

EKELAND'S VARIATIONAL PRINCIPLE FOR SET-VALUED MAPS WITH APPLICATIONS TO VECTOR OPTIMIZATION IN UNIFORM SPACES

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Abstract. In this paper, we introduce the concept of a weak q -distance and for this distance we derive a set-valued version of Ekeland's variational principle in the setting of uniform spaces. By using this principle, we prove the existence of solutions to a vector optimization problem with a set-valued map. Moreover, we define the (p, ε) -condition of Takahashi and the (p, ε) -condition of Hamel for a set-valued map. It is shown that these two conditions are equivalent. As an application, we discuss the relationship between an ε -approximate solution and a solution of a vector optimization problem with a set-valued map. Also, a well-posedness result for a vector optimization problem with a set-valued map is given.

1. INTRODUCTION

In 1972, Ekeland [12] discovered a variational principle which states that if a real-valued function defined on a complete metric space is bounded below and lower semicontinuous, then a slight perturbation of this function has a strict minimum. It is known as Ekeland's variational principle (EVP) and is one of the most important results from nonlinear functional analysis. It has significant applications in optimization, control theory and several other areas of science, social science, management and engineering, see, for example, [3, 4, 9, 12] and the references therein. It is well known that the Ekeland's variational principle is equivalent to several important results in nonlinear analysis, see, for example, [1, 2, 3, 9, 13, 19, 26, 40] and the references therein. To weaken the lower semicontinuity assumption in the EVP, several different kinds

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of distance functions, namely, w -distance, τ -distance, τ -function, weak τ -distance, q -distance, etc, have been investigated and the EVP is obtained for these distances, see, for example, [1, 2, 4, 23, 25, 30, 35, 36] and the references therein. Recently, Włodarczyk and Plebaniak [39] introduced the concept of a \mathcal{J} -families of generalized pseudodistances in a uniform space which generalizes several distance functions, namely, distance of Tataru [38], w -distance of Kada et al. [23], τ -distance of Suzuki [36], τ -function of Lin et al. [30] and weak τ -function of Khanh et al. [25] in metric space setting. Very recently, Qiu et al. [35] introduced the notion of p -distance and more generally q -distance in a uniform space and a new type of completeness for uniform spaces. By using q -distance and the new type of completeness, they obtained a new version of EVP which includes several known versions of EVP. We note that the q -distance on a metric space generalizes pseudodistance but the converse is not true, see, Example 1.3 in [14].

During the last decade, the EVP has been extended and generalized for vector-valued functions and for set-valued maps, see, for example, [2, 6, 9, 13, 18, 25, 26, 31, 34] and the references therein. In 1998, Chen and Huang [7] introduced the concept of approximate solutions for set-valued maps and provided a sufficient condition for the existence of approximate solutions for set-valued maps. They also obtained a set-valued version of Ekeland's variational principle. Since then, several papers have appeared in the literature on this topic, see, for example, [7, 8, 11, 18, 20, 26, 31] and the references therein.

Motivated by the concept of a q -distance in a uniform space and a generalized pseudodistance in a metric space, we introduce the concept of weak q -distance in a uniform space and sequential completeness with respect to a weak q -distance. We establish an EVP for set-valued maps in a Hausdorff uniform space which is a sequential complete with respect to a weak q -distance. By using this EVP, we derive the existence of solutions to vector optimization problems with set-valued maps. We introduce the concepts of (p, ε) -condition of Takahashi and the (p, ε) -condition of Hamel. These concepts generalize the Takahashi's condition in [37] and Hamel's condition in [19], respectively. As a consequence of our EVP, we study the equivalence between (p, ε) -condition of Takahashi and (p, ε) -condition of Hamel. Furthermore, we discuss the relationship between an ε -approximate solution and a solution of a vector optimization problem with a set-valued map. Also, a well-posedness result for a vector optimization problem with a set-valued map is given.

2. EKELAND'S VARIATIONAL PRINCIPLE FOR SET-VALUED MAPS

Definition 2.1. A nonempty subset A of a uniform space X is said to be *sequentially closed* if for any sequence $\{x_n\}$ in A converges to x , we have $x \in A$.

Note that if A is closed, then A is sequentially closed but the converse need not be

true. Also, if some subsets of a uniform space are sequentially closed, then we cannot conclude that the space is metrizable.

Definition 2.2. Let X be a Hausdorff uniform space. A function $p : X \times X \rightarrow [0, \infty)$ is said to be a *weak q -distance* on X if the following conditions hold:

- (P1) $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$;
- (P2) If $\{x_n\}$ is a sequence in X and $\limsup_{n \rightarrow \infty} \{p(x_n, x_m) : m > n\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X ;
- (P3) If $\{x_n\}$ is a sequence in X and $p(x_n, x) \rightarrow 0$, then $x_n \rightarrow x$ in X .

Remark 2.1. If in addition to conditions (P1)–(P3), the following condition is also satisfied:

- (P4) for all $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$,

then a weak q -distance becomes q -distance defined in [35].

Example 2.1. [14, Example 1.3]. Let $p : [0, 2] \times [0, 2] \rightarrow [0, \infty)$ be a function defined by

$$p(x, y) = \begin{cases} 0, & \text{if } x - y = -2; \\ |x - y|, & \text{if } -2 < x - y \leq 0; \\ x - y + 2, & \text{if } 0 < x - y \leq 2. \end{cases}$$

Then, it can be easily seen that p is a weak q -distance but not q -distance.

Definition 2.3. Let X be a Hausdorff uniform space and p be a weak q -distance on X . Then, X is said to be *sequentially complete* with respect to p if for any sequence $\{x_n\}$ in X with $\limsup_{n \rightarrow \infty} \{p(x_n, x_m) : m > n\} = 0$, there exists $\bar{x} \in X$ such that $p(x_n, \bar{x}) \rightarrow 0$ as $n \rightarrow \infty$.

Several examples of sequentially complete spaces with respect to a q -distance can be found in [35]. Note that there are some examples of uniform spaces which are not sequentially complete, but they are sequentially complete with respect to a weak q -distance, see, for example, Example 3.5 in [35].

The following result is an extension of Theorem 1 in [27] from complete metric space to Hausdorff uniform space.

Theorem 2.1. *Let X be a Hausdorff uniform space and p be a weak q -distance on X . Assume that X is sequentially complete with respect to p , $T : X \rightrightarrows X$ is a set-valued map with nonempty sequentially closed values, and the following conditions hold.*

- (i) $T(T(x)) \subseteq T(x)$, for all $x \in X$;

- (ii) For all $x \in X$ and $\varepsilon > 0$, there exists $y \in T(x)$ such that $r_y(T(y)) < \varepsilon$, where $r_y(T(y)) = \sup\{p(y, v) : v \in T(y)\}$.

Then, there exists $\bar{x} \in X$ such that $T(\bar{x}) = \{\bar{x}\}$.

Proof. Let $x_0 \in X$ be arbitrary. By condition (ii), there exists $x_1 \in T(x_0)$ such that $r_{x_1}(T(x_1)) < 1$. Continuing this process, we can choose a sequence $\{x_n\}_{n \geq 0}$ in X such that $r_{x_n}(T(x_n)) < \frac{1}{n}$ and $x_n \in T(x_{n-1})$, for all $n \in \mathbb{N}$. On the other hand, from condition (i), we obtain

$$T(x_0) \supseteq T(x_1) \supseteq \cdots$$

Therefore, $p(x_n, x_m) < \frac{1}{n}$ for all $m > n > 0$, and so, $\limsup_{n \rightarrow \infty} \{p(x_n, x_m) : m > n\} = 0$. Since X is sequentially complete w. r. t. p , there exists $\bar{x} \in X$ such that $p(x_n, \bar{x}) \rightarrow 0$ as $n \rightarrow \infty$. Thus, $x_n \rightarrow \bar{x}$, and sequentially closedness of $T(x_n)$ implies $\bar{x} \in T(x_n)$ for all $n \in \mathbb{N}$. Also, if $y \in \bigcap_{n \in \mathbb{N}} T(x_n)$, then $p(x_n, y) < \frac{1}{n}$. Thus, $p(x_n, y) \rightarrow 0$, and so, $x_n \rightarrow y$. Since X is Hausdorff, $\bar{x} = y$. Therefore, $\bigcap_{n \in \mathbb{N}} T(x_n) = \{\bar{x}\}$. Furthermore, from condition (i), it follows that $T(\bar{x}) \subseteq \bigcap_{n \in \mathbb{N}} T(x_n) = \{\bar{x}\}$. Hence, $T(\bar{x}) = \{\bar{x}\}$. ■

Definition 2.4. Let Y be a locally convex space ordered by a closed convex cone C . A nonempty subset A of Y is said to be C -order-bounded from below if there exists $b \in Y$ such that $A \subseteq b + C$.

The *epigraph* of a set-valued map $F : X \rightrightarrows Y$ is defined as

$$\text{epi}F := \{(x, y) \in X \times Y : y \in F(x) + C\}.$$

The following result is a set-valued version of Ekeland's variational principle which extends Theorem 4.1 in [26] to uniform spaces.

Theorem 2.2. Let X be a Hausdorff uniform space and $p : X \times X \rightarrow [0, \infty)$ be a weak q -distance on X . Let Y be a locally convex space ordered by a closed convex cone C and D be a nonempty convex subset of C such that $0 \notin \text{cl}(C + D)$. Assume that $x_0 \in X$ and $F : X \rightrightarrows Y$ is a set-valued map such that the following conditions hold:

- (A1) The set $F(X)$ is C -order-bounded from below.
- (A2) The set $S = \{y \in X : F(x_0) \subseteq F(y) + p(x_0, y)D + C\}$ is nonempty and sequentially complete with respect to p .
- (A3) For any $x \in X$, the set $\{y \in X : F(x) \subseteq F(y) + p(x, y)D + C\}$ is sequentially closed.

Then, there exists $\bar{x} \in X$ such that

- (a) $F(x_0) \subseteq F(\bar{x}) + p(x_0, \bar{x})D + C$;
 (b) $F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C$, for all $x \neq \bar{x}$.

Proof. For all $x \in S$, define a set-valued map $T : S \rightrightarrows S$ by

$$T(x) := \{y \in S : F(x) \subseteq F(y) + p(x, y)D + C\}.$$

By condition (A3), $T(x)$ is sequentially closed for all $x \in S$. Assume that $T(x) \neq \emptyset$, for all $x \in S$. Then, T is a set-valued map with nonempty sequentially closed values. Now, we show that T satisfies conditions (i) and (ii) of Theorem 2.1.

Suppose that $x \in S$, $y \in T(x)$ and $z \in T(y)$. Then,

$$(2.1) \quad F(x) \subseteq F(y) + p(x, y)D + C \quad \text{and} \quad F(y) \subseteq F(z) + p(y, z)D + C.$$

By convexity of D , we have

$$\frac{p(x, y)}{p(x, y) + p(y, z)}D + \frac{p(y, z)}{p(x, y) + p(y, z)}D \subseteq D,$$

since $\frac{p(x, y)}{p(x, y) + p(y, z)} \geq 0$, $\frac{p(y, z)}{p(x, y) + p(y, z)} \geq 0$, and $\frac{p(x, y)}{p(x, y) + p(y, z)} + \frac{p(y, z)}{p(x, y) + p(y, z)} = 1$. Therefore,

$$(2.2) \quad p(x, y)D + p(y, z)D \subseteq (p(x, y) + p(y, z))D.$$

Clearly, for all $d \in D$,

$$(p(x, y) + p(y, z))d = p(x, z)d + (p(x, y) + p(y, z) - p(x, z))d.$$

But $p(x, z) \leq p(x, y) + p(y, z)$, $D \subseteq C$ and C is a cone, we have

$$(p(x, y) + p(y, z) - p(x, z))d \in C.$$

Therefore,

$$(2.3) \quad (p(x, y) + p(y, z))D \subseteq p(x, z)D + C.$$

From relations (2.2) and (2.3), we have

$$p(x, y)D + p(y, z)D \subseteq p(x, z)D + C.$$

Hence, by (2.1), we obtain

$$F(x) \subseteq F(z) + p(x, y)D + p(y, z)D + C \subseteq F(z) + p(x, z)D + C,$$

and thus, $T(T(x)) \subseteq T(x)$.

To prove condition (ii) of Theorem 2.1, assume to the contrary that there exist $\varepsilon > 0$ and $x \in S$ such that

$$(2.4) \quad r_y(T(y)) \geq \varepsilon, \quad \text{for all } y \in T(x).$$

Let $y_1 \in T(x)$ be arbitrary. Then, $r_{y_1}(T(y_1)) \geq \varepsilon$, and therefore, there exists $y_2 \in T(y_1)$ such that $p(y_1, y_2) \geq \frac{\varepsilon}{2}$. Since $y_2 \in T(y_1) \subseteq T(x)$, we have $r_{y_2}(T(y_2)) \geq \varepsilon$. Continuing in this way, we can choose a sequence $\{y_n\}$ in $T(x)$ such that for all $n \in \mathbb{N}$,

$$(2.5) \quad y_{n+1} \in T(y_n) \quad \text{and} \quad p(y_n, y_{n+1}) \geq \frac{\varepsilon}{2}.$$

Since $y_{n+1} \in T(y_n)$, we have

$$(2.6) \quad F(y_n) \subseteq F(y_{n+1}) + p(y_n, y_{n+1})D + C.$$

Hence,

$$\begin{aligned} F(y_1) &\subseteq F(y_2) + p(y_1, y_2)D + C \\ &\subseteq F(y_3) + p(y_2, y_3)D + C + p(y_1, y_2)D + C \\ &\quad \vdots \\ &\subseteq F(y_{n+1}) + p(y_n, y_{n+1})D + C + p(y_{n-1}, y_n)D + C + \cdots + p(y_1, y_2)D + C. \end{aligned}$$

Since C is a convex cone, we have $C + C \subseteq C$, and therefore, we deduce that

$$(2.7) \quad F(y_1) \subseteq F(y_{n+1}) + \sum_{i=1}^n p(y_i, y_{i+1})D + C.$$

Since $0 \notin \text{cl}((C + D))$, by Hahn-Banach separation theorem, there exists $y^* \in Y^*$ such that

$$\inf_{c \in C, d \in D} y^*(c + d) > 0.$$

Since $D \subseteq D + C$, we have

$$\inf_{d \in D} y^*(d) \geq \inf_{c \in C, d \in D} y^*(c + d) > 0.$$

Also, $y^*(c + d) > 0$, for all $d \in D$ and $c \in C$. Thus, $y^*(c) > -y^*(d)$, for all $d \in D$, $c \in C$. If $d \in D$ is arbitrary but fixed and $c \in C$, then $y^*(nc) > -y^*(d)$, for all $n \in \mathbb{N}$ because C is a cone. That is, $y^*(c) > \frac{-1}{n}y^*(d)$, for all $n \in \mathbb{N}$. If $n \rightarrow \infty$, then $y^*(c) \geq 0$, for all $c \in C$. Therefore,

$$(2.8) \quad \inf_{d \in D} y^*(d) > 0,$$

and

$$(2.9) \quad y^*(c) \geq 0, \quad \text{for all } c \in C.$$

Define $\varphi : 2^Y \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ by

$$\varphi(M) := \inf\{y^*(y) : y \in M\}, \quad \text{for all } M \in 2^Y \setminus \{\emptyset\}.$$

Then, from (2.7), we have

$$(2.10) \quad \begin{aligned} \varphi(F(y_1)) &\geq \inf \left\{ y^*(y) : y \in F(y_{n+1}) + \sum_{i=1}^n p(y_i, y_{i+1})D + C \right\} \\ &= \inf_{y \in F(y_{n+1})} y^*(y) + \inf_{v \in \sum_{i=1}^n p(y_i, y_{i+1})D} y^*(v) + \inf_{c \in C} y^*(c). \end{aligned}$$

Since $0 \in C$, by (2.9), we obtain

$$(2.11) \quad \inf_{c \in C} y^*(c) = 0.$$

If $v \in \sum_{i=1}^n p(y_i, y_{i+1})D$, then there exists $d_i \in D$ for $i = 1, 2, \dots, n$ such that $v \in \sum_{i=1}^n p(y_i, y_{i+1})d_i$. So,

$$(2.12) \quad \begin{aligned} y^*(v) &= y^* \left(\sum_{i=1}^n p(y_i, y_{i+1})d_i \right) \\ &= \sum_{i=1}^n p(y_i, y_{i+1})y^*(d_i) \\ &\geq \sum_{i=1}^n p(y_i, y_{i+1})\varphi(D), \end{aligned}$$

and therefore,

$$(2.13) \quad \inf_{v \in \sum_{i=1}^n p(y_i, y_{i+1})D} y^*(v) \geq \sum_{i=1}^n p(y_i, y_{i+1})\varphi(D).$$

Hence, by relations (2.10), (2.11) and (2.13), we have

$$(2.14) \quad \varphi(F(y_1)) \geq \varphi(F(y_{n+1})) + \sum_{i=1}^n p(y_i, y_{i+1})\varphi(D).$$

By condition (A1), there exists $z \in Y$ such that

$$F(y_{n+1}) \subseteq z + C, \quad \text{for all } n \in \mathbb{N}.$$

From (2.9), $\varphi(C) = 0$, and thus,

$$\varphi(F(y_{n+1})) \geq \varphi(z) + \varphi(C) = \varphi(z) = y^*(z).$$

Hence, by (2.14), we get

$$\sum_{i=1}^n p(y_i, y_{i+1})\varphi(D) \leq \varphi(F(y_1)) - \varphi(F(y_{n+1})) \leq \varphi(F(y_1)) - \varphi(z) < +\infty.$$

From (2.8), $\varphi(D) > 0$, and thus the sequence $\{\sum_{i=1}^n p(y_i, y_{i+1})\}_{n=1}^{\infty}$ is bounded. Also, $p(y_i, y_{i+1}) \geq 0$, then the series $\sum_{i=1}^{\infty} p(y_i, y_{i+1})$ is convergent. Hence,

$$p(y_n, y_{n+1}) \rightarrow 0.$$

This contradicts (2.5). Therefore, by Theorem 2.1, there exists $\bar{x} \in S$ such that $T(\bar{x}) = \{\bar{x}\}$. Hence,

$$(2.15) \quad F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C, \quad \text{for all } x \in S, x \neq \bar{x}.$$

If there exists $\bar{x} \in S$ such that $T(\bar{x}) = \emptyset$, then

$$F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C, \quad \text{for all } x \in S.$$

Thus, in any case, there exists $\bar{x} \in S$ such that

$$(2.16) \quad F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C, \quad \text{for all } x \in S, x \neq \bar{x}.$$

Now, we show that

$$(2.17) \quad F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C, \quad \text{for all } x \in X, x \neq \bar{x}.$$

Suppose contrary that (2.17) does not hold. Then, there exists $x \in X$ such that

$$x \neq \bar{x} \quad \text{and} \quad F(\bar{x}) \subseteq F(x) + p(\bar{x}, x)D + C.$$

By (2.16), $x \notin S$. Moreover,

$$F(x_0) \subseteq F(\bar{x}) + p(x_0, \bar{x})D \subseteq F(x) + p(x_0, \bar{x})D + p(\bar{x}, x)D + C.$$

Hence,

$$F(x_0) \subseteq F(x) + p(x_0, x)D + C,$$

and so, $x \in S$, a contradiction. ■

Remark 2.2. (a) The condition $0 \notin cl(C + D)$ in the above theorem was considered by Bednarczuk and Zagrodny [6].

(b) Liu and Ng [31, Theorem 3.5] obtained Theorem 2.2 in the case where (X, d) is a complete metric space and Y is a normed space under the following conditions:

- (i) $\text{epi}F = \{(x, y) \in X \times Y : y \in F(x) + C\}$ is closed.
- (ii) $\tilde{H}(\overline{\text{cone}}D, -C) := \inf\{d(b, -C) : b \in \overline{\text{cone}}D, \|b\| = 1\} > 0$,

where $d(y, A) = \inf\{d(y, a) : a \in A\}$ and $\overline{\text{cone}}(D)$ is a closed convex cone generated by the set D .

From the proof of Theorem 3.5 in [31], one can see that if $\tilde{H}(\overline{\text{cone}}D, -C) > 0$ and D is bounded, then $0 \notin \text{cl}(C + D)$.

The following example shows that the above condition (ii) is incomparable with the condition $0 \notin \text{cl}(C + D)$ in Theorem 2.2.

Example 2.2. Let $Y = \mathbb{R}^2$ be ordered by a closed convex cone $C = (-\infty, \infty) \times [0, \infty)$, and let $D = (-\infty, \infty) \times [1, \infty)$. Then, $\overline{\text{cone}}D = (-\infty, \infty) \times [0, \infty)$ and $\tilde{H}(\overline{\text{cone}}D, -C) = 0$. Hence, condition (ii) of Liu and Ng [31] does not hold. But $(0, 0) \notin \text{cl}(C + D)$.

Consider $Y = \mathbb{R}^2$ ordered by a closed convex cone $C = \mathbb{R}_+^2$, and $D = \text{int}\mathbb{R}_+^2$. Then, $\tilde{H}(\overline{\text{cone}}D, -C) = \tilde{H}(\mathbb{R}_+^2, -\mathbb{R}_+^2) > 0$. Therefore, condition (ii) of Liu and Ng [31] holds. Moreover, $(0, 0) \in \text{cl}(C + D)$.

It is a natural question that under what assumptions, condition (A3) in Theorem 2.2 holds? The following result gives an answer to this question.

Proposition 2.1. *Let X be a Hausdorff uniform space and $p : X \times X \rightarrow [0, \infty)$ be a weak q -distance on X . Let Y be a locally convex space ordered by a closed convex cone C , and D be a nonempty convex subset of C . Suppose that D is sequentially compact and p is lower semicontinuous in the second argument. If the epigraph of set-valued map $F : X \rightrightarrows Y$ is closed, then condition (A3) of Theorem 2.2 holds.*

Proof. For all $x \in X$, let

$$S(x) := \{y \in X : F(x) \subseteq F(y) + p(x, y)D + C\}.$$

Let $\{y_n\}$ be a sequence in $S(x)$ converging to \bar{y} . Then, we have

$$F(x) \subseteq F(y_n) + p(x, y_n)D + C \subseteq F(y_n) + C.$$

Since $\text{epi}F$ is closed, we have

$$F(x) \subseteq F(\bar{y}) + C.$$

If $p(x, \bar{y}) = 0$, then

$$F(x) \subseteq F(\bar{y}) + p(x, \bar{y})D + C.$$

Now, suppose that $p(x, \bar{y}) > 0$ and $m \in \mathbb{N}$ is an arbitrary point such that $0 < \frac{1}{m} < p(x, \bar{y})$. From the lower semicontinuity of p in the second argument, there exists $Q(m) \in \mathbb{N}$ such that

$$p(x, y_n) \geq p(x, \bar{y}) - \frac{1}{m}, \quad \text{for all } n > Q(m).$$

As $D \subseteq C$, we obtain

$$(2.18) \quad p(x, y_n)D \subseteq \left(p(x, \bar{y}) - \frac{1}{m} \right) D + C.$$

Since $y_n \in S(x)$, we have

$$F(x) \subseteq F(y_n) + p(x, y_n)D + C.$$

Thus,

$$F(x) \subseteq F(y_n) + \left(\left(p(x, \bar{y}) - \frac{1}{m} \right) D + C \right) + C,$$

and so,

$$F(x) \subseteq F(y_n) + \left(p(x, \bar{y}) - \frac{1}{m} \right) D + C.$$

Therefore, for any $y \in F(x)$, there exists $d_m \in D$ such that

$$y \in F(y_n) + \left(p(x, \bar{y}) - \frac{1}{m} \right) d_m + C.$$

Hence,

$$y - \left(p(x, \bar{y}) - \frac{1}{m} \right) d_m \in F(y_n) + C.$$

Since the above relation hold for sufficiently large n , the closedness of $\text{epi}F$ implies

$$y - \left(p(x, \bar{y}) - \frac{1}{m} \right) d_m \in F(\bar{y}) + C.$$

Since $0 < \frac{1}{m} < p(x, \bar{y})$ and D is sequentially compact, there exists a subsequence $\{d_{m_k}\}$ of $\{d_m\}$ that converges to $d \in D$. Since $\text{epi}F$ is closed, $(\bar{y}, y - p(x, \bar{y})d) \in \text{epi}F$, and therefore,

$$y \in F(\bar{y}) + p(x, \bar{y})d + C \subseteq F(\bar{y}) + p(x, \bar{y})D + C.$$

Hence,

$$F(x) \subseteq F(\bar{y}) + p(x, \bar{y})D + C,$$

that is, $\bar{y} \in S(x)$. Thus, condition (A3) holds. ■

A set-valued map $F : X \rightrightarrows Y$ is said to be C -lower semicontinuous on X if for any $y \in Y$, the set $\{x \in X : y \in F(x) + C\}$ is closed.

If $\text{epi}F$ is closed, then it is easy to see that F is C -lower semicontinuous. Conversely, if F is C -lower semicontinuous, then by Proposition 2.3 in [15] and Proposition 3.1 in [16], $\text{epi}F$ is closed provided that either $F(x)$ is weakly compact or $F(x) + C$ is closed. Notice that, in general, C -lower semicontinuity does not imply closedness of $\text{epi}F$, see Muselli [33].

Remark 2.3. Assume that the space X in Proposition 2.1 is sequentially complete with respect to weak q -distance p and all the conditions of Proposition 2.1 hold. If there exists $x_0 \in X$ such that the set $\{y \in X : F(x_0) \subseteq (F(y) + p(x_0, y)D) + C\}$ is nonempty, then condition (A2) of Theorem 2.2 holds.

The following example shows that a weak q -distance can be used to consider the set-valued version of EVP.

Example 2.3. Let p be the same as in Example 2.1, $Y = \mathbb{R}$, $C = \mathbb{R}^+$ and $D = [1, 2]$. Let $F : [0, 2] \rightrightarrows \mathbb{R}$ be defined by

$$F(x) = \begin{cases} [1, 2]e^x, & \text{if } x \neq 0, \\ \{1 + e\}, & \text{if } x = 0. \end{cases}$$

If $\bar{x} = \ln 3$, then

$$(2.19) \quad F(\bar{x}) \not\subseteq F(x) + p(x, \bar{x})D + C, \quad \text{for all } x \neq \bar{x}.$$

In order to show our claim, we consider the following cases.

Case 1. Let $0 < \bar{x} - x \leq 2$. If $x \in (0, 2]$, then

$$e^{\bar{x}} = 3 \notin [1, 2]e^x + [1, 2](\bar{x} - x + 2) + \mathbb{R}^+.$$

If $x = 0$, then

$$e^{\bar{x}} = 3 \notin \{e + 1\} + [1, 2](\bar{x} + 2) + \mathbb{R}^+.$$

Therefore, the relation (2.19) is satisfied.

Case 2. If $-2 < \bar{x} - x < 0$, then $e^{\bar{x}} < e^x$ and

$$e^{\bar{x}} \notin [1, 2]e^x + [1, 2]|\bar{x} - x| + \mathbb{R}^+.$$

Case 3. If $\bar{x} - x = -2$, then $e^{\bar{x}} < e^x$ and

$$e^{\bar{x}} \notin [1, 2]e^x + \mathbb{R}^+.$$

Hence, in any case, the relation (2.19) holds. However, there is no $\bar{x} \in [0, 2]$ such that

$$(2.20) \quad F(\bar{x}) \not\subseteq F(x) + |\bar{x} - x| D + C, \quad \text{for all } x \neq \bar{x}.$$

It is well-known that $e^x - x$ is nondecreasing on $[0, 2]$, If $\bar{x} \neq 0$, then for $x < \bar{x}$,

$$e^{\bar{x}} - \bar{x} \geq e^x - x.$$

Thus,

$$[1, 2](e^{\bar{x}} - \bar{x}) \subseteq [1, 2](e^x - x) + \mathbb{R}^+.$$

By convexity of $[1, 2]$, we have

$$[1, 2]e^{\bar{x}} - [1, 2]\bar{x} \subseteq [1, 2]e^x - [1, 2]x + \mathbb{R}^+,$$

that is,

$$[1, 2]e^{\bar{x}} \subseteq [1, 2]e^x + [1, 2]|x - \bar{x}| + \mathbb{R}^+.$$

If $\bar{x} = 0$, then

$$\{e + 1\} \subseteq [1, 2]e^1 + [1, 2]|1 - 0| + \mathbb{R}^+.$$

Hence, for any $\bar{x} \in [0, 2]$, there exists $x \neq \bar{x}$ such that

$$F(\bar{x}) \subseteq F(x) + |\bar{x} - x| D + C.$$

3. APPLICATIONS TO VECTOR OPTIMIZATION

In this section, we present some applications of Theorem 2.2 to vector optimization for set-valued maps. We derive the existence of solutions to vector optimization problems for set-valued maps. We introduce (p, ε) -Takahashi's condition and (p, ε) -Hamel's condition for set-valued maps. By using Theorem 2.2, we show that these conditions are equivalent. Moreover, we introduce the concept of an ε -approximate solution of a vector optimization problem with set-valued maps and then we discuss the relationship between ε -approximate solutions of a vector optimization problem with set-valued function and its solution set. Furthermore, a well-posedness result for a vector optimization problem for a set-valued map is also given.

Let X be a Hausdorff uniform space, p be a weak q -distance on X , Y be a locally convex space ordered by a pointed closed convex cone C , and D be a nonempty convex subset of C such that $0 \notin cl(C + D)$.

Let A be a nonempty subset of Y and $a \in A$. We say that a is an *efficient point* of A with respect to C if $(A - a) \cap (-C \setminus \{0\}) = \emptyset$. The set of all efficient points of A is denoted by $Eff(A)$.

Let $F : X \rightrightarrows Y$ be a set-valued map with nonempty values. The vector optimization problem for a set-valued map F is defined as

$$(VOP) \quad \min F(x), \quad \text{subject to } x \in X,$$

which means to find $\bar{x} \in X$ with the property that there exists $\bar{y} \in F(\bar{x})$ such that

$$(F(X) - \bar{y}) \cap (-C \setminus \{0\}) = \emptyset.$$

In fact, \bar{x} is a solution of (VOP) if $F(\bar{x})$ contains an efficient point of $F(X)$.

The set of all solutions of (VOP) is denoted by $E(F)$. For further details on vector optimization and set-valued vector optimization, we refer to [22, 29, 32].

The following result is a set-valued version of Takahashi's nonconvex minimization theorem.

Theorem 3.1. *Suppose that all the conditions of Theorem 2.2 are satisfied and for each $x \in X$ with $x \notin E(F)$, there exists $y \in X, y \neq x$ such that $F(x) \subseteq F(y) + p(x, y)D + C$. Then, $E(F) \neq \emptyset$.*

Proof. By Theorem 2.2, there exists $\bar{x} \in X$ such that

$$(3.21) \quad F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C, \quad \text{for all } x \neq \bar{x}.$$

We show that $\bar{x} \in E(F)$. Assume to the contrary that $\bar{x} \notin E(F)$. Then from hypothesis, there exists $y \in X, y \neq \bar{x}$ such that

$$F(\bar{x}) \subseteq F(y) + p(\bar{x}, y)D + C,$$

which contradicts (3.21). Hence, $\bar{x} \in E(F)$. ■

The following nonlinear scalarization function due to Gerstewitz [17], is modified in [34].

Definition 3.1. Let $e \in C \setminus \{0\}$. The *nonlinear scalarization function* $\xi_e : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as follows: if there exists $r \in \mathbb{R}$ such that $y \in re - C$, define $\xi_e(y) := \inf\{r \in \mathbb{R} : y \in re - C\}$; or else define $\xi_e(y) = +\infty$.

The following lemma describes some properties of ξ_e .

Lemma 3.1. [34]. *The nonlinear scalarization function $\xi_e : Y \rightarrow \mathbb{R} \cup \{+\infty\}$ is sublinear with the following properties:*

- (a) $\xi_e(y) \leq r \Leftrightarrow y \in re - C$.
- (b) $\xi_e(y) > r \Leftrightarrow y \notin re - C$.
- (c) $\xi_e(y + \alpha e) = \xi_e(y) + \alpha$, for any $y \in Y, \alpha \in \mathbb{R}$.

As an application of Theorem 2.2, we derive the following result which is crucial in obtaining the existence of solutions of (VOP).

Theorem 3.2. *Let $X_0 \subseteq X$ be nonempty sequentially complete with respect to p and p be lower semicontinuous in the second argument. Assume that $\text{int}C \neq \emptyset$, $G : X \rightrightarrows Y$ has a continuous selection g , $G(X) \cap (G(X_0) - C) \subseteq G(X_0)$, and there exists $y^* \in Y^*$ such that*

- (i) $\sup\{y^*(g(x)) : x \in X_0\} < \infty$,
- (ii) $p(x, y) \leq y^*(g(y) - g(x))$ for all $x \in X_0$ and $y \in X$ with $G(y) \cap (-C + G(x)) \neq \emptyset$.

Then, there exists $\bar{x} \in X$ such that $G(X) \cap (-C + G(\bar{x})) \subseteq G(\bar{x})$.

Proof. Let $m = \sup\{y^*(g(x)) : x \in X_0\}$ and $e \in \text{int}C$. Define $F : X_0 \rightrightarrows Y$ by

$$F(x) = \{[m - y^*(g(x))]e\}.$$

Since g is continuous, then the epigraph F is closed. If we set $D = \{e\}$, then all the assumptions of Proposition 2.1 are satisfied. Therefore, condition (A3) of Theorem 2.2, holds. Also it is easy to see that condition (A1) of Theorem 2.2 holds. Moreover, X_0 is sequentially complete w.r.t. p , thus by Theorem 2.2, there exists $\bar{x} \in X_0$ such that

$$(3.22) \quad F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)e + C, \quad \text{for all } x \neq \bar{x}.$$

Hence,

$$(3.23) \quad [m - y^*(g(\bar{x}))]e \notin [m - y^*(g(x))]e + p(\bar{x}, x)e + C, \quad \text{for all } x \neq \bar{x}.$$

By parts (b) and (c) of Lemma 3.1, we have

$$\xi_e(-[m - y^*(g(\bar{x}))]e) > -[m - y^*(g(x))] - p(\bar{x}, x).$$

Therefore,

$$(3.24) \quad p(\bar{x}, x) > y^*(g(x) - g(\bar{x})), \quad \text{for all } x \neq \bar{x}.$$

Now, let $v \in G(X) \cap (-C + G(\bar{x}))$. Since $G(X) \cap (-C + G(\bar{x})) \subseteq G(X_0)$, there exists $x_0 \in X_0$ such that $v \in G(x_0)$. By assumption (ii), we have

$$(3.25) \quad p(\bar{x}, x_0) \leq y^*(g(x_0) - g(\bar{x})).$$

Therefore, from relations (3.24) and (3.25), we have $x_0 = \bar{x}$. Thus, $v \in G(\bar{x})$, and so, $G(X) \cap (-C + G(\bar{x})) \subseteq G(\bar{x})$. ■

As a consequence of the above theorem, we obtain the following existence result for a solution of vector optimization problem for single-valued maps.

Corollary 3.1. *Let X_0, p be the same as in the above theorem and $\text{int}C \neq \emptyset$. Assume that $g : X \rightarrow Y$ is a continuous vector-valued function, $g(X) \cap (g(X_0) - C) \subseteq g(X_0)$, and there exists $y^* \in Y^*$ such that*

$$(h1) \sup\{y^*(g(x)) : x \in X_0\} < \infty,$$

$$(h2) p(x, y) \leq y^*(g(y) - g(x)) \text{ for all } x \in X_0 \text{ and } y \in X \text{ with } g(y) \in -C + g(x).$$

Then, there exists $\bar{x} \in X$ such that $(g(X) - g(\bar{x})) \cap -C = \{g(\bar{x})\}$, that is, $E(g) \neq \emptyset$.

Proof. It follows from Theorem 3.2 by considering $G(x) = \{g(x)\}$ for all $x \in X$. ■

Definition 3.2. We say that $x_0 \in X$ is an ε -approximate solution with respect to D of (VOP) if there exists $y_0 \in F(x_0)$ such that

$$(F(x) - y_0 + \varepsilon D) \cap (-C \setminus \{0\}) = \emptyset, \quad \text{for all } x \in X.$$

Remark 3.1. Let x_0 be an ε -approximate solution with respect to D . Assume that the conditions (A1)-(A3) of Theorem 2.2 are satisfied. Then, there exists $\bar{x} \in X$ such that the conclusions (a) and (b) of Theorem 2.2 hold and

$$p(x_0, \bar{x}) < \varepsilon.$$

Now we generalize the p -condition of Takahashi and Hamel [30] and the (ψ, ε) -condition of Takahashi and Hamel [34].

Definition 3.3. Let $\varepsilon > 0$. A set-valued map $F : X \rightrightarrows Y$ is said to satisfy

(C1) (p, ε) -condition of Takahashi if for all ε -approximate solution $x_0 \in X$ with respect to D of (VOP) which is not a solution of (VOP), there exists $\hat{x} \in X$, $\hat{x} \neq x_0$ such that

$$F(x_0) \subseteq F(\hat{x}) + p(x_0, \hat{x})D + C.$$

(C2) (p, ε) -condition of Hamel if for all ε -approximate solution $x_0 \in X$ with respect to D of (VOP) which is not a solution of (VOP), there exists $\bar{x} \in E(F)$, $\bar{x} \neq x_0$ such that

$$F(x_0) \subseteq F(\bar{x}) + p(x_0, \bar{x})D + C.$$

It is clear that if F satisfies the (p, ε) -condition of Hamel, then F satisfies the (p, ε) -condition of Takahashi.

As an application of Theorem 2.2, the following result shows that the (p, ε) -condition of Takahashi implies the (p, ε) -condition of Hamel. This result extends Theorem 4.1 in [34] to set-valued maps.

Theorem 3.3. *Let F satisfy the assumptions (A1) and (A3) of Theorem 2.2 and (p, ε) -condition of Takahashi. If assumption (A2) holds for any ε -approximate solution x_0 , then F satisfies the (p, ε) -condition of Hamel.*

Proof. Let $x_0 \in X$ be an ε -approximate solution with respect to D of (VOP). Assume that x_0 is not a solution of (VOP). Then, there exists $y_0 \in F(x_0)$ such that

$$(F(x) - y_0 + \varepsilon D) \cap (-C \setminus \{0\}) = \emptyset, \quad \text{for all } x \in X.$$

By Theorem 2.2, there exists $\bar{x} \in X$ such that

- (a) $F(x_0) \subseteq F(\bar{x}) + p(x_0, \bar{x})D + C$,
- (b) $F(\bar{x}) \not\subseteq F(x) + p(\bar{x}, x)D + C$, for all $x \neq \bar{x}$.

By (a), $F(x_0) \subseteq F(\bar{x}) + p(x_0, \bar{x})D + C \subseteq F(\bar{x}) + C$, hence there exists $\bar{y} \in F(\bar{x})$, $c^* \in C$ such that $y_0 = \bar{y} + c^*$. We show that

$$(3.26) \quad (F(x) - \bar{y} + \varepsilon D) \cap (-C \setminus \{0\}) = \emptyset, \quad \text{for all } x \in X.$$

Assume to the contrary that there exists $x \in X$ such that $(F(x) - \bar{y} + \varepsilon D) \cap (-C \setminus \{0\}) \neq \emptyset$. Then, $(F(x) - \bar{y} + \varepsilon D - c^*) \cap (-C \setminus \{0\}) \neq \emptyset$. Therefore,

$$(F(x) - y_0 + \varepsilon D) \cap (-C \setminus \{0\}) \neq \emptyset,$$

a contradiction.

Now, assume that \bar{x} is not an efficient solution of (VOP). Since F satisfies (p, ε) -condition of Takahashi, by (3.26), there exists $\hat{x} \in X$, $\hat{x} \neq \bar{x}$ such that

$$F(\bar{x}) \subseteq F(\hat{x}) + p(\bar{x}, \hat{x})D + C$$

a contradiction of (b). ■

For a nonempty subset A of X and $x \in X$, we set

$$p(x, A) := \inf\{p(x, a) : a \in A\}.$$

In the sequel, we assume that $E(F)$ and $Eff(F(X))$ are nonempty. In the next result, we give a relation between ε -approximate solutions and the set of solutions of problem (VOP) under (p, ε) -condition of Takahashi. This theorem extends Theorem 4.2 in [34] to set-valued maps.

Theorem 3.4. *Let F satisfy all the conditions of Theorem 3.3 and $\{x_n\}$ be a sequence in X such that*

$$\limsup_{n \rightarrow \infty} \{p(x_n, x_m) : m > n\} = 0.$$

Suppose that $0 < \varepsilon_n \leq \varepsilon$ for all $n \in \mathbb{N}$ and $\varepsilon_n \rightarrow 0$. Further, assume that x_n is an ε_n -approximate solution with respect to D of (VOP) which is not a solution of (VOP). Then, there exists a sequence $\{\bar{x}_n\}$ in X such that \bar{x}_n is a solution of (VOP), $p(x_n, \bar{x}_n) < \varepsilon_n$ and $p(x_n, E(F)) \leq p(x_n, \bar{x}_n) \leq \varepsilon_n$ for all $n \in \mathbb{N}$.

Proof. Since x_n is an ε_n -approximate solution with respect to D of (VOP) and $0 < \varepsilon_n \leq \varepsilon$, then there exists $y_n \in F(x_n)$ such that

$$(3.27) \quad (F(x) - y_n + \varepsilon_n D) \cap (-C \setminus \{0\}) = \emptyset, \quad \text{for all } x \in X.$$

Since F satisfies the (p, ε) -condition of Takahashi, by Theorem 3.3, for all $n \in \mathbb{N}$, there exists a solution $\bar{x}_n \in X$ of (VOP) such that

$$F(x_n) \subseteq F(\bar{x}_n) + p(x_n, \bar{x}_n)D + C.$$

Therefore, there exist $\bar{y}_n \in F(\bar{x}_n)$ and $d_n \in D$ such that

$$y_n \in \bar{y}_n + p(x_n, \bar{x}_n)d_n + C.$$

By Lemma 3.1, we have

$$(3.28) \quad p(x_n, \bar{x}_n) \leq -\xi_{d_n}(\bar{y}_n - y_n).$$

But by (3.27), we get

$$(3.29) \quad \bar{y}_n - y_n + \varepsilon_n d_n \notin (-C \setminus \{0\}).$$

Hence,

$$(3.30) \quad \xi_{d_n}(\bar{y}_n - y_n) + \varepsilon_n > 0.$$

From (3.28) and (3.30), we get

$$p(x_n, \bar{x}_n) < \varepsilon_n.$$

Since \bar{x}_n is a solution of (VOP), we have

$$p(x_n, E(F)) \leq p(x_n, \bar{x}_n) \leq \varepsilon_n. \quad \blacksquare$$

The theory of numerical techniques to obtain the approximate solutions of optimization problems is one of the most important subject within optimization. One of the subjects which is important in the studies of convergence of the numerical methods is the well-posedness. It is closely related to the study of the stability of an optimization problem. Many authors studied the well-posedness of scalar optimization and vector optimization problems; see, for example, [5, 10, 21, 24, 28] and references therein. Motivated by the notion of well-posedness for vector optimization [24], we extend this concept to set-valued optimization.

Definition 3.4. [24]. Let Y be a Banach space. The problem (VOP) is said to be *well-posed* if and only if for any sequence $\{x_n\} \subseteq X$, $\mathcal{H}(F(x_n), \text{Eff}(F(X))) \rightarrow 0$ implies $p(x_n, E(F)) \rightarrow 0$, where \mathcal{H} is the Hausdorff metric.

The following result provides the well-posedness of (VOP).

Theorem 3.5. *Let $(Y, \|\cdot\|)$ be a normed space, $\text{int}C \neq \emptyset$, $r = \inf_{t \in D} d(t, \partial C) > 0$ and $p(x, x) = 0$ for all $x \in X$. Suppose that all the conditions of Theorem 3.3 hold and $\varepsilon > 0$. If $\mathcal{H}(F(x), E(F(X))) < r\varepsilon$, then $p(x, E(F)) < \varepsilon$ and the problem (VOP) is well-posed.*

Proof. Assume that $x \in X$. If $F(x) \subseteq \text{Eff}(F(X))$, then $x \in E(F)$, and so, $p(x, E(F)) = 0$. Assume that $F(x) \not\subseteq \text{Eff}(F(X))$. Then, there exists $y \in F(x)$ such that

$$(F(X) - y) \cap (-C \setminus \{0\}) \neq \emptyset.$$

If $\mathcal{H}(F(x), \text{Eff}(F(X))) < r\varepsilon$, then $d(y, \text{Eff}(F(X))) < r\varepsilon$. Hence, there exists $\bar{y} \in \text{Eff}(F(X))$ such that $\|y - \bar{y}\| < r\varepsilon$. Therefore, $y \in \bar{y} + r\varepsilon B_Y$.

We claim that

$$(3.31) \quad (F(X) - y + \varepsilon D) \cap (-C) = \emptyset.$$

Assume to the contrary that there exist $x' \in X$, $y' \in F(x')$ and $t \in D$ such that

$$y' - y + \varepsilon t \in -C.$$

Since $r = \inf_{t \in D} d(t, \partial C) > 0$, we have $(t - rB_Y) \cap \partial C = \emptyset$, for all $t \in D$.

On the other hand $t \in D \subseteq C$, $0 \in \partial C$, therefore

$$(3.32) \quad t - rB_Y \subseteq \text{int}C \subseteq C \setminus \{0\}.$$

Hence,

$$\begin{aligned} y' &\in y - \varepsilon t - C \\ &\subseteq \bar{y} - \varepsilon(t - rB_Y) - C \\ &\subseteq \bar{y} - \varepsilon C \setminus \{0\} - C \\ &\subseteq \bar{y} - C \setminus \{0\}, \end{aligned}$$

which contradicts with $\bar{y} \in \text{Eff}(F(X))$. So, (3.31) holds. By Theorem 3.3, we conclude that there exists $\tilde{x} \in E(F)$ such that

$$F(x) \subseteq F(\tilde{x}) + p(x, \tilde{x})D + C,$$

that is, there exist $\tilde{y} \in F(\tilde{x})$ and $\tilde{t} \in D$ such that

$$\tilde{y} - y + p(x, \tilde{x})\tilde{t} \in -C.$$

Therefore, by part (a) of Lemma 3.1 we have

$$\xi_{\tilde{t}}(\tilde{y} - y) + p(x, \tilde{x}) \leq 0.$$

From (3.31), we obtain

$$\tilde{y} - y + \varepsilon \tilde{t} \notin -C \setminus \{0\}.$$

Thus, by part (b) of Lemma 3.1 we have

$$\xi_{\tilde{t}}(\tilde{y} - y) + \varepsilon \geq 0,$$

which leads to

$$p(x, \tilde{x}) \leq -\xi_{\tilde{t}}(\tilde{y} - y) \leq \varepsilon.$$

Therefore,

$$p(x, E(F)) \leq \varepsilon.$$

If $\{x_n\} \subseteq X$ and $\mathcal{H}(F(x_n), \text{Eff}(F(X))) \rightarrow 0$, then

$$\text{for all } \varepsilon > 0, \quad \exists n_0 \in \mathbb{N}: \quad n > n_0 \quad \mathcal{H}(F(x_n), \text{Eff}(F(X))) < r\varepsilon.$$

Hence,

$$p(x_n, E(F)) < \varepsilon.$$

Thus,

$$p(x_n, E(F)) \rightarrow 0. \quad \blacksquare$$

Remark 3.2. In the above theorem the assumption $r = \inf_{t \in D} d(t, \partial C) > 0$ implies that $D \subseteq \text{int}C$. Furthermore, if D is compact and $D \subseteq \text{int}C$, then $\inf_{t \in D} d(t, \partial C) > 0$.

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