in E$ such that $I(u_0) = c_1 > 0$ and $I'(u_0) = 0$ if $\|h\|_{L^2} < m_0$.

Step 2. There exists $u_1 \in E$ such that $I(u_1) < 0$ and $I'(u_1) = 0$. Since $h \in L^2(\mathrm{R}^N)$ and $h \not\equiv 0$, we can choose a function $\phi \in E$ such that

$$\int_{\mathrm{R}^N} h(x)\phi(x)dx > 0. \tag{2.16}$$

Then, it follows from (1.2), (2.6) and (2.16) that

$$\begin{aligned} I(t\phi) \le \frac{t^2}{2}\|\phi\|^2 + \frac{bt^4}{4}(\int_{\mathrm{R}^N} |\nabla\phi|^2 dx)^2 - C_2 t^\mu \|\phi\|^\mu_{L^\mu} \\ + C_3 t^2 \|\phi\|^2_{L^2} - t\int_{\mathrm{R}^N} h(x)\phi dx < 0 \end{aligned}$$

for $t > 0$ small enough. Then, we get $c_0 = \inf\{I(u) : u \in \bar{B}_\rho\} < 0$, where ρ is given by Lemma 2.2, $B_\rho = \{u \in E, \|u\| < \rho\}$. It follows from Ekeland's variational principle [16] that there exists a sequence $\{u_n\} \subset \bar{B}_\rho$ such that $c_0 \le I(u_n) \le c_0 + \frac{1}{n}$ and $I(\omega) \ge I(u_n) - \frac{1}{n}\|\omega - u_n\|$ for all $\omega \in \bar{B}_\rho$. Then by a standard procedure, we can show that $\{u_n\}$ is a bounded Palais-Smale sequence of I. In view of Lemma 2.5, we obtain that there exists a function $u_1 \in E$ such that $I'(u_1) = 0$, $I(u_1) = c_0 < 0$. The proof is complete. ■

Theorem 2.2. *If we replace the conditions* (f_3)-(f_4) *by the following conditions:*
(f_3') *There exist* $\mu > 4$ *such that*

$$\mu F(x,u) \le u f(x,u),\ \forall (x,u) \in \mathrm{R}^N \times \mathrm{R}$$

and

(f_4')

$$\inf_{x\in\mathrm{R}^N |u|=1} F(x,u) > 0,$$

then the conclusion of Theorems 2.1 remains true.

Proof. Obviously, (f_3') and (f_4') imply (f_3) and (f_4) with $r = 1$. The proof of Theorem 2.2 is complete. ■

Theorem 2.3. *Assume that* $h(x) \in L^2(R^N)$ *and* $h(x) \ge (\not\equiv) 0$. *Let* (V), (f_1)-(f_3) *and the following conditions:*
(f_5) *There exists* $4 < \alpha < 2^*$ *such that*

$$\liminf_{|u|\to\infty} \frac{F(x,u)}{|u|^\alpha} > 0,\ \text{uniformly for } x \in \mathrm{R}^N$$

hold, then there exists a constant $m_0 > 0$ such that problem (1.1) possesses at least two nontrivial solutions $u_0 \in E$ and $u_1 \in E$ satisfying $I(u_0) < 0 < I(u_1)$ when $\|h\|_{L^2} < m_0$.

Proof. It is sufficient to prove (f_4). In fact, by (f_5), we can choose $\epsilon \in (0, \liminf\limits_{|u|\to\infty} \frac{F(x,u)}{|u|^\alpha})$ small enough such that

$$F(x,u) \geq \epsilon |u|^\alpha \text{ for } |u| \text{ large enough}, \tag{2.17}$$

then we obtain from (2.17) that (f_4) satisfies. This completes the proof. ■

Theorem 2.4. *The conclusions of Theorem 2.1, 2.2 and 2.3 hold if we replace (f_3) or (f_3') by the following condition:*
(f_6) There exists $\mu > 4$ such that $u \to \frac{f(x,u)}{|u|^{\mu-1}}$ is increasing on $(-\infty, 0)$ and $(0, +\infty)$.

Proof. It is sufficient to prove (f_6) implies (f_3) or (f_3'). Indeed, whenever $u < 0$,

$$\begin{aligned} F(x,u) &= \int_0^1 f(x,tu)u\,dt \\ &= -\int_0^1 \frac{f(x,tu)}{(-ut)^{\mu-1}}(-u)^\mu t^{\mu-1}dt \\ &= -\int_0^1 \frac{f(x,tu)}{|ut|^{\mu-1}}|u|^\mu t^{\mu-1}dt \\ &\leq -\int_0^1 \frac{f(x,u)}{|u|^{\mu-1}}|u|^\mu t^{\mu-1}dt = \frac{1}{\mu}f(x,u)u. \end{aligned}$$

Whenever $u > 0$,

$$\begin{aligned} F(x,u) &= \int_0^1 f(x,tu)u\,dt = \int_0^1 \frac{f(x,tu)}{(ut)^{\mu-1}}u^\mu t^{\mu-1}dt \\ &\leq \int_0^1 \frac{f(x,u)}{u^{\mu-1}}u^\mu t^{\mu-1}dt = \frac{1}{\mu}f(x,u)u. \end{aligned}$$

It shows that (f_3') holds and then (f_3) follows. This completes the proof.

Remark 2.1. To the best of our knowledge, it seems that Theorem 2.1, 2.2, 2.3 and 2.4 are the first results about the existence of multiple solutions for the nonhomogeneous fourth order elliptic equation of Kirchhoff type.

Remark 2.2. For $(f'3)$ and (f_4') imply (f_3) and (f_4), Theorem 2.1 generalizes Theorem 2.2. For (f_5) implies (f_4), Theorem 2.2 generalizes Theorem 2.3. Moreover, Theorem 2.3 generalizes Theorem 2.4 for (f_6) implies (f_3).

Remark 2.3. There are functions, which satisfy all conditions of Theorem 2.1, but not satisfy Theorem 2.2. For example, set

$$f(x,t) = \begin{cases} |t|^{q-2}t(q\ln|t|+1), & |t| \geq 1, \\ -|t|^3 t, & |t| \leq 1, \end{cases}$$

where $4 < q < 2^*$. Simple computation shows that

$$F(x,t) = \begin{cases} |t|^q \ln|t| - \dfrac{1}{5}, & |t| \geq 1, \\ -\dfrac{1}{5}|t|^5, & |t| \leq 1, \end{cases}$$

and

$$tf(x,t) - \mu F(x,t) = (q-\mu)|t|^q \ln|t| + |t|^q - \frac{1}{5}\mu, \ \forall x \in \mathrm{R}^N, \ |t| \geq 1.$$

Setting $4 < \mu < \min\{q, 5\}$, it is easy to check that $f(x,t)$ satisfies all the conditions in Theorems 2.1, but not satisfy (f_3') for $tf(x,t) - \mu F(x,t) < 0$ when $|t| \leq 1$. So $f(x,t)$ does not satisfy Theorem 2.2. Moreover, set $f(x,t) = |t|^{q-2}t, 4 < q < 2^*$. Then $f(x,t)$ satisfies all the conditions in Theorems 2.1, 2.2 and 2.3, but not satisfy (f_6). So not satisfy Theorem 2.4.

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