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Periodic Solutions of Sublinear Impulsive Differential Equations

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Abstract. In this paper, we consider sublinear second order differential equations with impulsive effects. Basing on the Poincaré-Bohl fixed point theorem, we first will prove the existence of harmonic solutions. The existence of subharmonic solutions is also obtained by a new twist fixed point theorem recently established by Qian etc in 2015 [18].

1. Introduction

We are concerned in this paper with the existence of periodic solutions for the sublinear impulsive differential equation

$$\begin{cases} x'' + g(x) = p(t, x, x') & t \neq t_j, \\ \Delta x|_{t=t_j} = ax(t_j-), \\ \Delta x'|_{t=t_j} = ax'(t_j-) & j = \pm 1, \pm 2, \ldots, \end{cases} \tag{1.1}$$

where $0 \leq t_1 < \cdots < t_k < 2\pi$, $a > 0$ is a constant, $\Delta x|_{t=t_j} = x(t_j+) - x(t_j-)$ and $\Delta x'|_{t=t_j} = x'(t_j+) - x'(t_j-)$. In addition, we assume that the impulsive time is 2π-periodic, that is, $t_{j+k} = t_j + 2\pi$ for $j = \pm 1, \pm 2, \ldots$, and g is a continuous function with the sublinear growth condition

$$(g_0): \lim_{|x| \to +\infty} \frac{g(x)}{x} = 0.$$

This problem comes from Duffing's equation

$$x'' + g(x) = p(t), \tag{1.2}$$

and there is a wide literature dealing with this problem not only because its physical significance, but also the application of various mathematical techniques on it, such as

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Poincaré-Birkhoff twist theorem in [6, 8, 9], variational method in [1, 11] and topological degree or index theories in [3, 4]. Under different assumptions on the function g, for example being superlinear, subliner, superquadratic potential and so on, there are many interesting results on the existence and multiplicity of periodic solutions of (1.2), see [5, 16, 17, 20, 21] and the references therein.

Recently, as impulsive equations widely arise in applied mathematics, they attract a lot of attentions and many authors study the general properties of them in [2, 10], along with the existence of periodic solutions of impulsive differential equations via fixed point theory in [13, 14], topological degree theory in [7, 19], and variational method in [15, 22]. However, different from the extensive study for second order differential equations without impulsive terms, there are only a few results on the existence and multiplicity of periodic solutions for impulsive second order differential equations.

In [18], Qian etc considered the superlinear impulsive differential equation

$$\begin{cases} x'' + g(x) = p(t, x, x') & t \neq t_j, \\ \Delta x|_{t=t_j} = I_j(x(t_j-), x'(t_j-)), & \\ \Delta x'|_{t=t_j} = J_j(x(t_j-), x'(t_j-)) & j = \pm 1, \pm 2, \ldots, \end{cases} \tag{1.3}$$

where $0 \le t_1 < \cdots < t_k < 2\pi$, $\Delta x|_{t=t_j} = x(t_j+) - x(t_j-)$, $\Delta x'|_{t=t_j} = x'(t_j+) - x'(t_j-)$, and $I_j, J_j \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous maps for $j = \pm 1, \pm 2, \ldots$. In addition, they assume that the impulsive time is 2π-periodic, that is, $t_{j+k} = t_j + 2\pi$ for $j = \pm 1, \pm 2, \ldots$, and g is a continuous function with the superlinear growth condition

$$(g_0') : \lim_{|x| \to \infty} \frac{g(x)}{x} = +\infty.$$

The authors proved via the Poincaré-Birkhoff twist theorem the existence of infinitely many periodic solutions of (1.3) with $p = p(t)$ and also the existence of periodic solutions for non-conservative case with degenerate impulsive terms by developing a new twist fixed point theorem.

In this article, we discuss (1.1) with the sublinear condition (g_0), which is different from the superlinear case and there is few papers studying on it up to now. As we all know, the existence of impulses, even the most simple impulsive functions, may cause complicated dynamic phenomenons and bring difficulties to study. Here, we start with linear impulsive functions and investigate the existence of periodic solutions completely. The rest is organized as follows. In Section 2, we obtain the existence of harmonic solutions of (1.1) by Poincaré-Bohl fixed point theorem. In Section 3, some properties of Poincaré map of (1.1), including its derivative and rotation property, are considered and then the existence of subharmonic solutions is obtained according to a fixed point theorem coming from [18].

2. Harmonic solutions

It is well known that Duffing's equation (1.2) with the sublinear condition has at least one harmonic solution, see [12]. We will find that the existence of harmonic solutions also remains under the influence of special impulses, such as impulsive differential equation (1.1).

Theorem 2.1. *Suppose that in* (1.1) *the external force p is a bounded and continuous function with 2π-periodic in the first variable, and g is a locally Lipschitz continuous function satisfying the following conditions*

$$(g_0): \lim_{|x|\to+\infty} \frac{g(x)}{x} = 0$$

and

$$(g_1): \lim_{|x|\to+\infty} g(x)\,\mathrm{sign}(x) = +\infty, \qquad \lim_{x^+\to+\infty} \int_{x^-}^{x^+} \frac{ds}{\sqrt{G(x^+) - G(s)}} = +\infty,$$

where $x^- < 0 < x^+$ are two zeros of the potential $G(x) = \int_0^x g(s)\,ds$. Then there is at least one 2π-periodic solution of (1.1).

Before giving the proof of Theorem 2.1, we do some preparations. At first, some basic properties of impulsive differential equations will be given.

Consider the following initial value problem

$$\begin{cases} u' = f(t,u) & t \neq t_j, \\ \Delta u|_{t=t_j} = I_j(u(t_j-)) & j = \pm 1, \pm 2, \ldots, \\ u(t_0+) = u_0, \end{cases} \tag{2.1}$$

where $\Delta u|_{t=t_j} = u(t_j+) - u(t_j-)$, $j = \pm 1, \pm 2, \ldots$, and assume that

(1) $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in $(t_j, t_{j+1}] \times \mathbb{R}^n$, locally Lipschitz in the second variable and the limits $\lim_{t\to t_j+, v\to u} f(t,v)$, $j = \pm 1, \pm 2, \ldots$ exist,

(2) $I_j\colon \mathbb{R}^n \to \mathbb{R}^n$, $j = \pm 1, \pm 2, \ldots$, are continuous,

(3) f is 2π-periodic in the first variable, $0 \le t_1 < \cdots < t_k < 2\pi$, $t_{j+k} = t_j + 2\pi$ and $I_{j+k} = I_j$ for $j = \pm 1, \pm 2, \ldots$.

Then the following lemma holds.

Lemma 2.2. [2] *Assume that the conditions above hold. Then for any $x_0 \in \mathbb{R}^n$, there is a unique solution $u(t) = u(t; t_0, u_0)$ of* (2.1)*. Moreover, $P_t\colon u_0 \to u(t; t_0, u_0)$ is continuous in u_0 for $t \neq t_j$, $j = \pm 1, \pm 2, \ldots$.*

According to Lemma 2.2, we will further state some properties of the solutions of (2.1) and P_t in the next lemma, the proof can be found in [18].

Lemma 2.3. *Assume that the conditions of Lemma* 2.2 *hold. Then all solutions of* (2.1) *exist for* $t \in \mathbb{R}$ *provided that all solutions of* $u' = f(t,u)$ *exist for* $t \in \mathbb{R}$. *Moreover, if* $\Phi_j \colon u \to u + I_j(u)$, $j = 1,2,\ldots,k$, *are global homeomorphisms of* $\mathbb{R}^n$, *then* P_t *is a homeomorphism for* $t \neq t_j$, $j = \pm 1, \pm 2, \ldots$. *Furthermore, all solutions of* (2.1) *have elastic property, that is, for any* $b > 0$, *there is* $r_b > 0$ *such that the inequality* $|u_0| \geq r_b$ *implies* $|u(t;t_0,u_0)| \geq b$ *for* $t \in (t_0, t_0 + 2\pi]$.

In what follows, we write (1.1) as an equivalent system of the form

$$\begin{cases} x' = y, \quad y' = -g(x) + p(t,x,y) & t \neq t_j, \\ \Delta x|_{t=t_j} = a x(t_j-), \\ \Delta y|_{t=t_j} = a y(t_j-) & j = \pm 1, \pm 2, \ldots. \end{cases} \tag{2.2}$$

Let $x = x(t) = x(t;x_0,y_0)$, $y = y(t) = y(t;x_0,y_0)$ be the solution of (2.2) satisfying the initial value condition $x(0) = x_0$, $y(0) = y_0$. By Lemma 2.3, the solution $(x(t),y(t))$ exists for all $t \in \mathbb{R}$, and for any $b > 0$, there is r_b such that the inequality $r_0 = \sqrt{x_0^2 + y_0^2} \geq r_b$ implies $|r(t)| = \sqrt{x(t)^2 + y(t)^2} \geq b$ for $t \in [0,2\pi]$. Denote by

$$P \colon (x_0,y_0) \mapsto (x(2\pi;x_0,y_0), y(2\pi;x_0,y_0))$$

the Poincaré map of (2.2), then it is continuous in (x_0,y_0) and its fixed points are harmonic solutions of (2.2) correspondingly.

By the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

(2.2) has the following form

$$\begin{cases} \theta' = -\sin^2\theta - (g(r\cos\theta) - p(t,r\cos\theta,r\sin\theta))\cos\theta/r, \\ r' = r\cos\theta\sin\theta - (g(r\cos\theta) - p(t,r\cos\theta,r\sin\theta))\sin\theta & t \neq t_j, \\ \Delta\theta|_{t=t_j} = 0, \\ \Delta r|_{t=t_j} = a r(t_j-) & j = \pm 1, \pm 2, \ldots, \end{cases} \tag{2.3}$$

from which we can find $\theta(t)$ is continuous in $t \in \mathbb{R}$ actually. Next in the phase plane, we give some general descriptions of the trajectory of (2.2).

Lemma 2.4. *Assume that the conditions of Theorem* 2.1 *hold. If the initial value* r_0 *is large enough, then*

$$\theta'(t;\theta_0,r_0) < 0, \quad t \in [0,2\pi],\ t \neq t_j,\ j = 1,2,\ldots,k$$

with $(\theta_0,r_0) = (\theta(0),r(0))$.

Proof. From the discussion above, we know that for any $c > 0$, there is $d = d(c) > 0$ such that for $t \in [0, 2\pi]$,

$$r(t) = \sqrt{x(t)^2 + y(t)^2} \geq c \tag{2.4}$$

if $r_0 = \sqrt{x_0^2 + y_0^2} \geq d$. Suppose that (2.4) holds and c is a large number to be determined later.

In (2.3), we consider the first equation

$$\theta' = -\sin^2\theta - (g(r\cos\theta) - p(t, r\cos\theta, r\sin\theta))\cos\theta/r, \quad t \neq t_j,\ j = 1, 2, \dots, k.$$

Due to the condition (g_1) and the boundedness of p, there exists a constant $N > 0$ such that

$$\begin{cases} g(x) - p(t, x, y) > \frac{1}{2}g(x) & x \geq N, \\ g(x) - p(t, x, y) < \frac{1}{2}g(x) & x \leq -N, \end{cases}$$

and $x^{-1}g(x) > 0$ for $|x| \geq N$. Then

$$\theta' < -\sin^2\theta - \frac{1}{2}x^{-1}g(x)\cos^2\theta < 0, \quad |x| \geq N. \tag{2.5}$$

On the other hand, when $|x| \leq N$, since g and p are bounded, there is a $K > 0$ such that

$$-[g(x) - p(t, x, y)]\cos\theta \leq K, \quad t \in [0, 2\pi],$$

which implies that for $t \in [0, 2\pi]$ and $|x| \leq N$,

$$\theta' \leq K/r - \sin^2\theta.$$

We choose $c > N$ large enough in (2.4) satisfying $\pi/4 < \arccos(N/c) < \pi/2$. Denote

$$\alpha = \arccos\frac{N}{c},$$

then for $r(t) > c$ and $|x| \leq N$, we have $|\cos\theta| \leq N/c = \cos\alpha$ and

$$\sin^2\theta \geq \sin^2\alpha \geq \frac{1}{2},$$

which implies

$$\theta' \leq K/r - \frac{1}{2} < 0 \tag{2.6}$$

if $c > 2K$.

Combining (2.5) and (2.6) yields $\theta'(t; \theta_0, r_0) < 0$, $t \in [0, 2\pi]$, $t \neq t_j$, $j = 1, 2, \dots, k$, for r_0 large enough. Thus we have finished the proof of the lemma. $\square$

Lemma 2.5. *Assume that the condition (g_1) holds. Then*

$$-2\pi < \theta(2\pi;\theta_0,r_0) - \theta(0;\theta_0,r_0) < 0$$

with the initial value r_0 large enough.

Proof. Denote $\Gamma_{z_0} : z(t;z_0) = (x(t;z_0), y(t;z_0))$ the trajectory of (2.2) with the initial value condition $z_0 = (x_0, y_0)$ and $(\theta(t;\theta_0,r_0), r(t;\theta_0,r_0))$ the polar coordinates of $z(t;z_0)$.

As we know for $c_0 > 0$, there is $d_0 > 0$ such that the inequality $|z_0| \ge d_0$ implies $|z(t;z_0)| \ge c_0$ for $t \in [0,2\pi]$. Let d_0 large and by Lemma 2.4, one has $\Gamma_{z_0} \in D$, where

$$D = \left\{(x,y) \in \mathbb{R}^2 : |(x,y)| = \sqrt{x^2+y^2} \ge c_0\right\},$$

and

$$\theta'(t;\theta_0,r_0) < 0, \quad t \in [0,2\pi],\ t \ne t_j,\ j = 1,2,\ldots,k.$$

In what follows, we divide D into six subregion

$$\begin{aligned}
&D_1 = \{(x,y) \in D : |x| \le c_0, y > 0\}, && D_2 = \{(x,y) \in D : x \ge c_0, y \ge 0\},\\
&D_3 = \{(x,y) \in D : x \ge c_0, y \le 0\}, && D_4 = \{(x,y) \in D : |x| \le c_0, y < 0\},\\
&D_5 = \{(x,y) \in D : x \le -c_0, y \le 0\}, && D_6 = \{(x,y) \in D : x \le -c_0, y \ge 0\}.
\end{aligned}$$

Since for $t \in [0,2\pi]$ and $t \ne t_j$, $\theta'(t) < 0$ and for $t = t_j$, $\Delta\theta|_{t=t_j} = 0$, $j = 1,2,\ldots,k$, Γ_{z_0} moves by the following order:

$$D_1 \to D_2 \to D_3 \to D_4 \to D_5 \to D_6 \to D_1.$$

We first verify

$$-2\pi < \theta(t_j-;\theta_0,r_0) - \theta(t_{j-1}+;\theta_0,r_0) < 0, \quad j = 1,2,\ldots,k,k+1,$$

where $t_0 = 0$ and $t_{k+1} = 2\pi$. If it is not true, Γ_{z_0} moves at least one turn around the origin in D when $t \in (t_{j-1}, t_j)$ for some $j = 1,2,\ldots,k,k+1$. Therefore, there exists at least an interval $[s_1,s_2] \subset (t_{j-1},t_j)$, during which Γ_{z_0} goes through the region D_2. For the fixed c_0 and d_0 mentioned above, choose $d > d_0$ and there is a large constant R_0 such that the inequality $|z_0| > d > d_0$ implies $|z(t)| \ge R_0$, $t \in [0,2\pi]$. When $t \in (s_1,s_2)$,

$$x(t) > c_0, \quad \frac{dx(t)}{dt} = y(t) > 0,$$

and

$$x_1 = x(s_1) = c_0, \quad y(s_2) = 0, \quad x_2 = x(s_2) > R_0.$$

Since p is bounded, denote

$$m = \inf_{(t,x,y)\in[0,2\pi]\times\mathbb{R}\times\mathbb{R}} |p(t,x,y)|,$$

and then by (2.2),

$$y(t)\frac{dy(t)}{dt} = -g(x(t))\frac{dx(t)}{dt} + p(t,x(t),y(t))\frac{dx(t)}{dt} \geq -(g(x(t)) - m)\frac{dx(t)}{dt}.$$

Integrating the inequality above on the interval $[t, s_2] \subset [s_1, s_2]$, we have

$$\begin{aligned}\frac{1}{2}y^2(t) &\leq G(x_2) - G(x(t)) - m(x_2 - x(t))\\ &= [G(x_2) - G(x(t))]\left(1 - \frac{m}{g(\xi)}\right),\end{aligned}$$

where $\xi \in (x_1, x_2)$ by the mean value theorem. From the condition (g_1), one has

$$0 < \left(1 - \frac{m}{g(\xi)}\right) < 1$$

for R_0 large enough. Therefore

$$\frac{dx(t)}{dt} = y(t) \leq \sqrt{2(G(x_2) - G(x(t))}, \quad s_1 < t < s_2,$$

from which we deduce

$$s_2 - s_1 \geq \int_{c_0}^{x_2} \frac{dx}{\sqrt{2}\cdot\sqrt{(G(x_2) - G(x(t)))}}.$$

Notice that c_0 is a fixed constant and $x_2 > R_0$. By the condition (g_1), for R_0 sufficient large, $s_2 - s_1$ can be large enough, which implies $t_j - t_{j-1}$ cannot be bounded for $j = 1, 2, \ldots, k, k+1$. That is a contradiction. Then

$$-2\pi < \theta(t_j-; \theta_0, r_0) - \theta(t_{j-1}+; \theta_0, r_0) < 0, j = 1, 2, \ldots, k, k+1,$$

where $t_0 = 0$ and $t_{k+1} = 2\pi$.

Finally, from the proof above, together with the condition $\Delta\theta|_{t=t_j} = 0$, $j = 1, 2, \ldots, k$, Γ_{z_0} cannot go through the region D_2 for $t \in [0, 2\pi]$. Therefore

$$-2\pi < \theta(2\pi; \theta_0, r_0) - \theta(0; \theta_0, r_0) < 0$$

with the initial value r_0 large enough. □

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. By Lemma 2.5, for $|z_0| = d$ large enough, the argument of Γ_{z_0} satisfies

$$-2\pi < \theta(2\pi; z_0) - \theta(0; z_0) < 0.$$

Then on $D_d \triangleq \{z \in \mathbb{R}^2 : |z| \leq d\}$, the Poincaré map P of (2.2) has at least one fixed point ζ_0 by Poincaré-Bohl fixed point theorem. Correspondingly, $z(t; \zeta_0)$ is the 2π-periodic solution of (2.2). □

3. Subharmonic solutions

In this section, we discuss the existence of subharmonic solutions of (1.1), and we always assume that the conditions (g_0), (g_1) hold and

$$(g_2) : g'(x) > 0,\ x \in \mathbb{R}.$$

We now state the existence of subharmonic solutions of (1.1).

Theorem 3.1. *Assume that the conditions above hold. Then there is a positive integer sequence*

$$0 < n_1 < n_2 < \cdots < n_k < \cdots\ (\to \infty)$$

such that (1.1) *has at least one $2n_k\pi$-periodic solution for each n_k, $k = 1, 2, \ldots$.*

By Theorem 2.1, (2.2) has at least one harmonic solution: $x = \overline{x}(t)$, $y = \overline{y}(t)$. Let

$$x = u + \overline{x}(t), \quad y = v + \overline{y}(t),$$

then (2.2) can be transformed into

$$\begin{cases} u' = v, \quad v' = -H(t,u)u & t \neq t_j, \\ \Delta u|_{t=t_j} = au(t_j-), & \\ \Delta v|_{t=t_j} = av(t_j-) & j = \pm 1, \pm 2, \ldots, \end{cases} \tag{3.1}$$

where

$$H(t,u) = \int_0^1 g'(\overline{x}(t) + su)\, ds$$

is 2π-periodic in the first variable. By (g_2), $H(t,u) > 0$.

It is easy to see that $(u,v) = (0,0)$ is a trivial solution of (3.1). Let $(u_0, v_0) = (u(0), v(0)) \neq 0$, then the solution $(u(t), v(t))$ of (3.1) with the initial value (u_0, v_0) cannot intersect with the origin. Denote the solution by polar coordinates

$$u = \rho \cos\varphi, \quad v = \rho \sin\varphi.$$

If $\rho_0 = \sqrt{u_0^2 + v_0^2} \neq 0$, then $\rho = \rho(t) = \rho(t; \rho_0, \varphi_0) > 0$, where φ_0 is the argument of (u_0, v_0) with $\varphi_0 \in [0, 2\pi)$. From (3.1), it follows that

$$\Delta\varphi|_{t=t_j} = 0, \quad j = \pm 1, \pm 2, \ldots,$$

which implies that $\varphi = \varphi(t) = \varphi(t; \rho_0, \varphi_0)$ is continuous in $(t, \rho_0, \varphi_0) \in \mathbb{R} \times \mathbb{R}^+ \times [0, 2\pi)$.

Under polar coordinates,

$$\varphi' = -(H(t, \rho\cos\varphi)\cos^2\varphi + \sin^2\varphi), \quad t \neq t_j,\ j = \pm 1, \pm 2, \ldots. \tag{3.2}$$

Then for $\rho_0 > 0$,

$$\varphi'(t) < 0, \quad t \in \mathbb{R},\ t \neq t_j,\ j = \pm 1, \pm 2, \dots. \tag{3.3}$$

Next, we prove the following lemma.

Lemma 3.2. *The equality*

$$\lim_{t \to +\infty} \varphi(t) = -\infty \tag{3.4}$$

holds for $\rho_0 > 0$.

Proof. We first assume that $\rho = \rho(t)$ is bounded as $t \to +\infty$. Since $g'(\overline{x} + su)$ is bounded on $s \in [0, 1]$, H has a minimum H_0 such that

$$H(t, u) \geq H_0 > 0, \quad t \geq 0.$$

By (3.2), we have the estimate

$$\varphi'(t) \leq -(H_0 \cos^2 \varphi + \sin^2 \varphi) \leq -a,$$

where $a \triangleq \inf_{\varphi \in \mathbb{R}}(H_0 \cos^2 \varphi + \sin^2 \varphi) > 0$. This implies (3.4).

Secondly, we assume that $\rho = \rho(t)$ is unbounded as $t \to +\infty$. It follows from (3.3) that

$$\lim_{t \to +\infty} \varphi(t) = \varphi_1 \geq -\infty.$$

Then we just need to prove $\varphi_1 = -\infty$. Otherwise, we assume that $\varphi_0 > \varphi_1 > -\infty$. First, by (3.2), we have $\varphi_1 = k\pi$ (k is some integer). Indeed, if $\varphi_1 \neq k\pi$, then $\sin^2 \varphi_1 \neq 0$. By (3.2), for large enough t, we have

$$\varphi'(t) < -\sin^2 \varphi(t) < -\frac{1}{2} \sin^2 \varphi_1,$$

which means $\varphi(t) \to -\infty$ as $t \to +\infty$, and contradicts with the assumption.

Without loss of generality, let $k = 0$ and

$$\lim_{t \to +\infty} \varphi(t) = 0.$$

Then for (u_0, v_0) belonging to the first quadrant,

$$\frac{du}{dt} = v > 0, \quad \Delta u|_{t=t_j} = au(t_j-) > 0,$$

from which we know that

$$\lim_{t \to +\infty} u(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} v(t) = 0+.$$

We notice that $(\overline{x}(t), \overline{y}(t))$ is the harmonic solution of (2.2), therefore

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} (u(t) + \overline{x}(t)) = +\infty.$$

By (2.2) and the condition (g_1), as well as the boundedness of p, one has for t large enough, there is a constant $M > 0$ such that

$$y'(t) = (-g(x(t)) + p(t, x, y)) < -M,$$

which implies

$$\lim_{t\to+\infty} y(t) = -\infty. \tag{3.5}$$

On the other hand, since $\lim_{t\to+\infty} v(t) = 0+$, it follows that $y(t) = v(t) + \overline{y}(t)$ is bounded as $t \to +\infty$, which contradicts with (3.5). Thus $\varphi_1 = -\infty$ and the proof is finished. □

Generally speaking, if the external force p in (1.2) depends on the derivative x', then equation (1.2) is not conservative, and hence the Poincaré map of (1.2) is not area-preserving and the method via the Poincaré-Birkhoff twist theorem is invalid. To overcome this difficulty, Qian etc [18] established a new fixed point theorem, which is a partial extension of the Poincaré-Birkhoff twist theorem. As for the non-conservative equation (1.1), we will find that this theorem can be used to obtain the existence of subharmonic solutions. The fixed point theorem is stated as follows.

Lemma 3.3. [18] *Let Γ_- and Γ_+ be two convex closed curves surrounding the origin, $\operatorname{int}(\Gamma_-)$ and $\operatorname{int}(\Gamma_+)$ be the interior domain of Γ_- and Γ_+, respectively. Denote by $\mathcal{A}$ the annulus bounded by Γ_- and Γ_+. Consider a continuous map $F\colon \overline{\operatorname{int}(\Gamma_+)} \to \mathbb{R}^2$. Let*

$$E = \{z \in \mathcal{A} : |F(z)| \le |z|\},$$
$$J = \{z \in \mathcal{A} : F(z) \in \mathbb{R}^2 \setminus U(O), \langle Lz, F(z)\rangle = 0\},$$

where $U(O)$ is a small neighborhood of the origin O and L is a real orthogonal matrix with $\det(L) = 1$.

If for any curve γ connecting Γ_- and Γ_+, one has $\gamma \cap (J \cup E) \ne \emptyset$, then F has at least one fixed point in $\overline{\operatorname{int}(\Gamma_+)}$.

In the uv plane, let

$$B_r = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < r^2\},$$
$$C_r = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 = r^2\},$$

and $\Lambda_{\omega_0} : w(t; w_0) = (u(t), v(t))$ be the trajectory of (3.1) with the initial value $w_0 = (u_0, v_0) = (u(0), v(0))$. By Lemma 2.3, for $b > 0$, there is $c > 0$ such that $|w_0| \ge c$ implies

$|w(t)| \ge b$, $t \in [0, 2\pi]$. In order to give a description of the motion of Λ_{ω_0}, we consider the region

$$\Omega : |\varphi| < \frac{\pi}{4}, \quad \rho \ge b,$$

and estimate the time of Λ_{ω_0} going through Ω with $w_0 \in C_c$.

In Ω, one has

$$u = \rho \cos\varphi \ge b \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} b.$$

According to the condition (g_0), for any $\delta > 0$, there exists some b large enough such that

$$H(t, u) = \frac{g(\overline{x} + u) - g(\overline{x})}{u} < \delta^2 \quad \text{if } u \ge \frac{\sqrt{2}}{2} b.$$

Then in Ω and for $t \ne t_j$, $j = 1, 2, \dots, k$, by (3.2) we have

$$\begin{aligned} 0 > \varphi' &= -(H(t, \rho\cos\varphi)\cos^2\varphi + \sin^2\varphi) \\ &> -\delta^2 \cos^2\varphi - \sin^2\varphi. \end{aligned}$$

As for $t = t_j$, $j = 1, 2, \dots, k$, one has $\Delta\varphi|_{t=t_j} = 0$ which implies that $\varphi(t)$ is continuous in $t \in \mathbb{R}$. Then the time of Λ_{ω_0} going through Ω outside B_b is

$$\tau > \int_{\pi/4}^{-\pi/4} \frac{-d\varphi}{\delta^2 \cos^2\varphi + \sin^2\varphi} = \frac{2}{\delta} \arctan\frac{1}{\delta}.$$

Because

$$\lim_{\delta \to 0+} \frac{2}{\delta} \arctan\frac{1}{\delta} = +\infty,$$

then for any $n \in \mathbb{Z}^+$, we have

$$\tau > 2n\pi \tag{3.6}$$

if δ is sufficiently small (or the initial value c mentioned above is sufficiently large).

Lemma 3.4. *Let the Poincaré map of* (3.1) *be*

$$P_0 \colon (u_0, v_0) \to (u(2\pi; u_0, v_0), v(2\pi; u_0, v_0)).$$

Then for any $n \in \mathbb{Z}^+$, there are $c_n > 0$ and $n^* > n$ such that P_0^k $(k \le n)$ has no fixed point while $P_0^{n^*}$ has at least one fixed point outside B_{c_n}.

Proof. For any $n \in \mathbb{Z}^+$, we choose c_n large enough so that (3.6) holds, which implies that P_0^k $(k \le n)$ has no fixed point outside B_{c_n}.

Fix the c_n above. By (3.4), for $(u_0, v_0) \in C_{c_n}$, there is a positive integer $n^* > n$ such that

$$\varphi(2n^*\pi) - \varphi(0) < -2\pi + \frac{\pi}{2}.$$

On the other hand, (3.6) guarantees that there exists e_n ($e_n > c_n$) such that

$$-2\pi + \frac{\pi}{2} < \varphi(2n^*\pi) - \varphi(0) < 0, \quad (u_0, v_0) \in C_{e_n}.$$

Denote by $\mathcal{A} = \{(u,v) \in \mathbb{R}^2 : c_n^2 \leq u^2 + v^2 \leq e_n^2\}$ the annulus bounded by C_{c_n} and C_{e_n}. Let β be a curve connecting C_{c_n} and C_{e_n}, then there is $\omega^* \in \beta$ such that

$$\varphi(2n^*\pi; \varphi^*, \rho^*) - \varphi(0; \varphi^*, \rho^*) = -2\pi + \frac{\pi}{2},$$

where (φ^*, ρ^*) are the polar coordinates of ω^*. Therefore, choosing $L = \mathrm{id}$ in Lemma 3.3, we have

$$\langle L\omega^*, P_0^{n^*}(\omega^*)\rangle = 0,$$

which implies $\beta \cap J \neq \emptyset$. The continuous map $P_0^{n^*}$ meets all assumptions of Lemma 3.3. Thus $P_0^{n^*}$ has at least one fixed point ω_* in $\mathcal{A}$. □

Actually, Theorem 3.1 is the corollary of Lemma 3.4, since the fixed point of $P_0^{n^*}$ is also the $2n^*\pi$-periodic solutions of (3.1). Then we have finished the proof of Theorem 3.1.

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