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ONE PARAMETER FAMILY OF UNIVALENT BIHARMONIC MAPPINGS

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Abstract. In this paper, we provide sufficient conditions to construct sense-preserving and univalent biharmonic mappings that arises from analytic functions which are not necessarily univalent in the unit disk $|z| < 1$. Also we state and prove several theorems under different weaker hypothesis in each case, leading to an affirmative answer to the radius problem posed by Y. Abu Muhanna in 2008.

1. INTRODUCTION

A complex-valued function $F = u + iv$ which is four times continuously differentiable in a simply connected domain $D \subseteq \mathbb{C}$ is biharmonic if $\triangle F$, the Laplacian of F, is harmonic in D (see [5, 8]). Note that $\triangle F$ is harmonic in D, if F satisfies the biharmonic equation $\triangle(\triangle F) = 0$, where

$$\triangle = 4\frac{\partial^2}{\partial z \partial \overline{z}}.$$

Biharmonic mappings are widely used in engineering fields and applied mathematics (cf. [11, 13]). Throughout we consider harmonic and biharmonic functions defined on the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Every biharmonic function F on $\mathbb{D}$ has the form

$$F(z) = |z|^2 G(z) + H(z) \tag{1}$$

where G and H are harmonic in $\mathbb{D}$, see [1, 2, 3]. A biharmonic function F on $\mathbb{D}$ is said to be sense-preserving if the Jacobian $J_F(z)$ of F is positive in the punctured disk $\mathbb{D} \setminus \{0\}$, where

$$J_F(z) = |F_z(z)|^2 - |F_{\overline{z}}(z)|^2.$$

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A continuously differentiable function $f\colon \mathbb{D} \to \mathbb{C}$ which fixes the origin is called starlike in $\mathbb{D}$ if it is univalent in $\mathbb{D}$ and the range $f(\mathbb{D})$ is a starlike (with respect to the origin) domain.

In the case of analytic functions, $f(|z| = r)$ is a starlike curve for each $r \in (0,1)$ if f is starlike in $\mathbb{D}$. This property is not true in general in the case of harmonic starlike mappings (see [4, 12]).

Definition 1. A continuously differentiable function F on $\mathbb{D}$ is said to be *fully starlike* (see [16]) in $\mathbb{D}$ if it is sense-preserving, $F(0) = 0$, $F(z) \neq 0$ in $\mathbb{D}\backslash\{0\}$ and the curve $F(re^{it})$ is starlike (with respect to the origin) for each $r \in (0,1)$. The last condition is same as saying that

$$\frac{\partial}{\partial t}\big(\arg F(re^{it})\big) = \mathrm{Re}\left(\frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)}\right) > 0$$

for $z = re^{it} \in \mathbb{D}\backslash\{0\}$ (see also [4, 12] in order to distinguish the starlikeness property of harmonic mappings from conformal mappings).

Let $\mathcal{A}$ denote the class of normalized functions G of the form

$$G(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

which are analytic in $\mathbb{D}$. Set $\mathcal{S} = \{f \in \mathcal{A} :\ f \text{ is univalent in } \mathbb{D}\}$ (see [7, 10]). It has been proved by the first author that, if $G \in \mathcal{S}$, then $F = |z|^2 G$ is not necessarily univalent in $\mathbb{D}$ (see [2, Example 2.2] and [3, Example 1]). In fact, F is not even sense-preserving in $\mathbb{D}$ if $G \in \mathcal{S}$. On the other hand, it is easy to see that, $F = |z|^2 G$ is univalent in $\mathbb{D}$ whenever G is harmonic and starlike in $\mathbb{D}$, see [1] and [15, Theorem 1.3]. Therefore, it is natural to determine the radius of univalence of $|z|^2 G$ when $G \in \mathcal{S}$. This problem has been solved by Muhanna [3, Theorem 2].

For a given analytic function G consider the one parameter family of biharmonic functions $W_\alpha(z)$ defined by

$$W_\alpha(z) = |z|^2 G(z) + \left[-G(z) + \alpha\int_0^z \frac{G(\zeta)}{\zeta}\, d\zeta\right]. \tag{3}$$

In [3] Muhanna proved two theorems concerning the sense-preserving property and univalency of biharmonic functions $W_2(z)$, i.e. $\alpha = 2$ in (3). Indeed for $G \in \mathcal{S}$, the sharp inequality $(1-|z|^2)\left|\frac{zG'(z)}{G(z)}\right| \leq 4$ holds for $z \in \mathbb{D}$. The number 4 in this inequality cannot be replaced by a smaller number. As pointed out in [17] a closer examination of the proof of [3, Theorem 3 and Proposition 3] shows that these results are valid for $W_4(z)$ rather than for $W_2(z)$. Correct formulation of these results follows.

Theorem A. *If $G \in \mathcal{S}$, then $W_4(z)$ defined by* (3) *has Jacobian $J_{W_4}(z) > 0$ except at $z = 0$. Moreover, if G is starlike, then the biharmonic function $W_4(z)$ is univalent in $\mathbb{D}$.*

It is natural to ask whether Theorem A continues to hold when G is not necessarily univalent in $\mathbb{D}$. In Section 2 we identify a family $\mathcal{F}$ which contains also non-univalent functions $G \in \mathcal{F}$ such that $W_4(z)$ is sense-preserving and univalent in $\mathbb{D}$. In fact we prove something more than this. However, in the same paper, Muhanna [3, Problem 1] posed the following question.

Problem 1. *What is the radius of univalence of $W(= W_4(z))$ if $G \in \mathcal{S}$?*

In [17] Ponnusamy and Qiao discussed this problem and obtained for example the following results.

Theorem B. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq k$ for $k \geq 2$. Then the biharmonic function $W = W_4(z)$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_1\}$, where $r_1 \approx 0.34195$ is the root of the equation*

$$6r^5 - 10r^4 - 5r^3 + 15r^2 - 13r + 3 = 0$$

in the interval $(0, 1)$. Moreover, W is fully starlike for $|z| < r_1$ and also by all its sections

$$K_n(z) = |z|^2 G_n(z) + \left[-G_n(z) + 4 \int_0^z \frac{G_n(\zeta)}{\zeta}\, d\zeta \right], \tag{4}$$

where $G_n(z) = z + \sum_{k=2}^{n} a_k z^k$ denotes the section of $G(z)$.

We remark that if $G \in \mathcal{S}$ then, by de Branges theorem, $|a_k| \leq k$ for all $k \geq 2$ and therefore, the radius univalence r_S of $W_4(z)$ is bigger than or equal to $r_1 \approx 0.34195$. This answers Problem 1 affirmatively.

It is interesting to solve Problem 1 for many other geometric subclasses of univalent functions that are not necessarily starlike in $\mathbb{D}$. This is done in Section 3. These theorems may be obtained using some well-known sufficient conditions for univalence, namely, that G is close-to-convex in $\mathbb{D}$. Here $G \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{D}$, denoted by $G \in \mathcal{K}$, if there exists a function ϕ which is analytic, univalent and starlike in $\mathbb{D}$ for which

$$\mathrm{Re}\left(\frac{zG'(z)}{\phi(z)} \right) > 0, \quad z \in \mathbb{D}.$$

Interesting particular considerations are the classes corresponding to the functions

$$\phi_1(z) = z,\ \phi_2(z) = \frac{z}{1-z},\ \phi_3(z) = \frac{z}{1-z^2},\ \phi_4(z) = \frac{z}{(1-z)^2} \text{ and } \phi_5(z) = \frac{z}{1-z+z^2}.$$

Corresponding to the above choices, the family $\mathcal{K}$ of close-to-convex functions is denoted respectively by $\mathcal{K}_j$ $(j = 1, 2, 3, 4, 5)$. If G is given by (2), then one has the following. For example, the following implications are well-known (see [7, 10]):

(i) $G \in \mathcal{K}_1$, i.e. $\mathrm{Re}\,(G'(z)) > 0$ in $\mathbb{D}$, implies that $|a_n| \leq 2/n$ for $n \geq 2$

(ii) $G \in \mathcal{K}_4$, i.e. $\mathrm{Re}\,((1-z)^2 G'(z)) > 0$ in $\mathbb{D}$, implies that $|a_n| \leq 2 - 1/n$.

We recall that each function in $\mathcal{K}_4$ maps the unit disk $\mathbb{D}$ onto a domain that is convex in the direction of real axis. It is easy to see that there are non-starlike and non-univalent analytic functions G satisfying each of these necessary coefficient conditions stated above.

In this paper, we show that Theorem A continues to hold without G being univalent in $\mathbb{D}$. This is done in Section 2, by identifying the family $\mathcal{F}$ containing non-univalent functions for which Theorem A holds with $G \in \mathcal{F}$. In Section 3, we state and prove a number of theorems which give affirmative answers to Problem 1 under different geometric conditions.

2. Main Theorems

Now, we introduce the following notations. For $\lambda > 0$, we consider the family

$$\mathcal{R}(\lambda) = \{G \in \mathcal{A}\colon\, |G'(z) - 1| < \lambda,\ z \in \mathbb{D}\}.$$

Set $\mathcal{R}(1) = \mathcal{R}$. Functions in $\mathcal{R}$ are known to be univalent in $\mathbb{D}$, but functions in $\mathcal{R}(\lambda)$ are not necessarily univalent in $\mathbb{D}$ if $\lambda > 1$. Also, it is important to remark that functions in the class $\mathcal{R}$ are not necessarily starlike in $\mathbb{D}$. However, functions in $\mathcal{R}(\lambda)$ are known to be starlike in $\mathbb{D}$ provided $0 < \lambda \leq 2/\sqrt{5}$ (see [14, 18]) and $2/\sqrt{5}$ cannot be replaced by a larger one (see [9]). On the converse part, even a normalized convex function f is not necessarily satisfying the condition $\mathrm{Re}\,(f'(z)) > 0$ in $\mathbb{D}$ and hence need not belong to $\mathcal{R}$. Since functions in $\mathcal{R}(\lambda)$ are not necessarily univalent in $\mathbb{D}$ if $\lambda > 1$, the final case of the following theorem (see Table 1) clearly improves Theorem A in the case of $\alpha = 4$. Moreover, we prove a general result.

Theorem 1. *Let $G \in \mathcal{R}(\lambda)$ for some $2/\sqrt{5} < \lambda \leq 2$ and consider the biharmonic function W_α defined by* (3) *for $\alpha \in \mathbb{C}$.*

(i) If $G \in \mathcal{R}(1)$ and $\mathrm{Re}\,\alpha > 3/2$, then $J_{W_\alpha}(z) > 0$ in $\mathbb{D} \setminus \{0\}$.

(ii) If $G \in \mathcal{R}(2)$ and $\mathrm{Re}\,\alpha > 6$, then $J_{W_\alpha}(z) > 0$ in $\mathbb{D} \setminus \{0\}$.

(iii) For other values of λ, the range of $\mathrm{Re}\,\alpha$ for which $J_{W_\alpha}(z) > 0$ holds in $\mathbb{D} \setminus \{0\}$ is given in Table 1.

Proof. Let $W(z) = W_\alpha(z)$, where $W_\alpha(z)$ is defined by (3). We have, $W_{\overline{z}}(z) = zG(z)$ and

$$W_z(z) = \overline{z}G(z) + r^2G'(z) - G'(z) + \alpha\frac{G(z)}{z}.$$

If a is the dilatation of W then for $z \neq 0$,

$$\begin{aligned} a(z) = \frac{W_{\overline{z}}(z)}{W_z(z)} &= \frac{z}{\overline{z}}\left[\frac{1}{1+\frac{zG'(z)}{G(z)} - \frac{G'(z)}{\overline{z}G(z)} + \frac{\alpha}{|z|^2}}\right] \\ &= \frac{z}{\overline{z}}\left[\frac{1}{1+\frac{1}{|z|^2}\left[\alpha - (1-|z|^2)\frac{zG'(z)}{G(z)}\right]}\right]. \end{aligned}$$

In order to prove that $|a(z)| < 1$ in $\mathbb{D}\setminus\{0\}$, it suffices to show that

$$\operatorname{Re}\left(\alpha - (1-|z|^2)\frac{zG'(z)}{G(z)}\right) > 0 \quad \text{for } z \in \mathbb{D}.$$

Now we let $G \in \mathcal{R}(\lambda)$ for some $2/\sqrt{5} < \lambda \leq 2$. Then, there exists a function $\omega(z)$ analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$ such that

(5) $$G'(z) = 1 + \lambda\omega(z).$$

Thus, by integration we find that

$$G(z) = z + \lambda z\int_0^1 \omega(tz)\,dt,$$

that is,

(6) $$\frac{G(z)}{z} = 1 + \lambda\int_0^1 \omega(tz)\,dt.$$

By the Schwarz lemma, we obtain $|\omega(z)| \leq |z|$ in $\mathbb{D}$. Using this, we obtain that

$$\left|\frac{G(z)}{z} - 1\right| = \lambda\left|\int_0^1 \omega(tz)\,dt\right| \leq \lambda\int_0^1 t|z|\,dt = \lambda\frac{|z|}{2} < \frac{\lambda}{2},$$

showing that $|G(z)/z - 1| < 1$ whenever $0 < \lambda \leq 2$. In particular, $\operatorname{Re}(G(z)/z) > 0$ in $\mathbb{D}$ for $0 < \lambda \leq 2$ and hence, $G(z)/z \neq 0$ in $\mathbb{D}$ in this case. Also,

$$\frac{zG'(z)}{G(z)} = \frac{1+\lambda\omega(z)}{1+\lambda\int_0^1 \omega(tz)\,dt}$$

and, since $|\omega(z)| \leq |z|$, we see that

(7) $$(1-|z|^2)\left|\frac{zG'(z)}{G(z)}\right| \leq (1-|z|^2)\frac{1+\lambda|z|}{1-\frac{\lambda}{2}|z|} = f_\lambda(|z|),$$

where

$$f_\lambda(x) = \frac{2(1+\lambda x)}{2-\lambda x}(1-x^2), \quad x \in (0,1).$$

We need to determine the extrema of $f_\lambda(x)$ on $(0,1)$. Computationally, it looks like an uneasy situation for all the values of $\lambda \in (2/\sqrt{5}, 2]$ other than $\lambda = 1$ and $\lambda = 2$.

Let $\lambda = 1$. Then we have

$$f_1(x) = \frac{2(1+x)(1-x^2)}{(2-x)} = \frac{2(1+x-x^2-x^3)}{(2-x)}.$$

In order to determine $\max\limits_{0<x<1} f_1(x)$, a calculation shows that

$$f_1'(x) = \frac{2(x+1)(2x-1)(x-3)}{(2-x)^2}$$

from which we find that

$$\max_{x\in(0,1)} f_1(x) = f_1\left(\frac{1}{2}\right) = \frac{3}{2}.$$

Similarly, for $\lambda = 2$, we find that

$$f_2(x) = (1+2x)(1+x)$$

and therefore, $\max\limits_{x\in(0,1)} f_2(x) = f_2(1) = 6.$

Thus, by (7) in the case of $\lambda = 1$,

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{zG'(z)}{G(z)}\right| \le \frac{3}{2}$$

and for $\lambda = 2$, we see that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{zG'(z)}{G(z)}\right| \le 6.$$

These observations show that

$$\mathrm{Re}\left[\alpha - (1-|z|^2)\frac{zG'(z)}{G(z)}\right] \ge \begin{cases} \mathrm{Re}\,\alpha - \frac{3}{2} & \text{if } \lambda = 1, \\ \mathrm{Re}\,\alpha - 6 & \text{if } \lambda = 2. \end{cases}$$

Finally, we see that the dilatation $a(z)$ satisfies the condition $|a(z)| < 1$ in $0 < |z| < 1$ if either $\lambda = 1$ and $\mathrm{Re}\,\alpha > \frac{3}{2}$, or $\lambda = 2$ and $\mathrm{Re}\,\alpha > 6$. The proof of the theorem is complete. ■

Remark 1. In Theorem 1, we restrict λ such that $2/\sqrt{5} < \lambda \le 2$. But the result can be stated for $0 < \lambda \le 2/\sqrt{5}$, but in this case functions in $\mathcal{R}(\lambda)$ are starlike. However, one can prepare a table of values covering the case $0 < \lambda \le 2/\sqrt{5}$.

Table 1: Range for $\operatorname{Re}\alpha$ for certain values of $\lambda > 1$

S.No	Value of λ	Local maximum of $f_\lambda(x)$ occurs at	Lower bound for $\operatorname{Re}\alpha$
1	1.25	0.57224	1.79592
2	1.4	0.615521	2.03182
3	1.5	0.645909	2.22562
4	1.6	0.678804	2.46166
5	1.75	0.737174	2.94552

Theorem 2. *Suppose that G is an analytic function satisfying $|G'(z) - 1| < 1$, for $z \in \mathbb{D}$. Then the biharmonic function $W = W_\alpha$ defined by* (3) *is univalent in $\mathbb{D}$ for $\alpha > 3/2$.*

Proof. Fix $0 < \rho < 1$. Consider the function

$$V(\varphi) = \rho^2 G(\rho\varphi) + \left[-G(\rho\varphi) + \alpha \int_0^{\rho\varphi} \frac{G(\zeta)}{\zeta}\, d\zeta \right],$$

where $|\varphi| \leq 1$. Then a computation gives

$$V^{'}(\varphi) = \rho X(\rho\varphi)$$

where

$$X(\rho\varphi) = -(1-\rho^2)G'(\rho\varphi) + \alpha \frac{G(\rho\varphi)}{\rho\varphi}.$$

By assumption, $|G'(z) - 1| < 1$ in $\mathbb{D}$. Using the representations (5) and (6), $X(\rho\varphi)$ can be written in terms of the Schwarz function as

$$X(\rho\varphi) = \alpha - (1-\rho^2) - (1-\rho^2)\omega(\rho\varphi) + \alpha \int_0^1 \omega(t\rho\varphi)\, dt,$$

where ω is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{D}$. By the Schwarz lemma, $|\omega(z)| \leq |z|$ in $\mathbb{D}$ and using this, we have

$$\begin{aligned} \operatorname{Re}(X(\rho\varphi)) &= \operatorname{Re}\left(\alpha - (1-\rho^2) - (1-\rho^2)\omega(\rho\varphi) + \alpha \int_0^1 \omega(t\rho\varphi)\, dt \right) \\ &\geq \alpha - (1-\rho^2) - (1-\rho^2)\rho|\varphi| - \alpha\frac{\rho|\varphi|}{2} \\ &\geq \alpha - (1-\rho^2) - \left[(1-\rho^2)\rho + \frac{\alpha\rho}{2} \right]. \end{aligned}$$

It follows that $\operatorname{Re}(X(\rho\varphi)) > 0$ if and only if $\alpha > K(\rho)$, where

$$K(\rho) = \frac{2(1+\rho)^2(1-\rho)}{(2-\rho)}.$$

We see that $K(\rho)$ attains its maximum at $\rho = 1/2$ and the maximum value is $K(1/2) = 3/2$. Hence if $\alpha > 3/2$, then $\mathrm{Re}\,(X(\rho\varphi)) > 0$.

Thus, it follows that $V(\varphi)$ is close-to-convex and in particular, univalent in $\mathbb{D}$. This implies, as $V(\varphi) = W(\rho\varphi)$ for $|\varphi| = 1$, $W(z)$ is univalent on $|z| = \rho$.

Since, the Jacobian, $J_W(z) > 0$ except at $z = 0$, by Theorem 1 and the degree principle [6] implies that $W(z)$ is univalent on $|z| \leq \rho$.

As ρ is arbitrary, it follows that $W(z)$ is univalent in $\mathbb{D}$. ■

Theorem 3. *If $G \in \mathcal{R}(2)$, then the biharmonic function $W = W_\alpha$ defined by* (3) *is univalent in $\mathbb{D}$ for $\alpha > 6$.*

Proof. Follows as in Theorem 2, if we use Theorem 1(ii). So we omit its details. ■

3. Affirmative Solutions to Problem 1

A version of the following lemma is proved in [17]. Here we present a slightly different proof and we make its proof a self-contained one.

Lemma 1. *Let $F(z) = |z|^2G(z) + H(z)$ be a biharmonic mapping in $\mathbb{D}$, where $H(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $G(z) = \sum_{k=1}^{\infty} b_k z^k$ are analytic in $\mathbb{D}$, and satisfy the condition*

$$\sum_{k=1}^{\infty}(k+2)|b_k|r^{k+1} + \sum_{k=2}^{\infty} k|a_k|r^{k-1} \leq 1 \quad (0 < r \leq 1). \tag{8}$$

Then F is sense-preserving, univalent and fully starlike in $\mathbb{D}_r := \{z : |z| < r\}$.

Proof. Consider $F(z) = |z|^2 \sum_{k=1}^{\infty} b_k z^k + z + \sum_{k=2}^{\infty} a_k z^k$. Then it is easily seen that

$$F_z(z) = |z|^2 \sum_{k=1}^{\infty}(k+1)b_k z^{k-1} + 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \quad \text{and} \quad F_{\overline{z}}(z) = \sum_{k=1}^{\infty} b_k z^{k+1}$$

and therefore, $J_F(0) = |F_z(0)|^2 - |F_{\overline{z}}(0)|^2 = 1$. Next, we fix $r_0 \in (0, 1]$ and assume that (8) is satisfied for $r = r_0$. For $z \neq 0$,

$$J_F(z) = \Big(|F_z(z)| + |F_{\overline{z}}(z)|\Big)\Big(|F_z(z)| - |F_{\overline{z}}(z)|\Big) > 0$$

because for $0 < |z| < r_0$, the triangle inequality and (8) give

$$\begin{aligned} |F_z(z)| - |F_{\overline{z}}(z)| &= \left| |z|^2 \sum_{k=1}^{\infty}(k+1)b_k z^{k-1} + 1 + \sum_{k=2}^{\infty} k a_k z^{k-1} \right| - \left| \sum_{k=1}^{\infty} b_k z^{k+1} \right| \\ &> 1 - \sum_{k=1}^{\infty}(k+2)|b_k|r_0^{k+1} - \sum_{k=2}^{\infty} k|a_k|r_0^{k-1} \geq 0. \end{aligned}$$

Thus, F is sense-preserving in $\mathbb{D}_{r_0}$. Finally, fix an $r_0 \in (0, r]$ and consider the circle $C_{r_0} = \{z : |z| = r_0\}$. For $z \in C_{r_0}$, we have

$$\begin{aligned} zF_z(z) - \overline{z}F_{\overline{z}}(z) - F(z) &= |z|^2 \sum_{k=1}^{\infty}(k+1)b_k z^k + \sum_{k=2}^{\infty}(k-1)a_k z^k. \\ |zF_z(z) - \overline{z}F_{\overline{z}}(z) - F(z)| &\leq \sum_{k=1}^{\infty}(k+2-1)|b_k|\,|z|^{k+2} + \sum_{k=2}^{\infty}(k-1)|a_k|\,|z|^k \\ &\leq |z| \left(\sum_{k=1}^{\infty}(k+2)|b_k|\,|z|^{k+1} + \sum_{k=2}^{\infty} k|a_k|\,|z|^{k-1} \right) \\ &\quad - \sum_{k=2}^{\infty}|a_k|\,|z|^k - |z|^2 \sum_{k=1}^{\infty}|b_k|\,|z|^k \\ &\leq |z| - \sum_{k=2}^{\infty}|a_k|\,|z|^k - |z|^2 \sum_{k=1}^{\infty}|b_k|\,|z|^k \\ &\leq |H(z)| - |z|^2|G(z)| \\ &\leq |F(z)| \end{aligned}$$

which shows that for $|z| = r_0$

$$\left| \frac{zF_z(z) - \overline{z}F_{\overline{z}}(z)}{F(z)} - 1 \right| < 1$$

and hence, F is univalent on C_{r_0} and it maps C_{r_0} onto a starlike curve. By the sense-preserving property and the degree principle, it follows from [12, Lemma 2.1] that F is univalent and fully starlike in $\mathbb{D}_r$, since $r_0 \in (0, r]$ is arbitrary. The proof of the theorem is complete. ■

Theorem 4. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq (k+1)/2$ for $k \geq 2$. Then the biharmonic function $W = W_4$ defined by* (3) *has positive Jacobian in $\mathbb{D}_{r_S} \setminus \{0\}$ except at $z = 0$ and univalent in $\mathbb{D}_{r_S}$. Here $r_S \approx 0.38853$ is the root of the equation $\phi(r) = 0$, where*

$$\phi(r) = 8r^5 - 14r^4 - 6r^3 + 26r^2 - 24r + 6 \tag{9}$$

in the interval $(0, 1)$. Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

Proof. Consider the function $W(z)$ defined by

$$W(z) = |z|^2 G(z) + \left[-G(z) + 4 \int_0^z \frac{G(\zeta)}{\zeta}\, d\zeta \right].$$

Let $W_r(z) = \frac{1}{r}W(rz)$ for $0 < r < 1$, and $G(z) = \sum_{k=1}^{\infty} a_k z^k$ (with $a_1 = 1$) such that $|a_k| \leq (k+1)/2$ for all $k \geq 2$. Then a computation gives that

$$W_r(z) \,=\, |z|^2 \sum_{k=1}^{\infty} r^{k+1} a_k z^k + \sum_{k=1}^{\infty} \left(\frac{4}{k} - 1\right) r^{k-1} a_k z^k.$$

In order to apply Lemma 1, we consider

$$\frac{W_r(z)}{3} = |z|^2 \sum_{k=1}^{\infty} B_k z^k + z + \sum_{k=2}^{\infty} A_k z^k$$

where

$$B_k = \frac{1}{3} r^{k+1} a_k \quad \text{and} \quad A_k = \frac{4-k}{3k} r^{k-1} a_k.$$

Note that $W(z)$ is univalent and fully starlike in $|z| < r$ if and only if $W_r(z)$ is univalent and fully starlike in the unit disk $|z| < 1$. Thus, by Lemma 1, it suffices to show that

$$S(r) = \sum_{k=1}^{\infty} (k+2)|B_k| + \sum_{k=2}^{\infty} k|A_k| \leq 1$$

for $0 < r \leq r_S$. Now, since $|a_k| \leq (k+1)/2$ by assumption, it follows that $S(r) \leq \frac{1}{6}T(r)$, where

$$T(r) = \sum_{k=1}^{\infty} (k+1)(k+2) r^{k+1} + \sum_{k=2}^{\infty} (k+1)|k-4| r^{k-1}.$$

Clearly, $S(r) \leq 1$ holds if $T(r) \leq 6$. In order to prove the later inequality, we use the following equalities

$$\frac{r}{(1-r)^2} = \sum_{k=1}^{\infty} k r^k \quad \text{and} \quad \frac{r(1+r)}{(1-r)^3} = \sum_{k=1}^{\infty} k^2 r^k.$$

Now, by adjusting the second sum in $T(r)$ conveniently, it follows that

$$\begin{aligned}
T(r) \,&=\, \sum_{k=1}^{\infty} (k^2 + 3k + 2) r^{k+1} + 6r + 4r^2 + \sum_{k=4}^{\infty} (k+1)(k-4) r^{k-1} \\
&= \sum_{k=1}^{\infty} k^2 r^{k+1} + 3\sum_{k=1}^{\infty} k r^{k+1} + 2\sum_{k=1}^{\infty} r^{k+1} + \sum_{k=1}^{\infty} k^2 r^{k-1} - 3\sum_{k=1}^{\infty} k r^{k-1} \\
&\quad -4\sum_{k=1}^{\infty} r^{k-1} + 6 + 4r^2 + 6 + 6r + 4r^2 \\
&= \frac{(r^2+1)(1+r)}{(1-r)^3} + \frac{3(r^2-1)}{(1-r)^2} + \frac{2(r^2-2)}{1-r} + 6 + 12r + 8r^2.
\end{aligned}$$

A simple calculation gives that $T(r) \leq 6$ is equivalent to the inequality $\phi(r) \geq 0$, where $\phi(r)$ is given by (9). The inequality $\phi(r) \geq 0$ holds if $0 < r \leq r_S$, where $r_S \approx 0.38853$ is the root of the equation $\phi(r) = 0$ in the interval $(0,1)$. Thus, $S(r) \leq 1$ for $0 < r \leq r_S$ and so, by Lemma 1, $W_r(z)$ is univalent sense-preserving and fully starlike in $\mathbb{D}$ with $0 < r \leq r_S$. The proof of the theorem is complete. ∎

Remark 2. For $\lambda \in \mathbb{C}$, we consider $f_\lambda(z) = z + \lambda z^2$. Then we see that $f_\lambda(z)$ is not univalent in $\mathbb{D}$ if $|\lambda| > 1/2$. Thus, there are non-univalent functions satisfying the coefficient condition $|a_k| \leq (k+1)/2$.

Let $\mathcal{S}_1$ denote the set of all analytic functions $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in $\mathbb{D}$ satisfying the inequality

$$\operatorname{Re}\left(1 + z\frac{h''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in \mathbb{D}.$$

It is well-known that $\mathcal{S}_1 \subset \mathcal{S}$, but functions in $\mathcal{S}_1$ are not necessarily starlike in $\mathbb{D}$. In fact if we let

$$h_0(z) = \frac{1}{2}\left[\frac{z}{(1-z)^2} + \frac{z}{1-z}\right] = \frac{z - z^2/2}{(1-z)^2}$$

then

$$\operatorname{Re}\left(\frac{zh_0'(z)}{z/(1-z)^2}\right) = \operatorname{Re}\left(\frac{1}{1-z}\right) > 0$$

showing that h_0 is close-to-convex in $\mathbb{D}$. The fact that h_0 is not starlike in $\mathbb{D}$ can be easily seen by looking at $h_0(\mathbb{D})$. A calculation reveals that the boundary of $h_0(\mathbb{D})$ is the parabola $u + 2v^2 + 3/8 = 0$, see Figure 1. Moreover, if $h \in \mathcal{S}_1$ then, the Taylor coefficients of h are known to satisfy the inequality $|a_k| \leq (k+1)/2$ for each $k \geq 2$. Clearly, equality holds for the function $h_0(z)$.

Corollary 1. *Let $G \in \mathcal{S}_1$. Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ in $\mathbb{D}_{r_S} \setminus \{0\}$. Moreover, W (and each of its polynomial section) is univalent and fully starlike in $\{z : |z| < r_S\}$, where $r_S \approx 0.38853$.*

At this juncture, we would like to remark that analogue of Theorem 4 can be proved even for other values of α including $\alpha \in \mathbb{C}$. So it may be appropriate to state the theorem for W_α and outline a proof for the sake of completeness, for the case $\alpha \in (1,4]$.

Theorem 5. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq (k+1)/2$ for $k \geq 2$. Then the biharmonic function $W = W_\alpha$ where $\alpha \in (1,4]$ defined by* (3) *has Jacobian $J_W(z) > 0$ in $\mathbb{D}_{r_S} \setminus \{0\}$ and univalent in $\{z : |z| < r_S\}$. Here r_S is the smallest root of the equation $\psi(r) = 0$ on the interval $(0,1)$, where*

$$\psi(r) = Ar^5 + Br^4 + Cr^3 + Dr^2 + Er + F \tag{10}$$

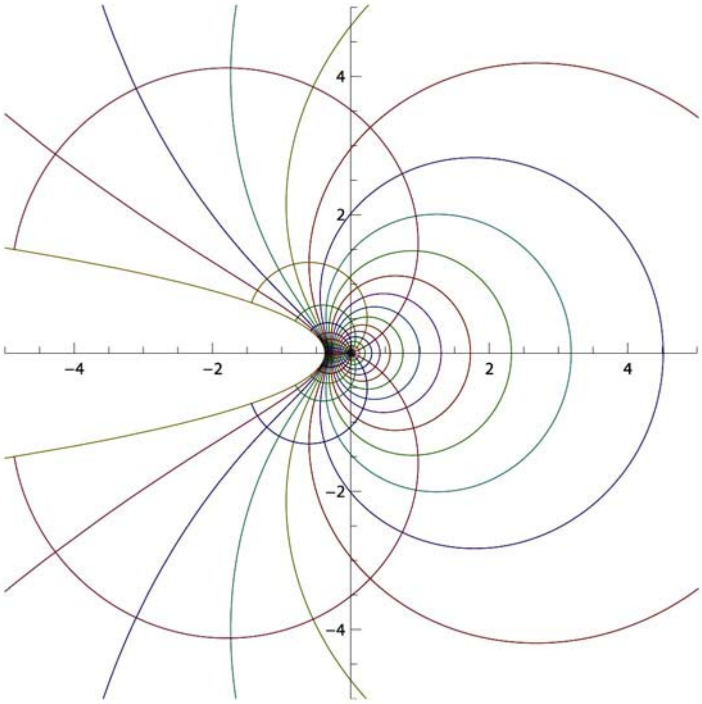

Fig. 1. The graph of the function $h_0(z) = \frac{z-z^2/2}{(1-z)^2}$.

with

$$\begin{aligned}
A &= 4|\alpha-3| + 4\alpha - 12,\\
B &= 3|\alpha-2| - 12|\alpha-3| - 9\alpha + 28,\\
C &= -2|\alpha-1| - 9|\alpha-2| + 12|\alpha-3| + 5\alpha - 14,\\
D &= 6|\alpha-1| + 9|\alpha-2| - 4|\alpha-3| - 6,\\
E &= -6|\alpha-1| - 3|\alpha-2| \ \ \textit{and} \ \ F = 2|\alpha-1|.
\end{aligned}$$

Moreover, W *is fully starlike for* $|z| < r_S$ *and also by all its sections.*

Proof. Proceeding exactly as in the proof of Theorem 4, we notice that it suffices to show that $L(r) \le 2|\alpha-1|$, where

$$\begin{aligned}
L(r) &= \sum_{k=1}^{\infty}(k+1)(k+2)r^{k+1} + \sum_{k=2}^{\infty}(k+1)|\alpha-k|r^{k-1}\\
&= \sum_{k=1}^{\infty}(k+1)(k+2)r^{k+1} + |\alpha-2|3r + |\alpha-3|4r^2 + \sum_{k=4}^{\infty}(k+1)(k-\alpha)r^{k-1}.
\end{aligned}$$

After a long computation, it follows that the inequality $L(r) \leq 2|\alpha - 1|$ is equivalent to $\psi(r) \geq 0$, where $\psi(r)$ is given by (10). We see that the inequality $\psi(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\psi(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. ■

We remark that if $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is analytic in $\mathbb{D}$ satisfying the condition $\mathrm{Re}\,(f(z)/z) > 1/2$ in $\mathbb{D}$, then $|a_k| \leq 1$ for $k \geq 2$ and moreover, f is not necessarily univalent in $\mathbb{D}$.

Theorem 6. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq c$ for $k \geq 2$, and for some $c > 0$. Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ in $\mathbb{D}_{r_S} \setminus \{0\}$ and univalent in $\{z : |z| < r_S\}$. Here r_S is the root of the equation $\psi_1(r) = 0$ on the interval $(0, 1)$ where*

$$\psi_1(r) = -2cr^4 + 2cr^3 + 3r^2 - (6 + 2c)r + 3. \tag{11}$$

Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

Proof. The proof goes in same lines with that of the proof of Theorem 4. For the sake of completeness, we indicate main steps. Thus, following the proof of Theorem 4, it is enough to show that $T_1(r) \leq 3/c$, where

$$\begin{aligned} T_1(r) &= \sum_{k=1}^{\infty} (k+2) r^{k+1} + \sum_{k=2}^{\infty} |k-4| r^{k-1} \\ &= \frac{-2r^3 + 3r^2 - 3 + 4r}{(1-r)^2} + 2r^2 + 4r + 3. \end{aligned}$$

A calculation shows that the inequality, $T_1(r) \leq 3/c$ is equivalent to $\psi_1(r) \geq 0$, where $\psi_1(r)$ is given by (11). We see that the inequality $\psi_1(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\psi_1(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. ■

Corollary 2. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ and consider the biharmonic function $W = W_4$ defined by* (3)*. Then*

(*i*) *If $|a_k| \leq 1$ for $k \geq 2$, then W has positive Jacobian in $\mathbb{D}_{r_1} \setminus \{0\}$ and univalent in $\mathbb{D}_{r_1}$, where $r_1 \approx 0.472727$ is the root of the equation*

$$2r^4 - 2r^3 - 3r^2 + 8r - 3 = 0$$

in the interval $(0, 1)$.

(*ii*) *If $|a_k| \leq 1$ for $k \geq 3$ and $a_2 = 0$, then W has positive Jacobian in $\mathbb{D}_{r_2} \setminus \{0\}$ and univalent in $\mathbb{D}_{r_2}$, where $r_2 \approx 0.553164$ is the root of the equation*

$$4r^5 - 10r^4 + 8r^3 - r^2 - 6r + 3 = 0$$

in the interval $(0, 1)$.

(*iii*) *If* $|a_k| \le 2$ *for* $k \ge 2$*, then* W *has positive Jacobian in* $\mathbb{D}_{r_3} \setminus \{0\}$ *and univalent in* $\mathbb{D}_{r_3}$*, where* $r_3 \approx 0.347051$ *is the root of the equation*

$$4r^4 - 4r^3 - 3r^2 + 10r - 3 = 0$$

in the interval $(0,1)$.

In all the three cases, W *is fully starlike in the corresponding disk and also by all its sections defined by* (4)*.*

Proof. Cases (i) and (iii) follow from Theorem 6. We omit the proof of Case (ii) as it follows after some easy computation. ■

As in the case of Theorem 5, Theorem 6 can be proved for even other values of α including complex values. For instance, we have

Theorem 7. *Let* $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ *be analytic in* $\mathbb{D}$ *such that* $|a_k| \le c$ *for* $k \ge 2$*. Then the biharmonic function* $W = W_\alpha$ *where* $\alpha \in (1,4]$ *defined by* (3) *has Jacobian* $J_W(z) > 0$ *except at* $z = 0$ *and univalent in* $\{z : |z| < r_S\}$*. Here* r_S *is the smallest root of the equation* $\psi_2(r) = 0$ *on the interval* $(0,1)$*, where*

$$\psi_2(r) = Pr^4 + Qr^3 + Rr^2 + Sr + T \tag{12}$$

with

$$\begin{aligned} P &= -c(\alpha + |\alpha - 3| - 3), \\ Q &= -c(-\alpha + |\alpha - 2| - 2|\alpha - 3| + 2), \\ R &= |\alpha - 1| - c(3 - 2|\alpha - 2| + |\alpha - 3|), \\ S &= -2|\alpha - 1| - c|\alpha - 2| \;\; \textit{and} \;\; T = |\alpha - 1|. \end{aligned}$$

Moreover, W *is fully starlike for* $|z| < r_S$ *and also by all its sections.*

Proof. According to a closer examination of the proof of Theorem 6, it suffices to show that $M(r) \le |\alpha - 1|/c$, where

$$\begin{aligned} M(r) &= \sum_{k=1}^{\infty} (k+2) r^{k+1} + \sum_{k=2}^{\infty} |\alpha - k| r^{k-1} \\ &= \sum_{k=1}^{\infty} (k+2) r^{k+1} + |\alpha - 2| r + |\alpha - 3| r^2 + \sum_{k=4}^{\infty} (k - \alpha) r^{k-1}. \end{aligned}$$

A long computation gives that the inequality $M(r) \le |\alpha - 1|/c$ is equivalent to $\psi_2(r) \ge 0$ where $\psi_2(r)$ is given by (12). It follows that the inequality $\psi_2(r) \ge 0$ holds if $0 < r \le r_S$, where r_S is the root of the equation $\psi_2(r) = 0$ in the interval $(0,1)$. The proof of the theorem is complete. ■

Theorem 8. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq c/k$ for $k \geq 2$. Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$. Here r_S is the root of the equation $\phi_1(r) = 0$ on the interval $(0, 1)$, where*

$$\phi_1(r) = \frac{3}{c} - \frac{2r^3 + r^2 + 3r - 12}{3(r-1)} + \frac{(2r^2 - 4)}{r} \log(1 - r). \tag{13}$$

Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

Proof. As with the proof of Theorem 6, it suffices to indicate only the main steps. By Lemma 1, it is enough to show that $T_2(r) \leq 3/c$, where

$$\begin{aligned} T_2(r) &= \sum_{k=1}^{\infty} \frac{(k+2)}{k} r^{k+1} + \sum_{k=2}^{\infty} \frac{|k-4|}{k} r^{k-1} \\ &= \frac{r^2}{1-r} + \frac{1}{1-r} + \frac{4}{r} \log(1-r) - 2r \log(1-r) + \frac{2r^2}{3} + 2r + 3. \end{aligned}$$

A calculation shows that the inequality, $T_2(r) \leq 3/c$ is equivalent to $\phi_1(r) \geq 0$, where $\phi_1(r)$ is given by (13). We see that the inequality $\phi_1(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\phi_1(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. ∎

Taking $c = 1$ in Theorem 8, we have the following.

Corollary 3. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq 1/k$ for $k \geq 2$. Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$, where $r_S \approx 0.612948$ is the root of the equation*

$$3 - \frac{2r^3 + r^2 + 3r - 12}{3(r-1)} + \frac{(2r^2 - 4)}{r} \log(1 - r) = 0$$

in the interval $(0, 1)$. Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

Taking $c = 2$ in Theorem 8 gives the following

Corollary 4. *Let $G \in \mathcal{K}_1$, i.e. $\mathrm{Re}\,(G'(z)) > 0$ in $\mathbb{D}$. Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$, where $r_S \approx 0.454048$ is the root of the equation*

$$\frac{3}{2} - \frac{2r^3 + r^2 + 3r - 12}{3(r-1)} + \frac{(2r^2 - 4)}{r} \log(1 - r) = 0$$

in the interval $(0, 1)$. Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

For $\alpha \in (1,4]$, we have the following generalization of Theorem 8.

Theorem 9. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq c/k$ for $k \geq 2$. Then the biharmonic function $W = W_\alpha$ where $\alpha \in (1,4]$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$, where r_S is the smallest root of the equation $\phi_2(r) = 0$ on the interval $(0,1)$ and*

$$\phi_2(r) = \frac{|\alpha - 1|}{c} + \frac{A_1 r^3 + B_1 r^2 + C_1 r + D_1}{6(r-1)} - \frac{(\alpha - 2r^2)}{r}\log(1-r) \tag{14}$$

with

$$\begin{aligned} A_1 &= 6 - 2|\alpha - 3| - 2\alpha, \\ B_1 &= 6 - 3|\alpha - 2| + 2|\alpha - 3| - \alpha, \\ C_1 &= 3|\alpha - 2| - 3|\alpha| \ \ \textit{and} \ \ D_1 = 6\alpha. \end{aligned}$$

Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections.

Proof. It is enough to show that $M_1(r) \leq |\alpha - 1|/c$, where

$$\begin{aligned} M_1(r) &= \sum_{k=1}^{\infty} \frac{(k+2)}{k} r^{k+1} + \sum_{k=2}^{\infty} \frac{|\alpha - k|}{k} r^{k-1} \\ &= \sum_{k=1}^{\infty} r^{k+1} + 2\sum_{k=1}^{\infty} \frac{r^{k+1}}{k} + \frac{|\alpha - 2|}{2} r + \frac{|\alpha - 3|}{3} r^2 + \sum_{k=4}^{\infty} \frac{(k-\alpha)}{k} r^{k-1}. \end{aligned}$$

A long-winded computation gives that the inequality $M_1(r) \leq |\alpha - 1|/c$ is equivalent to $\phi_2(r) \geq 0$, where $\phi_2(r)$ is as defined in (14). It follows that the inequality $\phi_2(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\phi_2(r) = 0$ in the interval $(0,1)$. The proof of the theorem is complete. ■

Theorem 10. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq 2 - 1/k$ for $k \geq 2$* (*which holds, for example if $G \in \mathcal{K}_4$*). *Then the biharmonic function $W = W_4$ defined by* (3) *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$, where $r_S \approx 0.407285$ is the root of the equation $\phi_3(r) = 0$ in the interval $(0,1)$ and*

$$\phi_3(r) = \frac{-10r^4 + 11r^3 + 11r^2 - 45r + 21}{3(r-1)^2} + \frac{4 - 2r^2}{r}\log(1-r). \tag{15}$$

Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections defined by (4).

Proof. We include the important steps. By the method of proof of Lemma 1, it is enough to show that $S_1(r) \leq 1$, where

$$\begin{aligned} S_1(r) &= \sum_{k=1}^{\infty} \frac{(k+2)(2k-1)}{3k} r^{k+1} + \sum_{k=2}^{\infty} \frac{|4-k|(2k-1)}{3k} r^{k-1} \\ &= \frac{2r^2}{3(1-r)^2} + \frac{2}{3(1-r)^2} + \frac{r^2}{(1-r)} - \frac{3}{(1-r)} + \frac{2r}{3}\log(1-r) \\ &\quad -\frac{4}{3r}\log(1-r) + \frac{10r^2}{9} + 2r + 1. \end{aligned}$$

A calculation shows that the inequality $S_1(r) \leq 1$ is equivalent to $\phi_3(r) \geq 0$, where $\phi_3(r)$ is given by (15). The inequality $\phi_3(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\phi_3(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. ∎

Again a generalization of Theorem 10 may be now stated.

Theorem 11. *Let $G(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be analytic in $\mathbb{D}$ such that $|a_k| \leq 2 - 1/k$ for $k \geq 2$. Then the biharmonic function $W = W_\alpha$, where $\alpha \in (1, 4]$ defined by* (3), *has Jacobian $J_W(z) > 0$ except at $z = 0$ and univalent in $\{z : |z| < r_S\}$. Here r_S is the smallest root of the equation $\phi_4(r) = 0$ in the interval $(0, 1)$ and*

$$\phi_4(r) = |\alpha - 1| - \frac{P_1 r^4 + Q_1 r^3 + R_1 r^2 + S_1 r + T_1}{6(1-r)^2} + \frac{\alpha - 2r^2}{r}\log(1-r) \tag{16}$$

with

$$\begin{aligned} P_1 &= 10|\alpha - 3| + 10\alpha - 30, \\ Q_1 &= 9|\alpha - 2| - 20|\alpha - 3| - 11\alpha + 24, \\ R_1 &= -18|\alpha - 2| + 10|\alpha - 3| - 2\alpha + 30, \\ S_1 &= 9|\alpha - 2| + 9\alpha \quad \textit{and} \quad T_1 = -6\alpha. \end{aligned}$$

Moreover, W is fully starlike for $|z| < r_S$ and also by all its sections.

Proof. It suffices to prove that $S_2(r) \leq |\alpha - 1|$, where

$$\begin{aligned} S_2(r) &= \sum_{k=1}^{\infty} \frac{(k+2)(2k-1)}{k} r^{k+1} + \sum_{k=2}^{\infty} \frac{|\alpha-k|(2k-1)}{k} r^{k-1} \\ &= \frac{2r^2}{(1-r)^2} + \frac{3r^2}{(1-r)} + 2r\log(1-r) + |\alpha - 2|\frac{3}{2}r \\ &\quad + |\alpha - 3|\frac{5}{3}r^2 + \sum_{k=4}^{\infty}\left(\frac{\alpha}{k} + 2k - 2\alpha - 1\right) r^{k-1}. \end{aligned}$$

A computation reveals that the inequality $S_2(r) \leq |\alpha - 1|$ is equivalent to $\phi_4(r) \geq 0$, where $\phi_4(r)$ is given by (16). The inequality $\phi_4(r) \geq 0$ holds if $0 < r \leq r_S$, where r_S is the root of the equation $\phi_4(r) = 0$ in the interval $(0, 1)$. The proof of the theorem is complete. ■

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