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CONTACT CR-WARPED PRODUCT SUBMANIFOLDS OF NEARLY TRANS-SASAKIAN MANIFOLDS

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Abstract. Recently, many authors studied the relations between the squared norm of the second fundamental form (extrinsic invariant) and the warping function (intrinsic invariant) for warped product submanifolds (see [1, 7, 14]). Inspired by those relations we establish a general sharp inequality, namely $\|h\|^2 \geq 2s[\|\nabla lnf\|^2 + \alpha^2 - \beta^2]$, for contact CR-warped products of nearly trans-Sasakian manifolds. Our inequality generalizes all derived inequalities for contact CR-warped products either in any contact metric manifold. The equality case is also handled.

1. Introduction

A $(2n+1)$-dimensional differentiable manifold $\bar{M}$ of class C^∞ is said to have a contact structure (J.W. Gray [13]) if the structural group of its tangent bundle reduces to $U(n) \times 1$; equivalently (Sasaki and S. Hatakeyama [19]), an almost contact structure is given by a triple (ϕ, ξ, η) satisfying certain conditions. Many different types of almost contact structures are defined in the literature like cosymplectic, Sasakian, quasi-Sasakian, normal contact, Kenmotsu, trans-Sasakian. These type of structures bear sufficient resemblance to cosymplectic and Sasakian structures. Later on, D. Chinea and C. Gonzalez generalized these structures into a general system. They divided almost contact metric manifolds into twelve well known classes. An almost contact metric manifold is nearly trans-Sasakian if it belongs to the class $\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$. Recently, C. Gherghe introduced a nearly trans-Sasakian structure of type (α, β), which generalizes trans-Sasakian structure in the same sense as nearly Sasakian generalizes Sasakian ones. Moreover, a nearly trans-Sasakian of type (α, β) is nearly-Sasakian [4] or nearly

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Kenmotsu [20] or nearly cosymplectic [3] according to $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$, respectively.

On the other hand, the idea of warped product submanifolds was introduced by Chen in [7] (see also [6, 8]). He studied the warped product CR-submanifolds of a Kaehler manifold. He proved many interesting results on the existence of warped products and established general sharp inequalities for the second fundamental form in terms of the warping function f. Later on, many articles have been appeared for the same inequalities in almost Hermitian as well as almost contact metric manifolds (see [1, 5, 17]). In this paper, we study the warped product contact CR-submanifolds of nearly trans-Sasakian manifolds. In the beginning, we prove some existence and non-existence results and then obtain a general sharp inequality for the second fundamental form in terms of the warping function f and the smooth functions $\alpha,\ \beta$ on a nearly trans-Sasakian manifold. The inequality obtained in this paper is more general as it generalizes all inequalities obtained for contact CR-warped products in contact metric manifolds.

2. Preliminaries

A $(2n+1)-$dimensional C^∞ manifold $\bar{M}$ is said to have an *almost contact structure* if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$, a vector field ξ and a $1-$form η satisfying [3]

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1. \tag{2.1}$$

There always exists a Riemannian metric g on an almost contact manifold $\bar{M}$ satisfying the following compatibility condition

$$\eta(X) = g(X, \xi),\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

where X and Y are vector fields on $\bar{M}$.

There are two known classes of almost contact metric manifolds, namely Sasakian and Kenmotsu manifolds. Sasakian manifolds are characterized by the tensorial relation $(\bar{\nabla}_X\phi)Y = g(X, Y)\xi - \eta(Y)X$, while the Kenmotsu manifolds are given by the tensor equation $(\bar{\nabla}_X\phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$.

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a trans-Sasakian structure [18] if $(\bar{M} \times R, J, G)$ belongs to the class W_4 of the Gray-Hervella classification of almost Hermitian manifolds [12], where J is the almost complex structure on $\bar{M} \times R$ defined by $J(X, ad/dt) = (\phi X - a\xi, \eta(X)d/dt)$, for all vector fields X on $\bar{M}$ and smooth functions a on $\bar{M} \times R$ and G is the product metric on $\bar{M} \times R$. This may be expressed by the condition

$$(\bar{\nabla}_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions α and β on $\bar{M}$, and we say that the trans-Sasakian structure is of type (α, β).

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a nearly trans-Sasakian structure [11] if

$$(\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y). \tag{2.3}$$

Moreover, a nearly trans-Sasakian of type (α, β) is nearly-Sasakian [4] or nearly Kenmotsu [20] or nearly cosymplectic [3] according as $\beta = 0,\ \alpha = 1$; or $\alpha = 0,\ \beta = 1$; or $\alpha = \beta = 0$, respectively.

The covariant derivative of the tensor filed ϕ is defined as

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi\bar{\nabla}_X Y. \tag{2.4}$$

Let M be submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g and let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$-module of smooth sections of a vector bundle TM over M, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.5}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.6}$$

for each $X,\ Y \in \Gamma(TM)$ and $N \in \Gamma(T^{\perp}M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into $\bar{M}$. They are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as induced on M.

Bishop and O'Neill [2] introduced the notion of warped product manifolds. They defined these manifolds as: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f be a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times {}_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{2.8}$$

A warped product manifold $N_1 \times {}_f N_2$ is said to be *trivial* if the warping function f is constant.

We recall the following general result for later use.

Lemma 2.1. ([2]). *Let $M = N_1 \times_f N_2$ be a warped product manifold with the warping function f, then*

(i) $\nabla_X Y \in \Gamma(TN_1)$ *is the lift of* $\nabla_X Y$ *on* N_1,

(ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,

(iii) $\nabla_Z W = \nabla^{N_2}_Z W - g(Z, W)\nabla \ln f$

for each $X, Y \in \Gamma(TN_1)$ *and* $Z, W \in \Gamma(TN_2)$*, where* $\nabla \ln f$ *is the gradient of* $\ln f$ *and* ∇ *and* ∇^{N_2} *denote the Levi-Civita connections on* M *and* N_2*, respectively.*

For a Riemannian manifold M of dimension n and a smooth function f on M, we recall ∇f, the gradient of f which is defined by

$$g(\nabla f, X) = X(f), \tag{2.9}$$

for any $X \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^{n} (e_i(f))^2 \tag{2.10}$$

for an orthonormal frame $\{e_1, \cdots, e_n\}$ on M.

3. Contact CR-Warped Product Submanifolds

In this section first we recall the invariant, anti-invariant and contact CR-submanifolds. For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following:

(i) A submanifold M tangent to ξ is an invariant submanifold if ϕ preserves any tangent space of M, that is, $\phi(T_pM) \subset T_pM$, for every $p \in M$.

(ii) A submanifold M tangent to ξ is an anti-invariant submanifold if ϕ maps any tangent space of M into the normal space , that is, $\phi(T_pM) \subset T_p^{\perp}M$, for every $p \in M$.

Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $p \in M$ there exists a maximal invariant subspace denoted by D_p of the tangent space T_pM of M. If the dimension of D_p is same for all values of $p \in M$, then D_p gives an invariant distribution D on M.

A submanifold M of an almost contact manifold $\bar{M}$ is said to be a *contact CR-submanifold* if there exists on M a differentiable distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, that is;

(i) $TM = D \oplus D^{\perp} \oplus \langle\xi\rangle$

(ii) D is an invariant distribution, i.e., $\phi D \subseteq TM$

(iii) $D^{\perp}$ is an anti-invariant distribution, i.e., $\phi D^{\perp} \subseteq T^{\perp}M$.

A contact CR-submanifold is anti-invariant if $D_p = \{0\}$ and invariant if $D_p^{\perp} = \{0\}$ respectively, for every $p \in M$. It is a proper contact CR-submanifold if neither $D_p = \{0\}$ nor $D_p^{\perp} = \{0\}$, for each $p \in M$.

If ν is the $\phi-$invariant subspace of the normal bundle $T^{\perp}M$, then in case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as

$$T^{\perp}M = \phi D^{\perp} \oplus \nu, \tag{3.1}$$

where ν is the $\phi-$invariant normal subbundle of $T^{\perp}M$.

In this section, we investigate the warped products $M = N_{\perp} \times_f N_T$ and $M = N_T \times_f N_{\perp}$, where N_T and $N_{\perp}$ are invariant and anti-invariant submanifolds of a nearly trans-Sasakian manifold $\bar{M}$, respectively. First we discuss the warped products $M = N_{\perp} \times_f N_T$, here two possible cases arise:

(i) ξ is tangent to N_T,

(ii) ξ is tangent to $N_{\perp}$.

We start with the case (i).

Theorem 3.1. *Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist warped product submanifolds $M = N_{\perp} \times_f N_T$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ is an anti-invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly $\alpha-$Sasakian.*

Proof. Consider $\xi \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_{\perp})$, then by the structure equation of nearly trans-Sasakian, we have $(\bar{\nabla}_Z\phi)\xi + (\bar{\nabla}_\xi\phi)Z = -\alpha Z - \beta\phi Z$. Using (2.4), we obtain $-\phi\bar{\nabla}_Z\xi + \bar{\nabla}_\xi\phi Z - \phi\bar{\nabla}_\xi Z = -\alpha Z - \beta\phi Z$. Then from Lemma 2.1(ii) and (2.5), we derive

$$\bar{\nabla}_\xi\phi Z - 2\phi h(Z,\xi) = -\alpha Z - \beta\phi Z. \tag{3.2}$$

Taking the inner product with ϕZ in (3.2) and then using (2.2) and the fact that $\xi \in \Gamma(TN_T)$, we get $\beta\|Z\|^2 = 0$, for non zero function smooth function β on $\bar{M}$ and hence we conclude that M is invariant, which proves the theorem. ∎

Now, we will discuss the other case, when ξ is tangent to $N_{\perp}$.

Theorem 3.2. *Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist warped product submanifolds $M = N_{\perp} \times_f N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to ξ and N_T is an invariant submanifold of $\bar{M}$, unless $\bar{M}$ is nearly $\beta-$Kenmotsu.*

Proof. Consider $\xi \in \Gamma(TN_{\perp})$ and $X \in \Gamma(TN_T)$, then we have$(\bar{\nabla}_X\phi)\xi + (\bar{\nabla}_\xi\phi)X = -\alpha X - \beta\phi X$. Using (2.4), we get

$$-\phi\bar{\nabla}_X\xi + \bar{\nabla}_\xi\phi X - \phi\bar{\nabla}_\xi X = -\alpha X - \beta\phi X. \tag{3.3}$$

Taking the inner product with X in (3.3) and using (2.2), (2.5), Lemma 2.1 (ii) and the fact that ξ is tangent to $N_\perp$, we obtain $\alpha\|X\|^2 = 0$, for some smooth function α on $\bar{M}$. Thus, we conclude that M is anti-invariant submanifold of a nearly trans-Sasakian manifold $\bar{M}$ otherwise $\bar{M}$ is nearly $\beta-$Kenmotsu. This completes the proof. ∎

Now, we will discuss the warped product $M = N_T \times_f N_\perp$ such that the structure vector field ξ is tangent to $N_\perp$.

Theorem 3.3. *Let $\bar{M}$ be a nearly trans-Sasakian manifold. Then there do not exist the warped product submanifolds $M = N_T \times_f N_\perp$ such that $N_\perp$ is an anti-invariant submanifold tangent to ξ and N_T an invariant submanifold of $\bar{M}$.*

Proof. If we consider $X \in \Gamma(TN_T)$ and the structure vector field ξ is tangent to $N_\perp$, then by (2.3), we have $(\bar{\nabla}_X\phi)\xi + (\bar{\nabla}_\xi\phi)X = -\alpha X - \beta\phi X$. Using (2.4), we obtain $\bar{\nabla}_\xi\phi X - \phi\bar{\nabla}_X\xi - \phi\bar{\nabla}_\xi X = -\alpha X - \beta\phi X$. Then by (2.5) and Lemma 2.1 (ii), we derive

$$(\phi X \ln f)\xi - 2\phi h(X,\xi) + h(\phi X,\xi) = -\alpha X - \beta\phi X. \tag{3.4}$$

Hence, the result is obtained by taking the inner product with ξ in (3.4). ∎

If we consider the structure vector field ξ tangent to N_T for the warped product $M = N_T \times_f N_\perp$, then we prove the following result for later use.

Lemma 3.1. *Let $M = N_T \times {}_f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then, we have*

(i) $\xi(\ln f) = \beta$,

(ii) $g(h(X,Y),\phi Z) = 0$,

(iii) $g(h(X,W),\phi Z) = g(h(X,Z),\phi W) = -\{(\phi X \ln f) + \alpha\eta(X)\}g(Z,W)$,

(iv) $g(h(\xi,Z),\phi W) = -\alpha g(Z,W)$

for every $X, Y \in \Gamma(TN_T)$ and $Z, W \in \Gamma(TN_\perp)$.

Proof. If ξ is tangent to N_T, then for any $Z \in \Gamma(TN_\perp)$, we have $(\bar{\nabla}_\xi\phi)Z + (\bar{\nabla}_Z\phi)\xi = -\alpha Z - \beta\phi Z$. Then from (2.4), (2.5) and Lemma 2.1 (ii), we obtain

$$2(\xi \ln f)\phi Z + 2\phi h(Z,\xi) - \bar{\nabla}_\xi\phi Z = \alpha Z + \beta\phi Z. \tag{3.5}$$

Taking the inner product with ϕZ in (3.5) and using (2.2), we derive

$$2(\xi \ln f)\|Z\|^2 - g(\bar{\nabla}_\xi\phi Z,\phi Z) = \beta\|Z\|^2. \tag{3.6}$$

On the other hand, by the property of Riemannian connection, we have $\xi g(\phi Z, \phi Z) = 2g(\bar{\nabla}_\xi \phi Z, \phi Z)$. By (2.2) and the property of Riemannian connection, we get

$$g(\bar{\nabla}_\xi Z, Z) = g(\bar{\nabla}_\xi \phi Z, \phi Z). \tag{3.7}$$

Using this fact in (3.6) and then from (2.5) and Lemma 2.1 (ii), we deduce that $(\xi \ln f)\|Z\|^2 = \beta\|Z\|^2$, for any $Z \in \Gamma(TN_\perp)$, which gives (i). For the other parts of the lemma, we have $(\bar{\nabla}_X \phi)Z + (\bar{\nabla}_Z \phi)X = -\alpha\eta(X)Z - \beta\eta(X)\phi Z$, for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\perp)$. Using (2.4), (2.5) and (2.6), we derive

$$\begin{aligned} \alpha\eta(X)Z + \beta\eta(X)\phi Z = {} & A_{\phi Z}X - \nabla^\perp{}_X \phi Z + 2(X \ln f)\phi Z \\ & -(\phi X \ln f)Z - h(\phi X, Z) + 2\phi h(X, Z) \end{aligned} \tag{3.8}$$

Thus, the second part can be obtained by taking the inner product in (3.8) with Y, for any $Y \in \Gamma(TN_T)$. Again, taking the inner product in (3.8) with W for any $W \in \Gamma(TN_\perp)$, we get

$$\begin{aligned} \alpha\eta(X)g(Z, W) = {} & g(h(X, W), \phi Z) - (\phi X \ln f)g(Z, W) \\ & -2g(h(X, Z), \phi W). \end{aligned} \tag{3.9}$$

By polarization identity, we get

$$\begin{aligned} \alpha\eta(X)g(Z, W) = {} & g(h(X, Z), \phi W) - (\phi X \ln f)g(Z, W) \\ & -2g(h(X, W), \phi Z). \end{aligned} \tag{3.10}$$

Then from (3.9) and (3.10), we obtain

$$g(h(X, Z), \phi W) = g(h(X, W), \phi Z), \tag{3.11}$$

which is the first equality of (iii). Using (3.11) either in (3.9) or in (3.10), we get the second equality of (iii). Now, for the last part, replacing X by ξ in the third part of this lemma. This proves the lemma completely. ■

Now, we have the following characterization theorem.

Theorem 3.4. *Let M be a contact CR-submanifold of a nearly trans-Sasakian manifold $\bar{M}$ with integrable invariant and anti-invariant distribution $D \oplus \langle\xi\rangle$ and $D^\perp$. Then M is locally a contact CR-warped product if and only if the shape operator of M satisfies*

$$A_{\phi W}X = -(\phi X \mu)W - \alpha\eta(X)W, \ \forall \ X \in \Gamma(D \oplus \langle\xi\rangle), \ W \in \Gamma(D^\perp) \tag{3.12}$$

for some smooth function μ on M satisfying $V(\mu) = 0$ for every $V \in \Gamma(D^\perp)$.

Proof. Direct part follows from the Lemma 3.1 (iii). For the converse, suppose that M is contact CR-submanifold satisfying (3.12), then we have $g(h(X, Y)\phi W) =$

$g(A_{\phi W}X, Y) = 0$, for any $X, Y \in \Gamma(D \oplus \langle\xi\rangle)$ and $W \in \Gamma(D^{\perp})$. Using (2.2) and (2.5), we get $g(\bar{\nabla}_X Y, \phi W) = -g(\phi\bar{\nabla}_X Y, W) = 0$. Then from (2.4), we obtain

$$g((\bar{\nabla}_X\phi)Y, W) = g(\bar{\nabla}_X\phi Y, W). \tag{3.13}$$

Similarly, we have

$$g((\bar{\nabla}_Y\phi)X, W) = g(\bar{\nabla}_Y\phi X, W). \tag{3.14}$$

Then from (3.13) and (3.14), we derive

$$g((\bar{\nabla}_X\phi)Y + (\bar{\nabla}_Y\phi)X, W) = g(\bar{\nabla}_X\phi Y + \bar{\nabla}_Y\phi X, W). \tag{3.15}$$

Using (2.3) and the fact that ξ is tangent to N_T, then by orthogonality of two distributions, we obtain

$$g(\bar{\nabla}_X\phi Y + \bar{\nabla}_Y\phi X, W) = 0. \tag{3.16}$$

This means that $\nabla_X\phi Y + \nabla_Y\phi X \in \Gamma(D \oplus \langle\xi\rangle)$, for any $X, Y \in \Gamma(D \oplus \langle\xi\rangle)$, that is $D\oplus\langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. So far as the anti-invariant distribution $D^{\perp}$ is concerned, it is integrable on M (cf. [16], Theorem 8.1). Let $N_{\perp}$ be the leaf of $D^{\perp}$ and $h^{\star}$ be the second fundamental form of $N_{\perp}$ in M. Then for any $X \in \Gamma(D \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(D^{\perp})$, we have $g(h^{\star}(Z, W), \phi X) = g(\nabla_Z W, \phi X)$. Using (2.2), (2.4) and (2.5), we obtain $g(h^{\star}(Z, W), \phi X) = g((\bar{\nabla}_Z\phi)W, X) - g(\bar{\nabla}_Z\phi W, X)$. Then from (2.6) and (2.7), we get

$$g(h^{\star}(Z, W), \phi X) = g((\bar{\nabla}_Z\phi)W, X) + g(A_{\phi W}X, Z). \tag{3.17}$$

Using (3.12), we derive

$$g(h^{\star}(Z, W), \phi X) = g((\bar{\nabla}_Z\phi)W, X) + \{(\phi X)\mu - \alpha\eta(X)\}g(Z, W). \tag{3.18}$$

Similarly, we obtain

$$g(h^{\star}(Z, W), \phi X) = g((\bar{\nabla}_W\phi)Z, X) + \{(\phi X)\mu - \alpha\eta(X)\}g(Z, W). \tag{3.19}$$

Then from (3.18) and (3.19), we get

$$\begin{aligned} 2g(h^{\star}(Z, W), \phi X) = {} & g((\bar{\nabla}_Z\phi)W + (\bar{\nabla}_W\phi)Z, X) \\ & +2\{(\phi X)\mu - \alpha\eta(X)\}g(Z, W). \end{aligned} \tag{3.20}$$

Using the structure equation of nearly trans-Sasakian and the fact that ξ is tangent to N_T, we obtain

$$2g(h^{\star}(Z, W), \phi X) = 2\alpha g(Z, W)g(\xi, X) + 2\{(\phi X)\mu - \alpha\eta(X)\}g(Z, W). \tag{3.21}$$

That is

$$g(h^{\star}(Z,W),\phi X)=(\phi X)\mu g(Z,W). \tag{3.22}$$

Using (2.9), we derive

$$g(h^{\star}(Z,W),\phi X)=g(\nabla\mu,\phi X)g(Z,W). \tag{3.23}$$

From the last relation, we obtain that

$$h^{\star}(Z,W)=(\nabla\mu)g(Z,W). \tag{3.24}$$

The above relation shows that the leaves of $D^{\perp}$ are totally umbilical in M with mean curvature vector $\nabla\mu$. Moreover, the condition $V\mu=0$, for any $V\in\Gamma(D^{\perp})$ implies that the leaves of $D^{\perp}$ are extrinsic spheres in M, that is the integral manifold $N_{\perp}$ of $D^{\perp}$ is umbilical and its mean curvature vector field is non zero and parallel along $N_{\perp}$. Hence, by a result of [15] M is locally a warped product $M=N_T\times {}_fN_{\perp}$, where N_T and $N_{\perp}$ denote the integral manifolds of the distributions $D\oplus\langle\xi\rangle$ and $D^{\perp}$, respectively and f is the warping function. Thus, the theorem is proved completely. ■

4. Inequality for Contact CR-Warped Products

In the following section we obtain a general sharp inequality for the length of second fundamental form of warped product submanifold. We prove the following main result of this section.

Theorem 4.1. *Let $M=N_T\times {}_fN_{\perp}$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that N_T is an invariant submanifold tangent to ξ and $N_{\perp}$ an anti-invariant submanifold of $\bar{M}$. Then, we have*

(i) *The second fundamental form of M satisfies the inequality*

$$||h||^2\geq 2s[\|\nabla lnf\|^2+\alpha^2-\beta^2] \tag{4.1}$$

where s is the dimension of $N_{\perp}$ and $\nabla\ln f$ is the gradient of $\ln f$.

(ii) *If the equality sign of (4.1) holds identically, then N_T is a totally geodesic submanifold and $N_{\perp}$ is a totally umbilical submanifold of $\bar{M}$. Moreover, M is a minimal submanifold in $\bar{M}$.*

Proof. Let $\bar{M}$ be a $(2n+1)-$ dimensional nearly trans-Sasakian manifold and $M=N_T\times {}_fN_{\perp}$ be an $m-$dimensional contact CR-warped product submanifolds of $\bar{M}$. Let us consider $dimN_T=2p+1$ and $dimN_{\perp}=s$, then $m=2p+1+s$. Let $\{e_1\cdots,e_p;\phi e_1=e_{p+1},\cdots,\phi e_p=e_{2p},e_{2p+1}=\xi\}$ and $\{e_{(2p+1)+1},\cdots,e_m\}$ be the local orthonormal frames on N_T and $N_{\perp}$, respectively. Then the orthonormal frames in the normal bundle $T^{\perp}M$ of $\phi D^{\perp}$ and ν are $\{\phi e_{(2p+1)+1}, \cdots, \phi e_m\}$ and $\{e_{m+s+1}, \cdots, e_{2n+1}\}$, respectively. Then the length of second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2. \tag{4.2}$$

For the assumed frames, the above equation can be written as

$$\|h\|^2 = \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2 + \sum_{r=m+s+1}^{2n+1} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2. \tag{4.3}$$

The first term in the right hand side of the above equality is the $\phi D^{\perp}-$component and the second term is $\nu-$component. If we equate only the $\phi D^{\perp}-$component, then we have

$$\|h\|^2 \geq \sum_{r=m+1}^{m+s} \sum_{i,j=1}^{m} g(h(e_i, e_j), e_r)^2. \tag{4.4}$$

For the given frame of $\phi D^{\perp}$, the above equation will be

$$\|h\|^2 \geq \sum_{k=(2p+1)+1}^{m} \sum_{i,j=1}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

Let us decompose the above equation in terms of the components of $h(D, D)$, $h(D, D^{\perp})$ and $h(D^{\perp}, D^{\perp})$, then we have

$$\begin{aligned} \|h\|^2 \geq & \sum_{k=2p+2}^{m} \sum_{i,j=1}^{2p+1} g(h(e_i, e_j), \phi e_k)^2 \\ & +2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2 \\ & + \sum_{k=2p+2}^{m} \sum_{i,j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2. \end{aligned} \tag{4.5}$$

By Lemma 3.1 (ii), the first term of the right hand side of (4.5) is identically zero and we shall compute the next term and will left the last term

$$\|h\|^2 \geq 2 \sum_{k=2p+2}^{m} \sum_{i=1}^{2p+1} \sum_{j=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

As $j, k = 2p + 2, \cdots, m$, then the above equation can be written for one summation as

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} g(h(e_i, e_j), \phi e_k)^2.$$

Making use of Lemma 3.1 (iii), the above inequality will be

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f + \alpha\eta(e_i))^2 g(e_j, e_k)^2. \tag{4.6}$$

The above expression can be written as

$$\begin{aligned} \|h\|^2 \geq\; & 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 \\ & +2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\alpha\eta(e_i))^2 g(e_j, e_k)^2 \\ & +4\alpha \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)\eta(e_i) g(e_j, e_k)^2. \end{aligned} \tag{4.7}$$

The last term of (4.7) is identically zero for the given frames. Thus, the above relation gives

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + 2\alpha^2 s. \tag{4.8}$$

On the other hand, from (2.10), we have

$$\|\nabla \ln f\|^2 = \sum_{i=1}^{p} (e_i lnf)^2 + \sum_{i=1}^{p} (\phi e_i \ln f)^2 + (\xi \ln f)^2. \tag{4.9}$$

Now, the equation (4.8) can be modified as

$$\begin{aligned} \|h\|^2 \geq\; & 2 \sum_{i=1}^{2p} \sum_{j,k=2p+2}^{m} (\phi e_i \ln f)^2 g(e_j, e_k)^2 + 2 \sum_{j,k=2p+1}^{m} (\xi \ln f)^2 g(e_j, e_k)^2 \\ & +2\alpha^2 s - 2 \sum_{j,k=2p+1}^{m} (\xi \ln f)^2 g(e_j, e_k)^2, \end{aligned}$$

or

$$\|h\|^2 \geq 2\alpha^2 s - 2 \sum_{j,k=2p+2}^{m} (\xi \ln f)^2 g(e_j, e_k)^2$$

$$+2\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(\phi e_i \ln f)^2 g(e_j,e_k)^2$$
$$+2\sum_{i=1}^{p}\sum_{j,k=2p+2}^{m}(e_i \ln f)^2 g(e_j,e_k)^2$$
$$+2\sum_{j,k=2p+2}^{m}(\xi \ln f)^2 g(e_j,e_k)^2,\ \text{(since } \phi\xi\ln f = 0).$$

Therefore, using Lemma 3.1 (i) and (4.9), we arrive at

$$\|h\|^2 \geq 2s\alpha^2 - 2s\beta^2 + 2s\|\nabla \ln f\|^2,$$

which is the inequality (4.1). Let $h^\star$ be the second fundamental form of $N_\perp$ in M, then from (3.24), we have

$$h^\star(Z,W) = g(Z,W)\nabla \ln f, \tag{4.10}$$

for any $Z, W \in \Gamma(D^\perp)$. Now, assume that the equality case of (4.1) holds identically. Then from (4.3), (4.5) and (4.7), we obtain

$$h(D,D) = 0,\ h(D^\perp, D^\perp) = 0,\ h(D, D^\perp) \subset \phi D^\perp. \tag{4.11}$$

Since N_T is a totally geodesic submanifold in M (by Lemma 2.1 (i)), using this fact with the first condition in (4.11) implies that N_T is totally geodesic in $\bar{M}$. On the other hand, by direct calculations same as in the proof of Theorem 3.4, we deduce that $N_\perp$ is totally umbilical in M. Therefore, the second condition of (4.11) with (4.10) implies that $N_\perp$ is totally umbilical in $\bar{M}$. Moreover, all three conditions of (4.11) imply that M is minimal submanifold of $\bar{M}$. This completes the proof of the theorem. ■

References

1. K. Arslan, R. Ezentas, I. Mihai and C. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, *J. Korean Math. Soc.*, **42** (2005), 1101-1110.

2. R. L. Bishop and B. O'Neill, Manifolds of negative curvature, *Trans. Amer. Math. Soc.*, **145** (1969), 1-49.

3. D. E. Blair, Almost contact manifolds with Killing structure tensors I, *Pacific J. Math.*, **39** (1971), 285-292.

4. D. E. Blair, D. K. Showers and K. Yano, Nearly Sasakian structures, *Kodai Math. Sem. Rep.*, **27** (1976), 175-180.

5. V. Bonanzinga and K. Matsumoto, Warped product CR-submanifolds in locally conformal Kaehler manifolds, *Periodica Math. Hungar.*, **48** (2004), 207-221.

6. B. Y. Chen, On isometric minimal immersions from warped products into real space forms, *Proc. Edinb. Math. Soc.*, **45** (2002), 579-587.

7. B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds I, *Monatsh. Math.*, **133** (2001), 177-195.

8. B. Y. Chen, Geometry of warped product CR-submanifolds in Kaehler manifolds II, *Monatsh. Math.*, **134** (2001), 103-119.

9. B. Y. Chen and M. I. Munteanu, Geometry of $PR-$warped products in para-Kaehler manifolds, *Taiwan. J. Math.*, **16** (2012), 1293-1327.

10. D. Chinea and C. Gonzalez, A classification of almost contact metric manifolds, *Ann. Mat. Pura Appl.*, **156** (1990), 15-36.

11. C. Gherghe, Harmonicity of nearly trans-Sasaki manifolds, *Demonstratio. Math.*, **33** (2000), 151-157.

12. A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, *Ann. Mat. Pura Appl.*, **123** (1980), 35-58.

13. J. W. Gray, Some global properties of contact structures, *Ann. Math.*, **69** (1959), 421-450.

14. I. Hasegawa and I. Mihai, Contact CR-warped product submanifolds in Sasakian manifolds, *Geom. Dedicata*, **102** (2003), 143-150.

15. S. Hiepko, Eine inner kennzeichungder verzerrten produkte, *Math. Ann.*, **241** (1979), 209-215.

16. J. S. Kim, X. Liu and M. M. Tripathi, On semi-invariant submanifolds of nearly trans-Sasakian manifolds, *Int. J. Pure Appl. Math. Sci.*, **1** (2004), 15-34.

17. I. Mihai, Contact CR-warped product submanifolds in Sasakian space forms, *Geom. Dedicata*, **109** (2004), 165-173.

18. J. A. Oubina, New classes of almost contact metric structures, *Publ. Math. Debrecen*, **32** (1985), 187-193.

19. S. Sasaki and Y. Hatakeyama, On the differentiable manifolds with certain structures which are closely related to almost contact structure II, *Tohoku. Math. J.*, **13** (1961), 281-294.

20. M. M. Tripathi and S. S. Shukla, Semi-invariant submanifolds of nearly Kenmotsu manifolds, *Bull. Calcutta Math. Soc.*, **95** (2003), 17-30.

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