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GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A NONLINEAR TIMOSHENKO BEAM SYSTEM WITH A DELAY TERM

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Abstract. We consider the Timoshenko system in bounded domain with a delay term in the nonlinear internal feedback

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) \\ \qquad\qquad +\mu_1 g_1(\psi_t(x,t)) + \mu_2 g_2(\psi_t(x,t-\tau)) = 0, \end{cases}$$

and prove the global existence of its solutions in Sobolev spaces by means of the energy method combined with the Faedo-Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we establish a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

1. INTRODUCTION

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear Timoshenko system of the type

$$(P)\qquad \begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) & \\ \qquad +\mu_1 g_1(\psi_t(x,t)) + \mu_2 g_2(\psi_t(x,t-\tau)) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ \psi_t(x,t-\tau) = f_0(x,t-\tau) & \text{in }]0,1[\times]0,\tau[, \end{cases}$$

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where $\tau > 0$ is a time delay, μ_1 and μ_2 are positive real numbers, and the initial data (ψ_0, ψ_1, f_0) belong to a suitable function space.

A simple model describing the transverse vibration of a beam, which was developed in [25], is given by a system of coupled hyperbolic equations of the form

$$\begin{cases} \rho u_{tt}(x,t) = (K(u_x - \phi))_x & \text{in }]0, L[\times]0, +\infty[, \\ \tilde{\rho}\phi_{tt}(x,t) = (EI\psi_x)_x + K(u_x - \phi) & \text{in }]0, L[\times]0, +\infty[, \end{cases}$$

where t denotes the time variable, x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and ϕ is the rotation angle of the filament of the beam. The coefficients $\rho, \tilde{\rho}, E, I$ and K are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus.

In the absence of delay ($\mu_2 = 0$), the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see [2, 11, 16, 17, 18] and [21]). In addition, we would like to mention the most recent work in this direction due to Cavalcanti et al. [4] which is the pioneer in establishing very general explicit decay rate estimates for solutions to a wave equation with boundary damping-source.

In recent years, PDEs with time delay effects have become an active area of research and arise in many practical problems (see, for example, [1, 24]). The presence of delay may be a source of instability. For example, it was proved in [6] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize a hyperbolic system involving input delay terms, additional control terms are necessary (see [20, 8]). For instance, in [20] the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [20], is an observability inequality obtained with a Carleman estimate. Laskri and Said-Houari [13] examined problem (P) in the linear situation (that is $g_1(s) = g_2(s) = s$ for all $s \in \mathbb{R}$). Under the assumption $\mu_2 \leq \mu_1$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_2 < \mu_1$ an exponential decay result for the case of equal speed wave propagation. We also recall the result by Han and Xu [8], where the authors proved a result similar to the one in [13] for the case when both the damping and the delay act on the boundary and for the one-space dimension by adopting the spectral analysis approach.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term. We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system with the presence of nonlinear degenerate delay term.

To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [14]) with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [20] does not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by [5] and [12] and used by Liu and Zuazua [15] and Alabau-Boussouira [2].

2. Preliminaries and Main Results

First assume the following hypotheses:

(H1) $g_1 : \mathbf{R} \to \mathbf{R}$ is a non-decreasing function of the class $C(\mathbf{R})$ such that there exist $\epsilon_1, c_1, c_2 > 0$ and a convex and increasing function $H : \mathbf{R}_+ \to \mathbf{R}_+$ of the class $C^1(\mathbf{R}_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon']$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon']$), such that

$$c_1|s| \leq |g_1(s)| \leq c_2|s| \quad \text{if } |s| \geq \epsilon', \tag{1}$$

$$s^2 + g_1^2(s) \leq H^{-1}(sg_1(s)) \quad \text{if } |s| \leq \epsilon'. \tag{2}$$

$g_2 : \mathbf{R} \to \mathbf{R}$ is an odd non-decreasing function of the class $C^1(\mathbf{R})$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \leq c_3 \tag{3}$$

$$\alpha_1 \; sg_2(s) \leq G_2(s) \leq \alpha_2 \; sg_1(s), \tag{4}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr$$

and

$$\alpha_2\mu_2 < \alpha_1\mu_1. \tag{5}$$

We first state some Lemmas which will be needed later.

Lemma 2.1. (Sobolev-Poincaré's inequality). *Let q be a number with $2 \leq q < +\infty$ $(n = 1, 2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_*\|\nabla\psi\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

We introduce as in [20] the new variable

$$z(x,\rho,t)=\psi_t(x,t-\tau\rho), x\in(0,1),\ \rho\in(0,1),\quad t>0. \tag{6}$$

Then, we have

$$\tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0,\ \text{ in } (0,1)\times(0,1)\times(0,+\infty). \tag{7}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt}(x,t)-K(\varphi_x+\psi)_x(x,t)=0 & \text{in }]0,1[\times]0,+\infty[,\\ \rho_2\psi_{tt}(x,t)-b\psi_{xx}(x,t)+K(\varphi_x+\psi)(x,t) & \\ \qquad +\mu_1 g_1(\psi_t(x,t))+\mu_2 g_2(z(x,1,t))=0 & \text{in }]0,1[\times]0,+\infty[,\\ \tau z'(x,\rho,t)+z_\rho(x,\rho,t)=0 & \text{in }]0,1[\times]0,1[\times]0,+\infty[,\\ \varphi(0,t)=\varphi(1,t)=\psi(0,t)=\psi(1,t)=0 & t\geq 0,\\ z(x,0,t)=\psi_t(x,t) & \text{on }]0,1[\times[0,+\infty[,\\ \psi(x,0)=\psi_0(x),\quad \psi_t(x,0)=\psi_1(x) & x\in]0,1[,\\ \varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x) & x\in]0,1[,\\ z(x,\rho,0)=f_0(x,-\rho\tau) & \text{in }]0,1[\times]0,1[. \end{cases} \tag{8}$$

Let ξ be a positive constant such that

$$\tau\frac{\mu_2(1-\alpha_1)}{\alpha_1}<\xi<\tau\frac{\mu_1-\alpha_2\mu_2}{\alpha_2}. \tag{9}$$

We define the energy associated to the solution of the problem (8) by the following formula:

$$\begin{aligned} E(t)=E(t,z,\varphi,\psi)=&\frac{1}{2}\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2+b\psi_x^2\right\}dx\\ &+\xi\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx. \end{aligned} \tag{10}$$

We have the following theorem.

Theorem 2.1. *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1)\in(H^2(0,1)\cap H_0^1(0,1))\times H_0^1(0,1), f_0\in H_0^1((0,1);H^1(0,1))$ *satisfy the compatibility condition*

$$f_0(.,0)=\psi_1.$$

Assume that the hypothesis $(H1)$ *holds. Then the problem* (P) *admits a unique weak solution*

$$\psi,\varphi\in L^\infty_{loc}((-\tau,\infty);H^2(0,1)\cap H_0^1(0,1)),\quad \psi_t,\varphi_t\in L^\infty_{loc}((-\tau,\infty);H_0^1(0,1)),$$

$$\psi_{tt},\varphi_{tt}\in L^\infty_{loc}((-\tau,\infty);L^2(0,1))$$

and, for some constants ω_1, ω_2 *and* ω_3, ϵ_0 *we obtain the following decay property:*

(11)
$$E(t) \le \omega_1 H_1^{-1}(\omega_2 t + \omega_3), \quad \forall t > 0,$$

where

(12)
$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds$$

and

$$H_2(t) = \begin{cases} t & \textit{if } H \textit{ is linear on } [0, \epsilon'], \\ tH'(\epsilon_0 t) & \textit{if } H'(0) = 0 \textit{ and } H'' > 0 \textit{ on } \ ]0, \epsilon']. \end{cases}$$

Remark 2.1. 1.By the mean value Theorem for integrals and the monotonicity of g_2, we find that

$$G_2(s) = \int_0^s g_2(r)\, dr \le s g_2(s).$$

Then, $\alpha_1 \le \alpha_2 \le 1$.

2. We need the condition (3) only to prove global existence, so if we study the energy decay, we can replace the linear growth order of the function $g_2(s)$ for large $|s|$ by nonlinear polynomial growth.

Example. Let g be given by $g_1(s) = s^p(-\ln s)^q$, where $p \ge 1$ and $q \in \mathbb{R}$ on $(0, \epsilon_1]$. Then $g_1'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$ we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = cs^{\frac{p+1}{2}}(-\ln\sqrt{s})^q.$$

We have

$$H'(s) = cs^{\frac{p-1}{2}}\left(-\ln\sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right) \text{ when } s \text{ is near } 0.$$

Thus

$$H_2(s) = cs^{\frac{p+1}{2}}\left(-\ln\sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right) \text{ when } s \text{ is near } 0.$$

and

$$\begin{aligned} H_1(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2}}\left(-\ln\sqrt{s}\right)^{q-1}\left(\frac{p+1}{2}\left(-\ln\sqrt{s}\right) - \frac{q}{2}\right)}\, ds \\ &= c\int_1^{\frac{1}{\sqrt{t}}} \frac{z^{p-2}}{(\ln z)^{q-1}\left(\frac{p+1}{2}\ln z - \frac{q}{2}\right)}\, dz \text{ when } t \text{ is near } 0. \end{aligned}$$

We obtain in a neighborhood of 0

$$H_1(t) \equiv \begin{cases} c\dfrac{1}{t^{\frac{p-1}{2}}\,(-\ln t)^q} & \text{if } p>1, \\ c\,(-\ln t)^{1-q} & \text{if } p=1,\ q<1, \\ c\,(\ln(-\ln t)) & \text{if } p=1,\ q=1. \end{cases}$$

and then in a neighborhood of $+\infty$

$$H_1^{-1}(t) \equiv \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } p>1, \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p=1,\ q<1, \\ ce^{-e^t} & \text{if } p=1,\ q=1. \end{cases}$$

Then

$$E(t) \leq \begin{cases} ct^{-\frac{2}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } p>1, \\ ce^{-t^{\frac{1}{1-q}}} & \text{if } p=1,\ q<1, \\ ce^{-e^t} & \text{if } p=1,\ q=1. \end{cases}$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 2.2. *Let (φ,ψ,z) be a solution of the problem (8). Then, the energy functional defined by (10) satisfies*

$$\begin{aligned} E'(t) \leq & -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\ & -\left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ \leq & \ 0 \end{aligned} \tag{13}$$

Proof. Multiplying the first equation in (8) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\int_0^1 \{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2 + b\psi_x^2\}\,dx\right) \\ &= -\mu_1\int_0^1 \psi_t g_1(\psi_t)\,dx - \mu_2\int_0^1 \psi_t(x,t)g_2(z(x,1,t))\,dx = 0. \end{aligned} \tag{14}$$

We multiply the third equation in (8) by $\xi g_2(z(x,\rho,t))$ and integrate the result over $(0,1)\times(0,1)$, to obtain:

$$\begin{aligned} \xi\int_0^1\int_0^1 z' g_2(z(x,\rho,t))\,d\rho\,dx &= -\frac{\xi}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\ &= -\frac{\xi}{\tau}\int_0^1 (G_2(z(x,1,t)) - G_2(z(x,0,t)))\,dx. \end{aligned} \tag{15}$$

Then

$$\xi \frac{d}{dt}\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx = -\frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx + \frac{\xi}{\tau}\int_0^1 G_2(\psi_t)\,dx. \tag{16}$$

From (14), (16) and using Young inequality we get

$$\begin{aligned} E'(t) &= -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\ &\quad -\mu_2\int_0^1 \psi_t(t) g_2(z(x,1,t))\,dx. \end{aligned} \tag{17}$$

Let us denote G_2^* to be the conjugate function of the convex function G_2, i.e., $G_2^*(s) = \sup_{t\in\mathbf{R}^+}(st - G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by (see Arnold [3], p. 61-62, and Lasiecka [5])

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \geq 0 \tag{18}$$

and satisfies the following inequality

$$st \leq G_2^*(s) + G_2(t), \quad \forall s,t \geq 0. \tag{19}$$

Then, from the definition of G_2, we get

$$G_2^*(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$\begin{aligned} G_2^*(g_2(z(x,1,t))) &= z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)) \\ &\leq (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)). \end{aligned} \tag{20}$$

Making use of (17), (19) and (20), we have

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau}\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\ &\quad +\mu_2\int_0^1 \left(G_2(\psi_t) + G_2^*(g(z(x,1,t)))\right)\,dx \\ &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \frac{\xi}{\tau}\int_0^1 G_2(z(x,1,t))\,dx \\ &\quad +\mu_2\int_0^1 G_2^*(g_2(z(x,1,t)))\,dx. \end{aligned} \tag{21}$$

Using (4) and (9), we obtain

$$\begin{aligned} E'(t) &\leq -\left(\mu_1 - \frac{\xi\alpha_2}{\tau} - \mu_2\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\ &\quad -\left(\frac{\xi}{\tau}\alpha_1 - \mu_2(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ &\leq 0. \end{aligned}$$

■

3. Global Existence

We are now ready to prove Theorem 2.1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in \mathbb{N}\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(x,0) = w_j.$$

Then, we may extend $\phi_j(x,0)$ by $\phi_j(x,\rho)$ over $L^2((0,1)\times(0,1))$ and denote Z_k the space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, z_k), k = 1, 2, 3, \ldots,$ in the form

$$\varphi_k(t) = \sum_{j=1}^{k} g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^{k} \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk} \phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \ldots, k,$ are determined by the following ordinary differential equations:

$$\rho_1(\varphi_k''(t), w_j) + K(\varphi_{kx}(t), w_{jx}) - k(\psi_{kx}(t), w_j) = 0, \quad 1 \leq j \leq k, \tag{22}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{23}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{24}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi)(t), w_j) + \mu_1(g_1(\psi_k'), w_j) \\ +\mu_2(g_2(z_k(.,1)), w_j) = 0 \quad 1 \leq j \leq k, \\ z_k(x,0,t) = \psi_k'(x,t) \end{cases} \tag{25}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{26}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{27}$$

and

$$(\tau z_{kt} + z_{k\rho}, \phi_j) = 0, \quad 1 \leq j \leq k, \tag{28}$$

$$z_k(\rho,0)=z_{0k}=\sum_{j=1}^{k}(f_0,\phi_j)\phi_j\to f_0 \text{ in } H_0^1((0,1);H^1(0,1)) \text{ as } k\to+\infty. \tag{29}$$

By virtue of the theory of ordinary differential equations, the system (22)-(29) has a unique local solution which is extended to a maximal interval $[0,T_k[$ (with $0<T_k\leq+\infty$) by Zorn lemma since the nonlinear terms in (25) are locally Lipschitz continuous. Note that $(\varphi_k(t),\psi_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0,T_k[$ to obtain one solution defined for all $t>0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (φ_k,ψ_k,z_k).

The first estimate. Since the sequences $\varphi_{0k},\varphi_{1k},\psi_{0k},\psi_{1k}$ and z_{0k} converge, then standard calculations, using (22)-(29), similar to those used to derive (13), yield C independent of k such that

$$\begin{aligned}&E_k(t)+a_1\int_0^t\int_0^1\psi_k'g_1(\psi_k')dx\;ds\\&+a_2\int_0^t\int_0^1 z_k(x,1,t)g_2(z_k(x,1,t))dx\;ds\leq E_k(0)\leq C,\end{aligned} \tag{30}$$

where

$$\begin{aligned}E_k(t)=\frac{1}{2}\int_0^1&\{\rho_1{\varphi_k'}^2+\rho_2{\psi_k'}^2+K|\varphi_{kx}+\psi_k|^2+b\psi_{kx}^2\}dx\\&+\xi\int_0^1\int_0^1 G_2(z_k(x,\rho,t))\,d\rho\,dx.\end{aligned}$$

$$a_1=\mu_1-\frac{\xi}{\tau}\alpha_2-\mu_2\alpha_2 \text{ and } a_2=\frac{\xi}{\tau}\alpha_1-\mu_2(1-\alpha_1).$$

for some C independent of k. These estimates imply that the solution (φ_k,ψ_k,z_k) exists globally in $[0,+\infty[$.
Estimate (30) yields

$$\varphi_k,\psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty;H_0^1(0,1)) \tag{31}$$

$$\varphi_k',\psi_k' \text{ are bounded in } L^\infty_{loc}(0,\infty;L^2(0,1)) \tag{32}$$

$$\psi_k'(t)g_1(\psi_k'(t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{33}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty;L^1((0,1)\times(0,1))) \tag{34}$$

$$z_k(x,1,t)g_2(z_k(x,1,t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{35}$$

The second estimate. First, we estimate $\varphi''_k(0)$ and $\psi''_k(0)$. Testing (22) by $g''_{jk}(t)$, (25) by $\tilde{g}''_{jk}(t)$ and choosing $t=0$ we obtain

$$\rho_1\|\varphi''_k(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2\|\psi''_k(0)\|_2 \le b\|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1\|g_1(\psi_{1k})\|_2 + \mu_2\|g_2(z_{0k})\|_2.$$

Hence from (23), (24) and (29):

$$\|\varphi''_k(0)\|_2 \le C.$$

Since $g_1(\psi_{1k}), g_2(z_{0k})$ are bounded in $L^2(0,1)$ by $(\mathbf{H1})$, (23), (26), (27) and (29) yield

$$\|\psi''_k(0)\|_2 \le C.$$

Differentiating (22) and (25) with respect to t, we get

$$(\rho_1\varphi'''_k(t) - K\varphi'_{kxx}(t) - K\psi'_{kx}(t), w_j) = 0 \tag{36}$$

and

$$\begin{aligned}(\rho_2\psi'''_k(t) - b\psi'_{kxx}(t) + K\varphi'_{kx}(t) + K\psi'_k(t) + \mu_1{\psi''}^2_k(t)g'_1(\psi'_k(t))\\ +\mu_2 z'_k(x,1,t)g'_2(z_k(x,1,t)), w_j) = 0.\end{aligned} \tag{37}$$

Multiplying (36) by $g''_{jk}(t)$ and (37) by $\tilde{g}''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2\right) - K\int_0^1 (\varphi'_{kx} + \psi'_k)_x\varphi''_k\,dx = 0 \tag{38}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2\right)\\ &\quad +K\int_0^1(\varphi'_{kx}+\psi'_k)\psi''_k\,dx + \mu_1\int_0^1 {\psi''}^2_k(t)g'_1(\psi'_k(t))\,dx\\ &\quad +\mu_2\int_0^1 {\psi''}_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx = 0.\end{aligned} \tag{39}$$

Differentiating (28) with respect to t, we get

$$(\tau z''_k(t) + \frac{\partial}{\partial\rho}z'_k, \phi_j) = 0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z'_k(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z'_k(t)\|_2^2=0. \tag{40}$$

Taking the sum of (38), (39) and (40), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2\\&+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\&+\mu_1\int_0^1\psi''^2_k(t)g'_1(\psi'_k(t))\,dx+\frac{1}{2}\int_0^1|z'_k(x,1,t)|^2\,dx\\=&-\mu_2\int_0^1\psi''_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx+\frac{1}{2}\|\psi''_k(t)\|_2^2.\end{aligned}$$

Using (3), Cauchy-Schwarz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2\\&+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\&+\mu_1\int_0^1\psi''^2_k(t)g'_1(\psi'_k(t))\,dx+c\int_0^1|z'_k(x,1,t)|^2\,dx\le c'\|\psi''_k(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we obtain

$$\begin{aligned}&\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2\\&+K\|\varphi'_{kx}(t)+\psi'_k\|_2^2+\tau\|z'_k(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\\\le&\, e^{cT}\Big(\rho_1\|\varphi''_k(0)\|_2^2+\rho_2\|\psi''_k(0)\|_2^2+b\|\psi'_{kx}(0)\|_2^2+K\|\varphi'_{kx}(0)+\psi'_k(0)\|_2^2\\&+\tau\|z'_k(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\Big)\end{aligned}$$

for all $t\in\mathbf{R}_+$, therefore, we conclude that

$$\varphi''_k,\psi''_k \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2) \tag{41}$$

$$\varphi'_k,\psi'_k \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0) \tag{42}$$

$$z'_k \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1))) \tag{43}$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (22) and (25), multiplying the result by $g'_{jk}(t)$ and $\tilde{g}'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2+\right)+K\int_0^1(\varphi_x+\psi)_x\varphi'_{kxx}\,dx=0. \tag{44}$$

$$\begin{aligned}(45)\qquad &\frac{1}{2}\frac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2\right)\\ &-K\int_0^1(\varphi_x+\psi)\psi'_{kxx}\,dx+\mu_1\int_0^1|\psi'_{kx}(t)|^2g_1'(\psi'_k(t))\,dx\\ &+\mu_2\int_0^1{\psi'}_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx=0.\end{aligned}$$

Replacing ϕ_j by $-\phi_{jxx}$ in (28), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\tau\frac{d}{dt}\|z_{kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2=0. \tag{46}$$

From (44), (45) and (46), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2\\ &+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2+\tau\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)\times(0,1))}\Big)\\ &+\mu_1\int_0^1|\psi'_{kx}(t)|^2g_1'(\psi'_k(t))\,dx+\frac{1}{2}\int_0^1|z_{kx}(x,1,t)|^2\,dx\\ =&-\mu_2\int_0^1\psi'_{kx}(t)z_{kx}(x,1,t)g_2'(z_k(x,1,t))\,dx+\frac{1}{2}\|\nabla\psi'_k(t)\|_2^2.\end{aligned}$$

Using (3), Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2\\ &+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kx}(t)\|_2^2+\tau\|z_{kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\Big)\\ &+\mu_1\int_0^1|\psi'_{kx}(t)|^2g_1'(\psi'_k(t))\,dx+c\int_0^1|z_{kx}(x,1,t)|^2\,dx\le c'\|\psi'_{kx}(t)\|_2^2.\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}&\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2\\ &+\tau\|z_{kx}(x,\rho,t)\|^2_{L^2((0,1)\times(0,1))}\\ \le\ & e^{cT}\Big(\rho_1\|\varphi'_{kx}(0)\|_2^2+\rho_2\|\psi'_{kx}(0)\|_2^2+K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2+b\|\psi_{kx}(0)\|_2^2\\ &+\tau\|z_{kx}(x,\rho,0)\|^2_{L^2((0,1)\times(0,1))}\Big)\end{aligned}$$

for all $t\in\mathbf{R}_+$, therefore, we conclude that

$$\varphi_k,\psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1)), \tag{47}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0(0,1;L^2(0,1))). \tag{48}$$

Applying Dunford-Petti's theorem we conclude from (31), (32), (33), (34), (41), (42), (43), (47) and (48), after replacing the sequences φ_k, ψ_k and z_k with a subsequence if needed, that

$$\text{(49)} \qquad \begin{cases} \varphi_k \to \varphi \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^2 \cap H^1_0(0,1)) \\ \psi_k \to \psi \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^2 \cap H^1_0(0,1)), \end{cases}$$

$$\begin{cases} \varphi'_k \to \varphi' \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0(0,1)) \\ \psi_k \to \psi' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0(0,1)), \end{cases}$$

$$\text{(50)} \qquad \begin{cases} \varphi''_k \to \varphi'' \text{ weak-star in } \ L^\infty_{loc}(0,\infty; L^2(0,1)) \\ \psi''_k \to \psi'' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; L^2(0,1)), \end{cases}$$

$$\text{(51)} \qquad \begin{aligned} & g_1(\psi'_k) \to \chi \ \text{ weak-star in } \ L^2((0,1)\times(0,T)), \\ & z_k \to z \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; H^1_0((0,1); L^2(0,1)), \\ & z'_k \to z' \ \text{ weak-star in } \ L^\infty_{loc}(0,\infty; L^2((0,1)\times(0,1))), \\ & g_2(z_k(x,1,t)) \to \psi \ \text{ weak-star in } \ L^2((0,1)\times(0,T)) \end{aligned}$$

for suitable functions $\varphi, \psi \in L^\infty(0,T; H^2 \cap H^1_0(0,1)), z \in L^\infty(0,T; L^2((0,1)\times(0,1))), \chi \in L^2((0,1)\times(0,T)), \psi \in L^2((0,1)\times(0,T))$ for all $T \geq 0$. We have to show that (φ, ψ, z) is a solution of (8).

From (31) and (32) we have (ψ'_k) is bounded in $L^\infty(0,T; H^1_0(0,1))$. Then (ψ'_k) is bounded in $L^2(0,T; H^1_0)$. Since (ψ''_k) is bounded in $L^\infty(0,T; L^2(0,1))$, then (ψ''_k) is bounded in $L^2(0,T; L^2(0,1))$. Consequently (ψ'_k) is bounded in $H^1(Q)$,where $Q = (0,1)\times(0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [14] we can extract a subsequence (ψ_ν) of (ψ_k) such that

$$\psi'_\nu \to \psi' \text{ strongly in } \ L^2(Q).$$

Therefore

$$\text{(52)} \qquad \psi'_\nu \to \psi' \text{ strongly and a.e on } Q.$$

Similarly we obtain

$$\text{(53)} \qquad z_\nu \to z \text{ strongly and a.e on } Q.$$

Lemma 3.1. *For each $T > 0$, $g_1(\psi'), g_2(z(x,1,t)) \in L^1(Q)$ and $\|g_1(\psi')\|_{L^1(Q)}, \|g_2(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a constant independent of t.*

Proof. By (H1) and (52) we have

$$g_1(\psi_k'(x,t)) \to g_1(\psi'(x,t)) \text{ a.e. in } Q,$$

$$0 \le g_1(\psi_k'(x,t))\psi_k'(x,t) \to g_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q$$

Hence, by (33) and Fatou's lemma we have

$$\int_0^T \int_0^1 u'(x,t) g_1(\psi'(x,t))\, dx\, dt \le K \text{ for } T > 0. \tag{54}$$

By Cauchy-Schwarz inequality and using (54), we have

$$\begin{aligned}\int_0^T \int_0^1 |g_1(\psi'(x,t))|\, dx\, dt &\le c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \psi' g_1(\psi')\, dx\, dt \right)^{\frac{1}{2}} \\ &\le c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1\end{aligned}$$

■

Lemma 3.2. *$g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T))$ and $g_2(z_k) \to g_2(z)$ in $L^1((0,1)\times(0,T))$.*

Proof. Let $E \subset (0,1)\times[0,T]$ and set

$$E_1 = \left\{ (x,t) \in E;\ g_1(\psi_k'(x,t)) \le \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in \mathbf{R} \text{ and } |g_1(s)| \ge r\}$,

$$\int_E |g_1(\psi_k')|\, dxdt \le \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} |\psi_k' g_1(\psi_k')|\, dxdt.$$

Applying (33) we deduce that $\sup_k \int_E |g_1(\psi_k')|\, dxdt \to 0$ as $|E| \to 0$. From Vitali's convergence theorem we deduce that $g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T))$, hence

$$g_1(\psi_k') \to g_1(\psi') \text{ weak star in } L^2(Q).$$

Similarly, we have

$$g_2(z_k') \to g_2(z') \text{ weak star in } L^2(Q),$$

and this imply that

$$\int_0^T \int_0^1 g_1(\psi_k') v\, dx\, dt \to \int_0^T \int_0^1 g_1(\psi') v\, dx\, dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{55}$$

$$\int_0^T\int_0^1 g_2(z_k)v\,dx\,dt \to \int_0^T\int_0^1 g_2(z)v\,dx\,dt \text{ for all } v\in L^2(0,T;H_0^1) \tag{56}$$

as $k\to+\infty$. It follows at once from (49), (50), (55), (56) and (51) that for each fixed $u,v\in L^2(0,T;H_0^1(0,1))$ and $w\in L^2(0,T;H_0^1((0,1)\times(0,1)))$

$$\begin{aligned}
&\int_0^T\int_0^1(\rho_1\varphi_k''-K(\varphi_{kx}+\psi_k)_x)u\,dx\,dt\\
&\to\int_0^T\int_0^1(\rho_1\varphi''-K(\varphi_x+\psi)_x)u\,dx\,dt\\
&\int_0^T\int_0^1(\rho_2\psi_k''-b\psi_{kxx}+K(\varphi_{kx}+\psi_k)++\mu_1g_1(\psi_k')+\mu_2g_2(z_k))v\,dx\,dt\\
&\to\int_0^T\int_0^1(\rho_2\psi''-b\psi_{xx}+K(\varphi_x+\psi)+\mu_1g_1(\psi')+\mu_2g_2(z))v\,dx\,dt\\
&\int_0^T\int_0^1\int_0^1(\tau z_k'+\frac{\partial}{\partial\rho}z_k)w\,dx\,d\rho\,dt\to\int_0^T\int_0^1\int_0^1(\tau z'+\frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt
\end{aligned}$$

as $k\to+\infty$. Hence

$$\begin{aligned}
&\int_0^T\int_0^1(\rho_1\varphi''-K(\varphi_x+\psi)_x)u\,dx\,dt=0\\
&\int_0^T\int_0^1(\rho_2\psi''-b\psi_{xx}+K(\varphi_x+\psi)+\mu_1g_1(\psi')+\mu_2g_2(z))v\,dx\,dt=0\\
&\int_0^T\int_0^1\int_0^1(\tau u'+\frac{\partial}{\partial\rho}z)w\,dx\,d\rho\,dt=0,\qquad w\in L^2(0,T;H_0^1((0,1)\times(0,1))).
\end{aligned}$$

Thus the problem (P) admits a global weak solution (φ,ψ).

Uniqueness. Let (φ_1,ψ_1,z_1) and (φ_2,ψ_2,z_2) be two solutions of problem (8). Then $(w,\tilde w,\tilde{\tilde w})=(\varphi_1,\psi_1,z_1)-(\varphi_2,\psi_2,z_2)$ verifies

$$\left\{\begin{array}{ll}
\rho_1w_{tt}(x,t)-K(w_x+\tilde w)_x(x,t)=0 & \text{in }]0,1[\times]0,+\infty[,\\
\rho_2\tilde w''(x,t)-b\tilde w_{xx}(x,t)+K(w_x+\tilde w) & \\
\quad+\mu_1g_1(\psi_1'(x,t))-\mu_1g_1(\psi_2'(x,t)) & \\
\quad+\mu_2g_2(z_1(x,1,t))-\mu_2g_2(z_2(x,1,t))=0, & \text{in }]0,1[\times]0,+\infty[,\\
\tau\tilde{\tilde w}'(x,\rho,t)+\tilde{\tilde w}_\rho(x,\rho,t)=0, & \text{in } (0,1)\times]0,1[\times]0,+\infty[\\
w(0,t)=w(1,t)=\tilde w(0,t)=\tilde w(1,t)=0, & t\ge0\\
\tilde{\tilde w}(x,0,t)=\psi_1'(x,t)-\psi_2'(x,t) & \text{on }]0,1[\times[0,+\infty[\\
w(x,0)=w'(x,0)=\tilde w(x,0)=\tilde w'(x,0)=0, & \text{in }]0,1[\\
\tilde{\tilde w}(x,\rho,0)=0 & \text{in }]0,1[\times]0,1[
\end{array}\right. \tag{57}$$

Multiplying the first equation in (57) by w', integrating over $(0,1)$ and using an integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}(\rho_1\|w'\|_2^2)+K\int_0^1(w_x+\tilde{w})_x w'\,dx=0 \tag{58}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2)+K\int_0^1(w_x+\tilde{w})\tilde{w}'\,dx+\mu_1(g_1(\psi_1')-g_1(\psi_2'),\tilde{w}')\\ +\mu_2(g_2(z_1(x,1,t))-g_2(z_2(x,1,t)),\tilde{w}')=0.\end{aligned} \tag{59}$$

Multiplying the second equation in (57) by $\tilde{\tilde{w}}$, integrating over $(0,1)\times(0,1)$, we get

$$\tau\frac{1}{2}\frac{d}{dt}\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho+\frac{1}{2}(\|\tilde{\tilde{w}}(x,1,t)\|_2^2-\|\tilde{w}'\|_2^2)=0. \tag{60}$$

From (58), (59), (60 and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho\right)\\ &\quad+\mu_1(g_1(\psi_1')-g_1(\psi_2'),\tilde{w}')+\frac{1}{2}\|\tilde{\tilde{w}}(x,1,t)\|_2^2\\ &=-\mu_2(g_2(z_1(x,1,t))-g_2(z_2(x,1,t)),\tilde{w}')+\frac{1}{2}\|\tilde{w}'\|_2^2\\ &\le\frac{1}{2}\|\tilde{w}'\|_2^2+||g(z_1(x,1,t))-g(z_2(x,1,t))||_2||\tilde{w}'||_2.\end{aligned}$$

Using condition (3) and Young's inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho\right)\le c\|\tilde{w}'\|_2^2,$$

where c is a positive constant. Then integrating over $(0,t)$, using Gronwall's lemma, we conclude that

$$\rho_1\|w'\|_2^2+\rho_2\|\tilde{w}'\|_2^2+b\|\tilde{w}_x\|_2^2+K\|w_x+\tilde{w}\|_2^2+\tau\int_0^1\|\tilde{\tilde{w}}'\|_2^2\,d\rho=0.$$

4. Asymptotic Behavior

Now we construct a Lyapunov functional L equivalent to E. For this, we define several functionals which allow us to obtain the needed estimates.

Then we have the following estimate.

Lemma 4.1. *Let (φ,ψ,z) be the solution of (8). Then the functional F_1 defined by*

$$F_1(t)=-\int_0^1(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)\,dx \tag{61}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\frac{dF_1(t)}{dt} \le &-\int_0^1 (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx + K\int_0^1 |\varphi_x+\psi|^2 dx \\ &+c\int_0^1 \psi_x^2 dx + c\int_0^1 g_1^2(\psi_t)dx + c\int_0^1 g_2^2(z(x,1,t))dx.\end{aligned} \tag{62}$$

Proof. By taking the time derivative of (61)

$$\frac{dF_1(t)}{dt} = -\int_0^1 (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx - \int_0^1 (\rho_1\varphi_{tt}\varphi+\rho_2\psi_{tt}\psi)dx.$$

Therefore, by using the first and the second equations in (8) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned}\tfrac{dF_1(t)}{dt} = &-\int_0^1 (\rho_1\varphi_t^2+\rho_2\psi_t^2)dx + K\int_0^1 |\varphi_x+\psi|^2 dx \\ &+b\int_0^1 \psi_x^2 dx + \mu_1\int_0^1 \psi g_1(\psi_t)dx + \mu_2\int_0^1 \psi g_2(z(x,1,t))dx.\end{aligned} \tag{63}$$

By exploiting Young's inequality and Poincaré's inequality, then (62) holds. ∎

Lemma 4.2. *Let* (φ,ψ,z) *be the solution of (8). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b}. \tag{64}$$

Then the functional F_2 *defined by*

$$F_2(t) = \rho_2\int_0^1 \psi_t(\varphi_x+\psi)dx + \rho_2\int_0^1 \psi_x\varphi_t\, dx. \tag{65}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\tfrac{dF_2(t)}{dt} \le &[b\varphi_x\psi_x]_{x=0}^{x=1} - (K-\varepsilon)\int_0^1 (\varphi_x+\psi)^2 dx \\ &+\rho_2\int_0^1 \psi_t^2 dx + \frac{c}{\varepsilon}\int_0^1 g_1^2(\psi_t)dx + \frac{c}{\varepsilon}\int_0^1 g_2^2(z(x,1,t))dx\end{aligned} \tag{66}$$

for any $0<\varepsilon<1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned}\frac{dF_2(t)}{dt} = &\int_0^1 \rho_2\psi_{tt}(\varphi_x+\psi)dx + \int_0^1 \rho_2\psi_t(\varphi_x+\psi)_t dx \\ &+\rho_2\int_0^1 \psi_x\varphi_{tt}dx + \rho_2\int_0^1 \psi_{tx}\varphi_t dx.\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (\varphi_x + \psi)[b\psi_{xx} - k(\varphi_x + \psi) - \mu_1 g_1(\psi_t) - \mu_2 g_2(z(x,1,t)]dx \\
&\quad + \rho_2 \int_0^1 \psi_t^2\, dx + \frac{\rho_2}{\rho_1} \int_0^1 k(\varphi_x + \psi)_x \psi_x\, dx.
\end{aligned}
$$

Then, by using Eqs.(8) and (64) we find

$$
\begin{aligned}
\frac{dF_2(t)}{dt} &= [b\varphi_x \psi_x]_{x=0}^{x=1} - K \int_0^1 (\varphi_x + \psi)^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\
&\quad - \mu_1 \textstyle\int_0^1 (\varphi_x + \psi) g_1(\psi_t) dx - \mu_2 \int_0^1 (\varphi_x + \psi) g_2(z(x,1,t)) dx.
\end{aligned}
$$

By the Young inequality (66) is established. ■

Lemma 4.3. *Let* $m \in C^1([0,1])$ *be a function satisfying* $m(0) = -m(1) = 2$. *Then there exists* $c > 0$ *such that, for any* $0 < \varepsilon < 1$, *the functional* F_3 *defined by*

$$
F_3(t) = \frac{b}{4\varepsilon} \int_0^1 \rho_2 m(x) \psi_t \psi_x\, dx + \frac{\varepsilon}{k} \int_0^1 \rho_1 m(x) \varphi_t \varphi_x\, dx
$$

satisfies, along the solution, the estimate

$$
\begin{aligned}
F_3'(t) &\le -\frac{b^2}{4\varepsilon}((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \varepsilon((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) \\
&\quad + (\frac{k}{4} + \frac{c}{k}\varepsilon) \int_0^1 (\psi + \varphi_x)^2\, dx + c\varepsilon\rho_1 \int_0^1 \varphi_t^2\, dx + \frac{c}{\varepsilon^2} \int_0^1 \psi_x^2\, dx \\
&\quad + \frac{c}{\varepsilon} \int_0^1 \psi_t^2\, dx + c \int_0^1 g_1^2(\psi_t)\, dx + c \int_0^1 g_2^2(z(x,1,t))\, dx
\end{aligned} \tag{67}
$$

Proof. Using Eqs. (8) and integrating by parts, obtain

$$
\begin{aligned}
F_3'(t) &= \frac{b}{4\varepsilon} \left[-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2) - \int_0^1 \frac{b}{2} m'(x) \psi_x^2\, dx \right. \\
&\quad - k \int_0^1 m(x) \psi_x (\varphi_x + \psi)\, dx - \int_0^1 m(x) \mu_1 g_1(\psi_t) \psi_x\, dx \\
&\quad \left. - \int_0^1 m(x) \mu_2 g_2(z(x,1,t)) \psi_x\, dx - \int_0^1 \frac{\rho_2}{2} m'(x)(\psi_t)^2\, dx \right] \\
&\quad \frac{\varepsilon}{k} \left[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) - \int_0^1 \frac{k}{2} m'(x) \varphi_x^2\, dx \right. \\
&\quad \left. + \int_0^1 k m(x) \psi_x \varphi_x\, dx - \int_0^1 \frac{\rho_1}{2} m'(x)(\varphi_t)^2\, dx \right]
\end{aligned}
$$

Then by the Young and Poincaré inequalities and the fact that

$$\varphi_x^2 \le 2(\psi+\varphi_x)^2 + 2\psi^2$$

we obtain

$$\begin{aligned} F_3'(t) \le \frac{b}{4\varepsilon}\Big[&-b((\psi_x(1,t))^2 + (\psi_x(0,t))^2) \\ &+\frac{c}{\varepsilon}\int_0^1 \psi_x^2\,dx + \varepsilon\frac{k}{b}\int_0^1 (\psi+\varphi_x)^2\,dx \\ &+\varepsilon\int_0^1 g_1^2(\psi_t)\,dx + \varepsilon\int_0^1 g_2^2(z(x,1,t))\,dx + c\int_0^1 \psi_t^2\,dx\Big] \\ &\frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + c\int_0^1 \psi_x^2\,dx \\ &+c\int_0^1 (\psi+\varphi_x)^2\,dx + c\int_0^1 \varphi_t^2\,dx\Big] \end{aligned}$$

This gives (67). ■

Lemma 4.4. *Assume that* $(H1)$ *hold. Then, for sufficiently small* ε*, the functional* F *defined by*

$$F(t) = 2c\varepsilon F_1(t) + F_2(t) + F_3(t)$$

satisfies, along the solution, the estimate

$$\begin{aligned} F'(t) \le -\frac{k}{2}\int_0^1 (\psi+\varphi_x)^2\,dx - \tau\int_0^1 \varphi_t^2\,dx + c\int_0^1 \psi_t^2\,dx + +c\int_0^1 \psi_x^2\,dx \\ +c\int_0^1 g_1^2(\psi_t)\,dx + c\int_0^1 g_2^2(z(x,1,t))\,dx, \end{aligned} \tag{68}$$

where $\tau = c\varepsilon\rho_1$.

Proof. Using Lemmas 4.1, 4.2, 4.3 and the fact that

$$[b\varphi_x\psi_x]_{x=0}^{x=1} \le \varepsilon[\varphi_x^2(1) + \varphi_x^2(0)] + \frac{b^2}{4\varepsilon}[\psi_x^2(1) + \psi_x^2(0)] \tag{69}$$

for any $0 < \varepsilon < 1$, we obtain (68). ■

Next, we introduce the following functional

$$I(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx, \tag{70}$$

where w is the solution of

$$-\omega_{xx} = \psi_x, \qquad \omega(0) = \omega(1) = 0. \tag{71}$$

Then we have the following estimate.

Lemma 4.5. *Let* (φ, ψ, z) *be the solution of (8), then for any* $\delta > 0$*, we have the following estimate*

$$\begin{aligned} \frac{dI(t)(t)}{dt} &\leq \frac{-b}{2}\int_0^1 \psi_x^2(x,t)dx + \frac{c}{\delta}\int_0^1 \psi_t^2(x,t)dx \\ &+\delta \int_0^1 \varphi_t^2(x,t)dx + c\int_0^1 g_1^2(\psi_t)dx + c\int_0^1 g_2^2(z(x,1,t))dx. \end{aligned} \tag{72}$$

Proof. Using Eqs. (8), we have

$$\begin{aligned} \frac{dI(t)}{dt} &= -b\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx - K\int_0^1 \psi^2 dx \\ &+K\int_0^1 \omega_x^2 dx + \rho_1 \int_0^1 \psi_t \omega_t dx - \mu_1 \int_0^1 \psi g_1(\psi_t)dx \\ &-\mu_2 \int_0^1 \psi g_2(z(x,1,t))dx. \end{aligned} \tag{73}$$

It is clear that, from (71), we have

$$\begin{aligned} \int_0^1 \omega_x^2 dx \leq \int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx \\ \int_0^1 \omega_t^2 dx \leq \int_0^1 \omega_{tx}^2 dx \leq \int_0^1 \psi_t^2 dx \end{aligned} \tag{74}$$

By using Young's inequality and Poincaré's inequality, the last two terms in (73) can be estimated as

$$\begin{aligned} &\mu_1 \int_0^1 \psi g_1(\psi_t)dx + \mu_2 \int_0^1 \psi g_2(z(x,1,t))dx \\ &\leq \frac{b}{2}\int_0^1 \psi_x^2 dx + c\int_0^1 g_1^2(\psi_t)dx + c\int_0^1 g_2^2(z(x,1,t))dx. \end{aligned} \tag{75}$$

Consequently, from (73)-(75), we obtain (72). ■

Now, let us introduce the following functional

$$I_3(t) = \int_0^1 \int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\, d\rho\, dx. \tag{76}$$

Then the following result holds.

Lemma 4.6. *Let* (φ, ψ, z) *be the solution of (8). Then it holds*

$$\frac{d}{dt}I_3(t) \leq -2I_3(t) - \frac{e^{-2\tau}}{\tau}\int_0^1 G_2(z(x,1,t))dx + \frac{1}{\tau}\int_0^1 G_2(\psi_t(x,t))\, dx. \tag{77}$$

Proof. Differentiating (76) with respect to t and using the third equation in (8), we have

$$\begin{aligned}
\frac{d}{dt}I_3(t) &= \int_0^1\int_0^1 e^{-2\tau\rho}z_t(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}z_\rho(x,\rho,t)g_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}\frac{d}{d\rho}G_2(z(x,\rho,t))\,d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\int_0^1 \left[\frac{d}{d\rho}\left(e^{-2\tau\rho}G_2(z(x,\rho,t))\right)+2\tau e^{-2\tau\rho}G_2(z(x,\rho,t))\right]d\rho\,dx\\
&= -\frac{1}{\tau}\int_0^1\left[e^{-2\tau}G_2(z(x,1,t))-G_2(\psi_t(x,t))\right]dx\\
&\quad -2\int_0^1\int_0^1 e^{-2\tau\rho}G(z(x,\rho,t))\,d\rho\,dx\\
&\le -2\int_0^1\int_0^1 e^{-2\tau\rho}G_2(z(x,\rho,t))\,d\rho\,dx-\frac{1}{\tau}\int_0^1 e^{-2\tau}G_2(z(x,1,t))dx\\
&\quad +\frac{1}{\tau}\int_0^1 G_2(\psi_t(x,t))dx\,dx\\
&\le -2I_3(t)-\frac{e^{-2\tau}}{\tau}\int_0^1 G_2(z(x,1,t))dx+\frac{1}{\tau}\int_0^1 G_2(\psi_t(x,t))\,dx. \qquad\blacksquare
\end{aligned}$$

For $N_1, N_2 > 0$, let

$$L(t) = N_1E(t) + N_2I(t) + F(t) + I_3(t). \tag{78}$$

By combining (13), (68), (72), (77), we obtain

$$\begin{aligned}
\frac{d}{dt}L(t) \le &-\left(N_1a_1-\frac{\alpha_2}{\tau}\right)\int_0^1 \psi_t g_1(\psi_t(x,t))\,dx\\
&-\left(N_1a_2+\alpha_1\frac{e^{-2\tau}}{\tau}-(N_2c+c)c_3\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx\\
&-\left(N_2\frac{b}{2}-c\right)\int_0^1 \psi_x^2\,dx\\
&-(\tilde{\tau}-N_2\delta)\int_0^1\varphi_t^2\,dx+\left(N_2\frac{c}{\delta}+c\right)\int_0^1\psi_t^2\,dx\\
&-\frac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx+(N_2c+c)\int_0^1 g_1^2(\psi_t)\,dx.
\end{aligned}\tag{79}$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\left(N_2\frac{b}{2}-c\right) > 0.$$

Next, we choose δ sufficiently small such that

$$(\tilde{\tau} - N_2\delta) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\left(N_1 a_1 - \frac{\alpha_2}{\tau}\right)$$

and

$$\left(N_1 a_2 + \alpha_1 \frac{e^{-2\tau}}{\tau} - (N_2 c + c)c_3\right).$$

Thus, (79) becomes

$$\begin{aligned} \frac{d}{dt}L(t) &\leq -d_1 \int_0^1 \psi_x^2\,dx - d_2 \int_0^1 \varphi_t^2\,dx - \frac{k}{2}\int_0^1 (\psi + \varphi_x)^2\,dx \\ &\quad + c\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx \\ &\leq -dE(t) + c\int_0^1 ((\psi_t)^2 + g_1^2(\psi_t))\,dx. \end{aligned} \tag{80}$$

At this stage, we are in position to compare $L(t)$ with $E(t)$. We have the following Lemma.

Lemma 4.7. *For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that*

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t) \qquad \forall t \geq 0. \tag{81}$$

Proof. We consider the functional

$$\mathcal{H}(t) = N_2 I(t) + F(t) + I_3(t)$$

and show that

$$|\mathcal{H}(t)| \leq \hat{C}E(t), \qquad C > 0.$$

from (61),(70),(65) and (76), we obtain

$$\begin{aligned} |\mathcal{H}(t)| &\leq N_2 \left| \int_0^1 \rho_2\psi_t\psi + \rho_1\varphi_t\omega)(x,t)dx \right| + \left| -\int_0^1 (\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)dx \right| \\ &\quad + \left| \rho_2 \int_0^1 \psi_t(\varphi_x + \psi)dx + \rho_2 \int_0^1 \psi_x\varphi_t\,dx \right| \\ &\quad + \left| \frac{b}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t\psi_x\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx \right| \\ &\quad + \left| \int_0^1\int_0^1 e^{-2\tau\rho} G_2(z(x,\rho,t))\,d\rho\,dx \right|. \end{aligned} \tag{82}$$

By using (74),(71), the trivial relation

$$\int_0^1 \varphi^2(x,t)dx \le 2\int_0^1 (\varphi_x+\psi)^2(x,t)dx + 2\int_0^1 \psi_x^2(x,t)dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned} |\mathcal{H}(t)| \le\ & \alpha_1 \int_0^1 \varphi_t^2(x,t)dx + \alpha_2 \int_0^1 \psi_t^2(x,t)dx \\ & + \alpha_3 \int_0^1 (\varphi_x+\psi)^2(x,t)dx + \alpha_4 \int_0^1 \psi_x^2(x,t)dx \\ & + \int_0^1\int_0^1 G_2(z(x,\rho,t))dx\,d\rho \end{aligned} \tag{83}$$

where the positive constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are determined as follows:

$$\begin{cases} \alpha_1 = \dfrac{N_2\rho_1}{2} + \rho_2 + \dfrac{\epsilon\rho_1}{K}, \\ \alpha_2 = \dfrac{N_2\rho_2}{2} + \rho_2 + \dfrac{\rho_2 b}{2\epsilon}, \\ \alpha_3 = \rho_1 + \dfrac{\rho_2}{2} + \dfrac{2\epsilon\rho_1}{K}, \\ \alpha_4 = \rho_2 + \dfrac{N_2}{2}\rho_2 + \rho_1 + \dfrac{\rho_2 b}{2\epsilon} + \dfrac{2\epsilon\rho_1}{K}, \end{cases}$$

According to (83) , we have

$$|H(t)| \le \hat{C}E(t)$$

for

$$\hat{C} = 2\max\left\{\frac{\alpha_1}{\rho_1}, \frac{\alpha_2}{\rho_2}, \frac{\alpha_3}{k}, \frac{\alpha_4}{b}, \frac{1}{2\xi}\right\}.$$

Therefore, we obtain

$$|L(t) - N_1E(t)| \le \hat{C}E(t).$$

So, we can choose N_1 large enough so that $\beta_1 = N_1 - \hat{C} > 0, \beta_2 = N_1 + \hat{C} > 0$. Then (81) holds true. ■

Therefore, (80) takes the form

$$\frac{d}{dt}L(t) \le -C_3E(t) + C_5(\|u'\|_2^2 + \|g_1(u_t)\|_2^2), \tag{84}$$

where C_3, C_4 and C_5 are three positive constants.

Now, we estimate the last term in the right hand side of (84). We define

$$\Omega^+ = \{x \in (0,1): |u'| \ge \varepsilon'\}, \quad \Omega^- = \{x \in (0,1): |u'| \le \varepsilon'\}.$$

From (1) and (2), it follows that

$$\int_{\Omega^+} (|u'|^2 + |g_1(u')|^2)\, dx \leq \mu_1 \int_{\Omega^+} u' g_1(u')\, dx \leq -\mu_1 E'(t). \tag{85}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $\mu_1' > 0$, such that $|g_1(s)| \leq \mu_1' |s|$ for all $|s| \leq \varepsilon'$, and thus

$$\int_{\Omega^-} (|u'|^2 + |g_1(u')|^2)\, dx \leq \mu_1' \int_{\Omega^-} u' g_1(u')\, dx \leq -\mu_1' E'(t). \tag{86}$$

Substitution of (85) and (86) into (84) gives

$$(L(t) + \mu E(t))' \leq -c_1 H_2(E(t)) \tag{87}$$

where $\mu = C_5(\mu_1 + \mu_1')$ and here and in the sequel we take C_i to be a generic positive constant.

Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.

Since H is convex and increasing, H^{-1} is concave and increasing. By the virtue of (1), the reversed Jensen's inequality for concave function, and (13), it follows that

$$\begin{aligned} &\int_{\Omega^-} (|u'|^2 + |g_1(u')|^2)\, dx \\ &\leq \int_{\Omega^-} H^{-1}(u' g_1(u'))\, dx \\ &\leq |\Omega| H^{-1}\left(\frac{1}{|\Omega|} \int_{\Omega^-} u' g_1(u')\, dx\right) \leq C H^{-1}(-C' E'(t)). \end{aligned} \tag{88}$$

A combination of (84), (85) and (88) yields

$$(L(t) + C_5 \mu_1 E(t))' \leq -C_3 E(t) + \tilde{C}_5 H^{-1}(-C' E'(t)), \quad t \geq 0. \tag{89}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t \in \mathbf{R}_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{90}$$

and which satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{91}$$

The relation (90) and the fact that $H'(0) = 0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0. \tag{92}$$

Making use of $E'(t) \leq 0, H''(t) \geq 0$, (89) and (92) we derive for $\varepsilon_0 > 0$ small enough

$$\begin{aligned}
&[H'(\varepsilon_0 E(t))\{L(t) + C_5\mu_1 E(t)\} + \tilde{C}_5 C' E(t)]' \\
&= \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(L(t) + C_5\mu_1 E(t)) \\
&\quad + H'(\varepsilon_0 E(t))(L'(t) + C_5\mu_1 E'(t)) + \tilde{C}_5 C' E'(t) \\
&\leq -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H'(\varepsilon_0 E(t)) H^{-1}(-C'E'(t)) + \tilde{C}_5 C' E'(t) \\
&\leq -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H^*(H'(\varepsilon_0 E(t))) \\
&\leq -C_3 H'(\varepsilon_0 E(t))E(t) + \tilde{C}_5 H'(\varepsilon_0 E(t))\varepsilon_0 E(t) \\
&\leq -\tilde{C}_3 H'(\varepsilon_0 E(t))E(t) \\
&= -\tilde{C}_3 H_2(E(t)).
\end{aligned} \tag{93}$$

We note that in the second inequality, we have used (91) and $0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$.
Let

$$\tilde{L}(t) = \begin{cases} L(t)+\mu E(t) & \text{if } H \text{ is linear on } [0,\varepsilon'] \\ H'(\varepsilon_0 E(t))\{L(t)+C_5\mu_1 E(t)\}+\tilde{C}_5 C' E(t) & \text{if } H'(0)=0 \text{ and } H''>0 \text{ on }]0,\varepsilon']. \end{cases} \tag{94}$$

From (87) and (93), it follows

$$\tilde{L}'(t) \leq -c_4 H_2(E(t)), \quad \forall t \geq 0. \tag{95}$$

On the other hand, after choosing $M > 0$ larger if needed, we can observe from Lemma 4.7 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. By the fact that H_2 is increasing, we obtain

$$\tilde{L}'(t) \leq -\tilde{c}_4 H_2(\tilde{L}(t)), \quad \forall t \geq 0. \tag{96}$$

Noting that $H_1' = -1/H_2$ (see (12)), we infer from (96)

$$\tilde{L}'(t) H_1'(\tilde{L}(t)) \geq \tilde{c}_4, \quad \forall t \geq 0.$$

A simple Integration over $(0, t)$ yields

$$H_1(\tilde{L}(t)) \geq H_1(\tilde{L}(0)) + \tilde{c}_4 t.$$

Then, exploiting the fact that H_1^{-1} is decreasing, we infer

$$\tilde{L}(t) \leq H_1^{-1}\left(H_1(\tilde{L}(0)) + \tilde{c}_4 t\right)$$

Consequently, the equivalence of $L, \tilde{L}$ and E, yields the estimate

$$E(t) \leq \omega_1 H_1^{-1}\left(\omega_2 t + \omega_3\right).$$

References

1. C. Abdallah, P. Dorato, J. Benitez-Read and R. Byrne, *Delayed Positive Feedback Can Stabilize Oscillatory System*, ACC, San Francisco, 1993, pp. 3106-3107.
2. F. Alabau-Boussouira, Asymptotic behavior for Timoshenko beams subject to a single nonlinear feedback control, *Nonlinear Diff. Equa. Appl.*, **14** (2007), 643-669.
3. V. I. Arnold, *Mathematical Methods of Classical Mecanics*, Springer-Verlag, New York, 1989.
4. M. M. Cavalcanti, V. D. Cavalcanti and I. Lasiecka, Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction, *J. Diff. Equa.*, **236** (2007), 407-459.
5. M. Daoulatli, I. Lasiecka and D. Toundykov, Uniform energy decay for a wave equation with partially supported nonlinear boundary dissipation without growth restrictions, *Disc. Conti. Dyna. Syst.*, **2** (2009), 67-95.
6. R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, **24** (1986), 152-156.
7. J. Dieudonné, Calcul infinitésimal, Collection Methodes, Herman, Paris, 1968.
8. Z. J. Han and G. Q. Xu, Exponential stability of timoshenko beam system with delay terms in boundary feedbacks, *ESAIM Control Optim.*, **17** (2011), 552-574.
9. A. Haraux, *Two remarks on dissipative hyperbolic problems*, Research Notes in Mathematics, Vol. 122. Pitman: Boston, MA, 1985, pp. 161-179.
10. V. Komornik, *Exact Controllability and Stabilization, The Multiplier Method*, Masson-John Wiley, Paris, 1994.
11. J. U. Kim and Y. Renardy, Boundary control of the Timoshenko beam, *SIAM J. Control Optim.*, **25** (1987), 1417-1429.
12. I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equations with nonlinear boundary dampin, *Diff. Inte. Equa.*, **6** (1993), 507-533.
13. Y. Laskri and B. Said-Houari, A stability result of a Timoshenko system with a delay term in the internal feedback, *Appl. Math. Comput.*, **217(6)** (2010), 2857-2869.
14. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

15. W. J. Liu and E. Zuazua, Decay rates for dissipative wave equations, *Ricerche di Matematica*, **XLVIII** (1999), 61-75.

16. S. A. Messaoudi and M. I. Mustafa, On the stabilization of the Timoshenko system by a weak nonlinear dissipation, *Math. Meth. Appl. Sci.*, **32** (2009), 454-469.

17. J. E. Muñoz Rivera and R. Racke, Mildly dissipative nonlinear Timoshenko systems-global existence and exponential stability, *J. Math. Anal. Appl.*, **276** (2002), 248-276.

18. J. E. Muñoz Rivera and R. Racke, Global stability for damped Timoshenko systems, *Discrete Contin. Dyn. Syst.*, **9** (2003), 1625-1639.

19. M. Nakao, Decay ofsolutions ofsome nonlinear evolution equations, *J. Math. Anal. Appl.*, **60** (1977), 542-549.

20. S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **45(5)** (2006), 1561-1585.

21. J. Y. Park and S. H. Park, General decay for a nonlinear beam equation with weak dissipation, *J. Math. Phys.*, **51(7)** (2010), 073508, 8 pp.

22. C. A. Raposo, J. Ferreira, M. L. Santos and N. N. O. Castro, Exponential stability for the Timoshenko system with two weak dampings, *Applied Math Letters*, **18** (2005), 535-541.

23. W. Rudin, *Real and Complex Analysis*, 2nd edition, McGraw-Hill, Inc., New York, 1974.

24. I. H. Suh and Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Autom. Control*, **25** (1980), 600-603.

25. S. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismaticbars, *Philisophical Magazine*, **41** (1921), 744-746.

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