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On the Average Size of an $(\overline{s},\overline{t})$-Core Partition

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Abstract. Let s and t be two coprime integers. Bessenrodt and Olsson obtained the number of $(\overline{s},\overline{t})$-cores for odd s and odd t by establishing a bijection between the lattice paths in (s,t) Yin-Yang diagram and $(\overline{s},\overline{t})$-cores. In this paper, motivated by their results, we extend the definition of Yin-Yang diagram and the bijection to all possible coprime pairs (s,t), then obtain that the number of $(\overline{s},\overline{t})$-cores is $\binom{\lfloor s/2\rfloor+\lfloor t/2\rfloor}{\lfloor s/2\rfloor}$. Furthermore, based on the identities of Chen-Huang-Wang, we determine the average size of an $(\overline{s},\overline{t})$-core depending on the parity of s, which is $(s-1)(t-1)(s+t-2)/48$ if s and t are both odd, or $(t-1)(s^2+st-3s+2t+2)/48$ if s is even and t is odd.

1. Introduction

The main purpose of this paper is to determine the number of all $(\overline{s},\overline{t})$-cores and the average size of an $(\overline{s},\overline{t})$-core for any coprime integers s and t. To this end, it is necessary to recall some basic definitions and notations of partitions and bar-core partitions.

A *partition* [2] of n is a finite nonincreasing sequence of positive integers $(\lambda_1,\lambda_2,\ldots,\lambda_m)$ such that $n=\lambda_1+\lambda_2+\cdots+\lambda_m$. We write $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, say the *size* $|\lambda|$ of λ is n and the *length* $\ell(\lambda)$ of λ is m. For $1\leq i\leq m$, λ_i is a *part* of λ and denote by $\lambda_i\in\lambda$. A partition $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_m)$ of n is called a *bar partition* if $\lambda_1>\lambda_2>\cdots>\lambda_m>0$, which implies that a bar partition consists of distinct parts. For positive integer t, a *$\overline{t}$-core partition*, or *$\overline{t}$-core*, is defined by *$\overline{t}$-abacus* [8] as follows. The $\overline{t}$-abacus has t runners running from north to south such that the i-th runner contains positions numbered $i, t+i, 2t+i,\ldots$ in increasing order from north to south. For $1\leq i\leq\lfloor(t-1)/2\rfloor$, the runners numbered i and $t-i$ are called conjugate, and a pair of conjugate runners is simply referred to as a runner pair. The abacus configuration of a bar partition is obtained by placing the parts of the partition as beads on the $\overline{t}$-abacus. A bar partition is a $\overline{t}$-core if and only if its abacus configuration satisfies the following four constraints:

i. the 0-th runner is has no beads;

ii. beads cannot occur simultaneously on a runner pair;

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iii. if t is even, then there is no restriction on the number of beads on runner $t/2$;

iv. a nonempty runner contains only beads in the top positions.

Equivalently, a bar partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a $\bar{t}$-core if and only if the parts λ_i satisfy the following conditions:

i. $t \nmid \lambda_i$ for $1 \le i \le m$;

ii. if t is odd, then $\lambda_i + \lambda_j \not\equiv 0 \pmod{t}$ for $1 \le i, j \le m$;

iii. if t is even, then $\lambda_i + \lambda_j \not\equiv 0 \pmod{t}$ for $1 \le i, j \le m$ except for $\lambda_i, \lambda_j \equiv t/2 \pmod{t}$;

iv. if $\lambda_i \in \lambda$, then $\lambda_i - t \in \lambda$ for $1 \le i \le m$.

A partition is an $(\bar{s}, \bar{t})$-*core* if it is simultaneous an $\bar{s}$-core and a $\bar{t}$-core.

After the structures of $\bar{t}$-core and $(\bar{s}, \bar{t})$-core posed, many results were obtained by combinatorial researchers. A bar partition λ has a $\bar{t}$-core $\lambda_{\bar{t}}$, which is obtained from λ by removing as many $\bar{t}$ from it as possible. The number of removed t-bars is the $\bar{t}$-weight. Bessenrodt [3] gave a classification of bar partitions of n that are of maximal $\bar{t}$-weight for all odd primes $t \le n$. In [9], Nath and Sellers obtained many Ramanujan-like congruences of $\bar{t}$-core partitions. Olsson [10] proved that if s and t are coprime odd integers, the $\bar{s}$-core of a $\bar{t}$-core partition is again a $\bar{t}$-core partition. Later, Gramain and Nath [6] generalized the results for partitions and bar-partitions by removing the restriction that s and t are coprime. Moreover, some combinatorial results related to quotient and representation are obtained in [7, 8].

Let s and t be two coprime odd integers. In [4], by introducing the combinatorial structures Yin diagram, Yang diagram and Yin-Yang diagram related with s and t, Bessenrodt and Olsson proved that the number of $(\bar{s}, \bar{t})$-cores is finite.

Theorem 1.1. [4, Theorem 3.2] *Let s, t be coprime odd integers. The number of $(\bar{s}, \bar{t})$-cores is given by*

$$\binom{u+v}{u},$$

where $u = (s-1)/2$ and $v = (t-1)/2$.

Based on the bijection given in [4, Section 3.], by studying a rectangular diagram $A(s,t)$ arising from the Yin-Yang diagram, we can compute the sum of the sizes of all $(\bar{s}, \bar{t})$-cores for coprime odd integers s and t. Then combining with Theorem 1.1, we are led to the average size of an $(\bar{s}, \bar{t})$-core with the above restriction on s and t.

Theorem 1.2. *Let s and t be coprime odd integers, then the average size of an $(\bar{s}, \bar{t})$-core is*

$$\frac{(s-1)(t-1)(s+t-2)}{48}.$$

Inspired by the work of Bessenrodt and Olsson [4], we naturally extend the concepts of Yin diagram, Yang diagram and Yin-Yang diagram only defined on coprime odd integers s and t to all possible coprime integers s and t. Since $\gcd(s,t)=1$, without loss of generality, assume that s is even then t is odd automatically. By establishing the bijection between the set of lattice paths in a rectangular diagram $B(s,t)$ and the set of $(\overline{s},\overline{t})$-cores, we obtain the theorem of the total number of $(\overline{s},\overline{t})$-cores.

Theorem 1.3. *Let s and t be coprime integers with even s. The number of $(\overline{s},\overline{t})$-cores is given by*

$$\binom{u+v}{u},$$

where $u=s/2$ and $v=(t-1)/2$.

By Theorem 1.3, using the similar technique in the proof of Theorem 1.2, we acquire the average size of an $(\overline{s},\overline{t})$-core for coprime integers s and t with even s, which together with Theorem 1.2, completely determine the average size the of an $(\overline{s},\overline{t})$-core for any coprime integers s and t.

Theorem 1.4. *Let s and t be coprime positive integers with even s, then the average size of an $(\overline{s},\overline{t})$-core is*

$$\frac{(t-1)(s^2+st-3s+2t+2)}{48}.$$

Note that combining Theorems 1.1 and 1.3, we have the following unified result on the enumeration of $(\overline{s},\overline{t})$-cores.

Theorem 1.5. *Let s and t be any coprime positive integers. The number of $(\overline{s},\overline{t})$-cores is given by*

$$\binom{\lfloor s/2\rfloor+\lfloor t/2\rfloor}{\lfloor s/2\rfloor}.$$

The rest of this paper is organized as follows. In Section 2, considering the case of s and t are coprime odd integers, we recall the main results of the Yin-Yang diagram in [4], then obtain the average size of an $(\overline{s},\overline{t})$-core (Theorem 1.2). For coprime positive integers s and t with even s, the corresponding Yin-Yang diagram is generalized and researched in Section 3, and the number of $(\overline{s},\overline{t})$-cores (Theorem 1.3) is also computed in this section. Finally, in Section 4, we solve the average size of an $(\overline{s},\overline{t})$-core (Theorem 1.4) with even constraint on s.

2. The average size of an $(\overline{s},\overline{t})$-core for coprime odd integers s and t

Throughout this section, we assume that $t>s>1$ are coprime odd integers and set $u=(s-1)/2$, $v=(t-1)/2$. To obtain the average size of an $(\overline{s},\overline{t})$-core, we begin with a review of the works on the structure of $(\overline{s},\overline{t})$-cores.

To calculate the number of $(\overline{s},\overline{t})$-cores, Bessenrodt and Olsson [4] defined the *Yin diagram* and the *Yang diagram* corresponding to $(\overline{s},\overline{t})$-cores.

Definition 2.1. Given an $(\overline{s},\overline{t})$-core, the corresponding Yin diagram and Yang diagram are two lower triangular arrays with positive integer entries that are determined by the following rule: for the Yin (resp. Yang) diagram, start with the largest entry $(s-1)t/2-s$ (resp. $(t-1)s/2-t$) in the lower-left corner and subtract multiples of s along the rows and multiples of t along the columns as long as possible.

From the above definition, it is easy to see that the number of rows of the Yin diagram and the Yang diagram are both u. Figures 2.1 and 2.2 show the details of the Yin diagram and the Yang diagram corresponding to $(\overline{9},\overline{17})$-core.

$$\begin{array}{ccccccc}
8 & & & & & & \\
25 & 16 & 7 & & & & \\
42 & 33 & 24 & 15 & 6 & & \\
59 & 50 & 41 & 32 & 23 & 14 & 5
\end{array}$$

Figure 2.1: The Yin diagram of $(\overline{9},\overline{17})$-core.

$$\begin{array}{ccccccc}
4 & & & & & & \\
21 & 12 & 3 & & & & \\
38 & 29 & 20 & 11 & 2 & & \\
55 & 46 & 37 & 28 & 19 & 10 & 1
\end{array}$$

Figure 2.2: The Yang diagram of $(\overline{9},\overline{17})$-core.

Bessenrodt and Olsson [4] proved that by rotating the Yang diagram 180° and combining with the Yin diagram, we can assemble a $u \times v$-rectangular diagram called the *Yin-Yang diagram*. The original Yin (resp. Yang) diagram can be referred as the Yin (resp. Yang) part of the Yin-Yang diagram. For example, the $(9,17)$ Yin-Yang diagram is presented in Figure 2.3, where the solid line is the boundary splitting the Yin part and the Yang part.

$$\begin{array}{cccccccc}
8 & 1 & 10 & 19 & 28 & 37 & 46 & 55 \\
25 & 16 & 7 & 2 & 11 & 20 & 29 & 38 \\
42 & 33 & 24 & 15 & 6 & 3 & 12 & 21 \\
59 & 50 & 41 & 32 & 23 & 14 & 5 & 4
\end{array}$$

Figure 2.3: The $(9,17)$ Yin-Yang diagram.

The following proposition gives the sum of entries in the (s,t) Yin-Yang diagram.

Proposition 2.2. [4, Proposition 3.7] *The sum of entries in the (s,t) Yin-Yang diagram* *is*

$$\frac{1}{6}uv(ut+vs-2).$$

In the proof of Theorem 1.1 (see [4, Theroem 3.2]), Bessenrodt and Olsson [4] constructed a bijection concerning with the lattice paths in the Yin-Yang diagram, which plays a essential role in the rest of paper.

Theorem 2.3 (Bessenrodt and Olsson [4])**.** *Given an (s,t) Yin-Yang diagram, there exists a bijection between the set of $(\overline{s},\overline{t})$-cores and the set of lattice paths in the Yin-Yang diagram from the upper-left corner to the lower-right corner. Moreover, for any lattice path, the corresponding $(\overline{s},\overline{t})$-core consists of the entries in the area enclosed by the lattice path and the boundary of the Yin part and the Yang part.*

For example, the dashed line in Figure 2.3 is a lattice path in $(9,17)$ Yin-Yang diagram that encloses an area together with the boundary of the Yin part and the Yang part. Then the entries in this area are $\{1,10,19,2,6,5\}$, which comprise the corresponding $(\overline{9},\overline{17})$-core $(19,10,6,5,2,1)$.

To simplify the computation of the average size of an $(\overline{s},\overline{t})$-core, we need to slightly adjust the Yin-Yang diagram. Bessenrodt and Olsson [4] verified that the sum of two adjacent entries in the left and right side of the boundary is exactly s. Therefore, given any (s,t) Yin-Yang diagram, flipping the Yin-Yang diagram horizontally and taking all entries in the Yin part into negative, we obtain a rectangular $u\times v$-array $A(s,t)=(A_{i,j})_{1\le i\le u,1\le j\le v}$ with entries

$$A_{i,j}=-\frac{s+1}{2}t+js+it \quad \text{for } 1\le i\le u \text{ and } 1\le j\le v, \tag{2.1}$$

where i ranges from top to bottom and j ranges from left to right. Let $A(s,t)=A$ and $\mathcal{P}(A)$ be the set of lattice paths in A from the lower-left corner to the upper-right corner. For a lattice path $P\in\mathcal{P}(A)$, denote by $M_A(P)$ the set of positive entries $A_{i,j}$ above P and the absolute values of negative entries $A_{i,j}$ below P. See Figure 2.4 for an example of $A=A(9,17)$, where the solid lines represent a lattice path P in $\mathcal{P}(A)$ and $M_A(P)=\{1,10,19,2,6,5\}$. By these notations, it is evident that the following corollary is straightly from Theorem 2.3.

-59	-50	-41	-32	-23	-14	-5	4
-42	-33	-24	-15	-6	3	12	21
-25	-16	-7	2	11	20	29	38
-8	1	10	19	28	37	46	55

Figure 2.4: A lattice path in the array $A(9,17)$.

Corollary 2.4. *Let $A = A(s,t)$ be the array defined by* (2.1)*. Then there exists a bijection Φ between the set $\mathcal{P}(A)$ of lattice paths and the set of $(\overline{s},\overline{t})$-cores such that for $P \in \mathcal{P}(A)$, the set of the parts of $\Phi(P)$ is given by $M_A(P)$.*

For any lattice path P in the array $A(s,t)$, denote by $\overline{P}$ (resp. $\underline{P}$) the set of cells (i,j) in $A(s,t)$ that are above (resp. below) P. Based on the bijection Φ, we obtain the following lemma.

Lemma 2.5. *Let $A = A(s,t)$, then for any lattice path $P \in \mathcal{P}(A)$, we have*

$$|\Phi(P)| = \sum_{(i,j)\in\overline{P}} A_{i,j} - \frac{uv}{6}(u+1)(1-2v).$$

Proof. By Corollary 2.4, we deduce that

$$\begin{aligned}|\Phi(P)| &= \sum_{\substack{(i,j)\in\overline{P}\\ A_{i,j}>0}} A_{i,j} - \sum_{\substack{(i,j)\in\underline{P}\\ A_{i,j}<0}} A_{i,j}\\ &= \sum_{(i,j)\in\overline{P}} A_{i,j} - \sum_{(i,j):\, A_{i,j}<0} A_{i,j}.\end{aligned}$$

From (2.1), it is clear that

$$\sum_{\substack{1\le i\le u\\ 1\le j\le v}} A_{i,j} = \sum_{\substack{1\le i\le u\\ 1\le j\le v}} \left(-\frac{s+1}{2}t + js + it\right) = \frac{uv}{2}(u-v). \tag{2.2}$$

Since the sum of absolute values of entries in $A(s,t)$ is exactly the same as the sum of entries in the corresponding (s,t) Yin-Yang diagram, by Proposition 2.2, we have

$$\sum_{\substack{1\le i\le u\\ 1\le j\le v}} |A_{i,j}| = \frac{uv}{6}(4uv+u+v-2). \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$\sum_{(i,j):\, A_{i,j}<0} A_{i,j} = \frac{1}{2}\sum_{\substack{1\le i\le u\\ 1\le j\le v}} (A_{i,j} - |A_{i,j}|) = \frac{uv}{6}(u+1)(1-2v),$$

which completes the proof. □

Let m and n be positive integers, and D_{mn} be an $m\times n$ rectangular diagram whose row index ranges from up to bottom and column index ranges form left to right, i.e., the coordinates of the cells of the first row are $(1,1),(1,2),\ldots,(1,n)$, and so on. Denote by $\mathcal{P}(D_{mn})$ the set of lattice paths from the lower-left corner to the upper-right corner of B_{mn}. Let $f(i,j)$ be the number of lattice paths in $\mathcal{P}(D_{mn})$ that lie below the cell (i,j), possibly touching the right or lower border of cell (i,j). In the investigation of the average size of a self-conjugate (s,t)-core, Chen, Huang and Wang [5] concluded the following identities.

Lemma 2.6. [5, Lemmas 2.4, 2.5] *For positive integers m and n, we have*

$$\sum_{\substack{1\le i\le m\\1\le j\le n}} f(i,j)=\binom{m+n}{m}\frac{mn}{2},\qquad \sum_{\substack{1\le i\le m\\1\le j\le n}} if(i,j)=\binom{m+2}{3}\binom{m+n}{m+1}$$

and

$$\sum_{\substack{1\le i\le m\\1\le j\le n}} jf(i,j)=\binom{n+2}{3}\binom{m+n}{n+1}.$$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $BC(s,t)$ denote the set of $(\overline{s},\overline{t})$-cores. By Corollary 2.4, we find that

$$\sum_{\lambda\in BC(s,t)}|\lambda|=\sum_{P\in\mathcal{P}(A)}|\Phi(P)|. \tag{2.4}$$

It follows from Lemma 2.5 that

$$\sum_{P\in\mathcal{P}(A)}|\Phi(P)|=\sum_{\mathcal{P}\in P(A)}\left(\sum_{(i,j)\in\overline{P}}A_{i,j}-\frac{uv}{6}(u+1)(1-2v)\right). \tag{2.5}$$

Combining (2.4) and (2.5), we arrive at

$$\sum_{\lambda\in BC(s,t)}|\lambda|=\sum_{P\in\mathcal{P}(A)}\sum_{(i,j)\in\overline{P}}A_{i,j}-\frac{uv}{6}(u+1)(1-2v)\binom{u+v}{u}$$

since $|\mathcal{P}(A)|=\binom{u+v}{u}$. According to (2.1) and Lemma 2.6, we get

$$\begin{aligned}\sum_{P\in\mathcal{P}(A)}\sum_{(i,j)\in\overline{P}}A_{i,j}&=\sum_{P\in\mathcal{P}(A)}\sum_{(i,j)\in\overline{P}}\left(-\frac{s+1}{2}t+js+it\right)\\&=s\sum_{\substack{1\le i\le u\\1\le j\le v}}jf(i,j)+t\sum_{\substack{1\le i\le u\\1\le j\le v}}if(i,j)-\frac{s+1}{2}t\sum_{\substack{1\le i\le u\\1\le j\le v}}f(i,j)\\&=s\binom{v+2}{3}\binom{u+v}{v+1}+t\binom{u+2}{3}\binom{u+v}{u+1}-\frac{s+1}{2}t\binom{u+v}{v}\frac{uv}{2}\\&=\left(\frac{uv}{6}(v+2)s+\frac{uv}{6}(u+2)t-\frac{uv(s+1)t}{4}\right)\binom{u+v}{v}\\&=\frac{uv}{6}(2u+1)(1-v)\binom{u+v}{v}.\end{aligned}$$

Hence, we have

$$\sum_{\lambda\in BC(s,t)} |\lambda| = \frac{uv}{6}((2u+1)(1-v)-(u+1)(1-2v))\binom{u+v}{u} = \frac{uv}{6}(u+v)\binom{u+v}{u}. \tag{2.6}$$

Substituting u by $(s-1)/2$ and v by $(t-1)/2$ in (2.6), combining with Theorem 1.1, we finally obtain that the average size of an $(\overline{s},\overline{t})$-core is given by

$$\frac{\sum_{\lambda\in BC(s,t)}|\lambda|}{\binom{u+v}{u}} = \frac{(s-1)(t-1)(s+t-2)}{48}.$$

The proof is completed. □

3. The number of $(\overline{s},\overline{t})$-cores for coprime integers s and t with even s

To obtain the number of $(\overline{s},\overline{t})$-cores for coprime pairs (s,t) with even s, we begin with extending the concept of Yin-Yang diagram defined only for odd coprime integers s and t to all possible coprime integers s and t.

In the rest of the paper, unless otherwise stated, we always assume that s and t are coprime and s is even. Thus, we may set $u = s/2$, $v = (t-1)/2$. Similar as the Yin-Yang diagram and the array $A(s,t)$ in Section 2, we build a $u\times v$-rectangular integer array $B(s,t) = (B_{i,j})_{1\le i\le u, 1\le j\le v}$ such that

$$B_{i,j} = -\frac{s+2}{2}t + js + it \quad \text{for } 1\le i\le u \text{ and } 1\le j\le v, \tag{3.1}$$

where i ranges from top to bottom and j ranges from left to right. Following the approach of Bessenrodt and Olsson [4], we show that the absolute values of entries in $B(s,t)$ are exactly the possible parts of all $(\overline{s},\overline{t})$-cores.

Lemma 3.1. *The set of the absolute values of entries in $B(s,t)$ contains all integers which may occur as parts of an $(\overline{s},\overline{t})$-core λ.*

Proof. Let $\mathbb{N}$ (resp. $\mathbb{N}^+$) be the set of all nonnegative (resp. positive) integers. By the definition of an $(\overline{s},\overline{t})$-core, if a is a part of λ, then a cannot be represented in the form $ks+\ell t$, where $k,\ell\ge 0$. Thus the parts of λ are contained in the set

$$X_{s,t} = \mathbb{N}\setminus\{ks+\ell t \mid k,\ell\ge 0\}.$$

According to [1, 11], we can rewrite $X_{s,t}$ in the form

$$X_{s,t} = \mathbb{N}^+ \cap \{st-s-t-(ks+\ell t) \mid k,\ell\ge 0\}. \tag{3.2}$$

Depending on the values of s and t, the proof is divided into three cases.

Case 1*:* $t > s = 2$. By (3.2), we have $X_{2,t} = \mathbb{N}^+ \cap \{t-2k \mid k \geq 1\}$. Notice that the set of $\overline{2}$-cores consists of partitions $\lambda^{(k)} = (2k-1, 2k-3, \ldots, 1)$ for $k \geq 1$ and empty partition $\lambda^{(0)}$. Thus $X_{2,t}$ contains all integers which may occur as parts of an $(\overline{2}, \overline{t})$-core since $\lambda^{(k)}$ must be a $(\overline{2}, \overline{t})$-core provided $2k-1 < t$. On the other hand, $B(2,t) = (B_{1,j})_{1\leq j\leq v}$ with $B_{1,j} = 2j - t$. Hence, it is clear that the set of absolute values of $B_{1,j} = 2j - t$ coincident with $X_{2,t}$.

Before giving the detailed proof of Cases 2 and 3, for $s \geq 4$, note that

$$\frac{s}{2} + t = st - s - t - \left(\frac{t-3}{2}s + \frac{s-4}{2}t\right) \in X_{s,t},$$

we claim that $s/2 + t$ cannot be a part of any $(\overline{s}, \overline{t})$-core λ. Suppose that $s/2 + t$ is a part of an $(\overline{s}, \overline{t})$-core λ. If $s < 2t$, then $t - s/2 = s/2 + t - s$ is also a part of λ since λ is an $\overline{s}$-core. Adding these two parts together, we get $s/2 + t + t - s/2 = 2t$, which is a contradiction since λ is an $\overline{t}$-core. Otherwise $s > 2t$, then $s/2 - t = (s/2 + t) - 2t$ is also a part of λ since it is a $\overline{t}$-core. Similarly, we find $s/2 + t + s/2 - t = s$, contradicting with the fact that λ is also an $\overline{s}$-core.

Motivating by the constructions in [4, 11], we arrange the elements of $X_{s,t}$ in an (s,t)-diagram in the following way. Start with the largest element $st - s - t$ in the lower-left corner and subtract multiples of s along the rows and multiples of t along the columns as long as possible (see Figure 3.1).

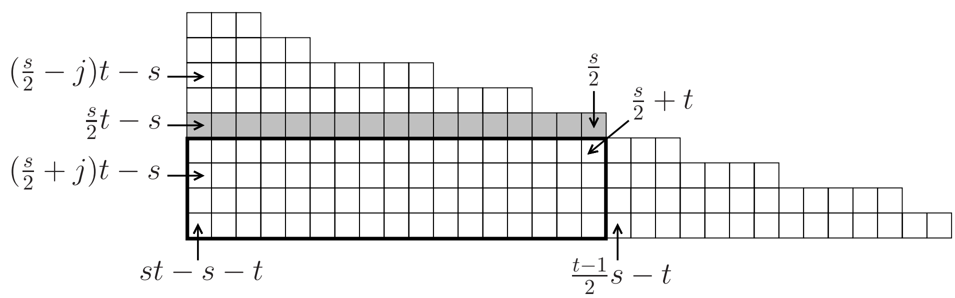

Figure 3.1: The (s,t)-diagram.

Recall that the set of parts of any $(\overline{s}, \overline{t})$-core is closed under the subtraction of multiples of s and t. Since $s/2+t$ cannot be a part of an $(\overline{s}, \overline{t})$-core λ, the entries in the rectangular which is determined by the lower-left corner element $st - s - t$ and the upper-right corner element $s/2+t$ cannot be parts of λ. Thus we can remove the rectangular from the (s,t)-diagram. It is evident that the size of this rectangular is $(u-1) \times v$. Here we also refer the remaining two smaller diagrams as the *Yin diagram* and the *Yang diagram*, which generalize the concept of the Yin diagram and the Yang diagram to all possible coprime pairs.

Case 2: $t > s \geq 4$. Clearly, the top Yin diagram has u rows with $st/2 - s$ as its largest element, and the bottom Yang diagram has $u-1$ rows with $(t-1)s/2 - t$ as its largest element. Recall that the entries in the rectangular which is determined by the lower-left corner element $st - s - t$ and the upper-right corner element $s/2 + t$ cannot be parts of λ. Consequently, we derive that for any positive integer a, it is a part of an $(\bar{s}, \bar{t})$-core if and only if a is in the Yin or Yang diagram. As an example, the $(8, 13)$-diagram is shown in Figure 3.2.

5										
18	10	2								
31	23	15	7							
44	36	28	20	12	4					
57	49	41	33	25	17	9	1			
70	62	54	46	38	30	22	14	6		
83	75	67	59	51	43	35	27	19	11	3

Figure 3.2: The $(8, 13)$-diagram.

Next we will prove some properties about the number c_i of elements of residue i modulo s in (s,t)-diagram. For $1 \leq i \leq s-1$, since c_i entries of residue i modulo s in (s,t)-diagram are $i, s+i, \ldots, s(c_i - 1) + i$, we deduce that $sc_i + i \equiv 0 \pmod{t}$. By Lemma 3.2 in [11], we have

$$c_i = \frac{t\sigma(i) - i}{s}, \tag{3.3}$$

where σ is the permutation on $\{1, 2, \ldots, s-1\}$ satisfying $t\sigma(i) \equiv i \pmod{s}$. Note that the definition of σ implies that for all $1 \leq i \leq s-1$,

$$t\sigma(i) + t\sigma(s-i) \equiv 0 \pmod{s}.$$

Since s and t are coprime and $1 \leq \sigma(i), \sigma(s-i) \leq s-1$, we see

$$\sigma(i) + \sigma(s-i) = s.$$

It follows from (3.3) that

$$c_i + c_{s-i} = t - 1. \tag{3.4}$$

Thus the number of entries in the (s,t)-diagram is

$$\frac{s-1}{2} \times (t-1) = (2u-1)v. \tag{3.5}$$

For accuracy in the rest of proof, we assign the row indices of the Yin diagram from bottom to top as $0, 1, \ldots, u-1$, and row indices of the Yang diagram from top to bottom as $1, 2, \ldots, u-1$. Note that the largest and smallest entry in the 0-th row of Yin diagram (the shadowed row in Figure 3.1) are $st/2 - s$ and $s/2$, respectively. Hence the length of the 0-th row is $(t-1)/2$, which implies that it consists of all entries of residue $s/2$ modulo s in the (s,t) diagram. Moreover, we know that for $1 \leq j \leq u-1$, the entries in the j-th row of the Yin diagram are of residue $(s/2 - j)t - s$ modulo s, and symmetrically, the entries in the j-th row of the Yang diagram are of residue $(s/2 + j)t - s$ modulo s (see Figure 3.1). Since $(s/2-j)t - s + (s/2+j)t - s \equiv 0 \pmod{s}$, by (3.4), the sum of lengths of j-th rows of the Yin and Yang diagrams is also $(t-1)/2$.

Therefore, the Yang diagram rotated 180°, together with the Yin diagram, can combine a $u \times v$-rectangular diagram by gluing the j-th rows of both diagrams for $1 \leq j \leq u-1$ and leaving the 0-th row of the Yin diagram as the bottom row of the rectangular diagram. Here we also refer this rectangular diagram as the (s,t) *Yin-Yang diagram*.

Note that in each row of (s,t) Yin-Yang diagram, the entries in the Yin (resp. Yang) part are of residue i (resp. $s-i$) modulo s for some $1 \leq i \leq s-1$. Let p_j (resp. q_j) be the smallest entry in the Yin (resp. Yang) part of the j-th row (from top to bottom) of the Yin-Yang diagram for $1 \leq j \leq u-1$. Based on the analysis above, we have $p_j + q_j = s$. Thus, flipping the Yin-Yang diagram by x-axis and turning the entries in the Yin part into negative, we can obtain a uniform expression for all entries. Computing directly, it is easy to deduce that the result rectangular diagram is exactly $B(s,t)$ (see Figure 3.3 for example of $s = 8$ and $t = 13$).

Case 3*:* $s > t \geq 3$. The proof goes similar as Case 2 except we consider the (s,t)-diagram in aspect of columns rather than rows. Since $s > t$, after removing the rectangle with corners $st - s - t$ and $s/2 + t$, and size $(u-1) \times v$, one can check that the Yin and Yang diagram both consist of v columns. Denote by d_i the number of elements of residue i modulo t in (s,t)-diagram for $1 \leq i \leq t-1$. By [11, Lemma 3.2] and same approach in Case 2, for each i we deduce $d_i + d_{t-i} = s - 1 = 2u - 1$.

We assign the column indices of the Yin diagram from left to right as $1, 2, \ldots, v$, and column indices of the Yang diagram from right to left as $1, 2, \ldots, v$. For $1 \leq j \leq v$, the entries in the j-th column of the Yin diagram are of residue $st/2 - js$ modulo t, and symmetrically, the entries in the j-th column of the Yang diagram are of residue $js - t$ modulo t. Since $st/2 - js + js - t \equiv 0 \pmod{t}$, then the sum of lengths of j-th columns of the Yin and Yang diagrams is u in view of $(2u-1) - (u-1) = u$. It follows that the Yang diagram rotated 180° and the Yin diagram also give the (s,t) Yin-Yang diagram with size $u \times v$. Note that in each column of (s, t) Yin-Yang diagram, the entries in the Yin (resp. Yang) part are of residue i (resp. $t - i$) modulo t for some $1 \leq i \leq t-1$. Let p_j (resp. q_j) be the smallest entry in the Yin (resp. Yang) part of the j-th column (from left to right) of the Yin-Yang diagram for $1 \leq j \leq v$. Then we have $p_j + q_j = t$. Therefore, flipping the (s, t) Yin-Yang diagram horizontally then taking the entries in Yin part into negative, we arrive at the array $B(s, t)$.

In conclusion, since the (s, t) Yin-Yang diagram contains all possible parts may occur in $(\overline{s}, \overline{t})$-cores, so dose the set of the absolute values of entries in $B(s, t)$. □

Let $\mathcal{P}(B)$ be the set of lattice paths in $B = B(s, t)$ from the lower-left corner to the upper-right corner (see the solid lines in Figure 3.3). For a lattice path P in $\mathcal{P}(B)$, denote by $M_B(P)$ the set of positive entries above P and the absolute values of negative entries below P. For example, $M_B(P) = \{1, 10, 2, 4\}$ in Figure 3.3.

-44	-36	-28	-20	-12	-4
-31	-23	-15	-7	1	9
-18	-10	-2	6	14	22
-5	3	11	19	27	35

Figure 3.3: A lattice path in the array $B(8, 13)$.

Based on Lemma 3.1, we find the similar bijective behavior as stated in Corollary 2.4 for coprime pairs (s, t) with even s.

Theorem 3.2. *Let $B = B(s, t)$ be the array defined by* (3.1). *Then there exists a bijection Ψ between the set $\mathcal{P}(B)$ of lattice paths and the set of $(\overline{s}, \overline{t})$-cores such that for $P \in \mathcal{P}(B)$, the set of the parts of $\Psi(P)$ is given by $M_B(P)$.*

Since the number of lattice paths in $B(s, t)$ is $\binom{u+v}{u}$, the proof of Theorem 1.3 is directly followed from Theorem 3.2.

Proof of Theorem 3.2. By Lemma 3.1, we know that the set of the absolute values of the array entries in B contains all possible parts in an $(\overline{s}, \overline{t})$-cores. Given any $P \in \mathcal{P}(B)$, we shall prove that $M_B(P)$ is the set of the parts of an $(\overline{s}, \overline{t})$-core λ. It is trivial for $s = 2$. Assuming $s \geq 4$ and $|B_{i,j}| \in M_B(P)$, we consider the following two cases.

Case 1. If $B_{i,j} > 0$, then $B_{i,j}$ is above the path P. Since P is a lattice path in B from the lower-left corner to the upper-right corner, the entry $B_{i-1,j}$ (resp. $B_{i,j-1}$) must be in $M_B(P)$ if $B_{i-1,j} > 0$ (resp. $B_{i,j-1} > 0$), which implies that $M_B(P) \cap \{B_{i,j} : B_{i,j} > 0\}$

is closed under subtraction of multiples of s and t. Meanwhile, if $B_{i-1,j} < 0$ (resp. $B_{i,j-1} < 0$), then $|B_{i-1,j}|$ (resp. $|B_{i,j-1}|$) cannot be in $M_B(P)$. In other words, the elements in the conjugate runner of the runner containing $B_{i,j}$ (both with respect to the $\overline{s}$- and $\overline{t}$-abacus) cannot be in $M_B(P)$.

Case 2. If $B_{i,j} < 0$, then $B_{i,j}$ is below the path P. Thus for $B_{i+1,j} < 0$ (resp. $B_{i,j+1} < 0$), $|B_{i+1,j}|$ (resp. $|B_{i,j+1}|$) must be in $M_B(P)$, and for $B_{i+1,j} > 0$ (resp. $B_{i,j+1} > 0$), $|B_{i+1,j}|$ (resp. $|B_{i,j+1}|$) cannot be in $M_B(P)$. With the same argument in Case 1, we derive that $|B_{i,j}|$ also satisfies the constraints of an $(\overline{s}, \overline{t})$-core.

By the definition of the $(\overline{s}, \overline{t})$-core, the elements in $M_B(P)$ indeed form an $(\overline{s}, \overline{t})$-core. For example in Figure 3.3, $(10, 4, 2, 1)$ is an $(\overline{8}, \overline{13})$-core.

On the other hand, if λ is an $(\overline{s}, \overline{t})$-core, denote by B_{i,j_i} the rightmost element in the i-th row of the array B whose absolute value is a part of λ. To prove all parts of λ form some lattice path P in B from the lower-left corner to the upper-right corner, it equivalent to show $j_i \geq j_{i+1}$. If $B_{i,j_i} > 0$ and $B_{i+1,j_{i+1}} < 0$, by the staircase shapes of the Yin and Yang diagrams, then clearly $j_i \geq j_{i+1}$. If $B_{i,j_i}, B_{i+1,j_{i+1}} > 0$ or $B_{i,j_i}, B_{i+1,j_{i+1}} < 0$, we also have $j_i \geq j_{i+1}$ since the part set of λ is closed under subtraction of multiples of s and t. If $B_{i,j_i} < 0$ and $B_{i+1,j_{i+1}} > 0$, then $|B_{i+1,j_i}|$ must be a part of λ since $|B_{i+1,j_i}| = |B_{i,j_i}| - t$ and λ is a $\overline{t}$-core. However, noting that $B_{i+1,j_i} < 0$, we are led to $|B_{i+1,j_i}| + |B_{i+1,j_{i+1}}| \equiv 0 \pmod{s}$, contracting λ is also a $\overline{s}$-core. □

4. The average size of an $(\overline{s}, \overline{t})$-core for coprime integers s and t with even s

As the same in Section 2, for any lattice path P in the array $B(s,t)$, denote by $\overline{P}$ (resp. $\underline{P}$) the set of cells (i, j) in $B(s,t)$ that are above (resp. below) P. Then in terms of the bijection Ψ, we obtain the following lemma.

Lemma 4.1. *Let $P \in \mathcal{P}(B)$, we have*

$$|\Psi(P)| = \sum_{(i,j)\in\overline{P}} B_{i,j} - \frac{(u+1)v}{6}(u - v - 2uv - 1). \tag{4.1}$$

Proof. By Theorem 3.2, we see that

$$\begin{aligned} |\Psi(B)| &= \sum_{\substack{(i,j)\in\overline{P}\\ B_{i,j}>0}} B_{i,j} - \sum_{\substack{(i,j)\in\underline{P}\\ B_{i,j}<0}} B_{i,j} \\ &= \sum_{(i,j)\in\overline{P}} B_{i,j} - \sum_{(i,j)\colon B_{i,j}<0} B_{i,j}. \end{aligned}$$

To prove (4.1), it is sufficient to show that $\sum_{(i,j)\colon B_{i,j}<0} B_{i,j} = \frac{v}{6}(u+1)(u-v-2uv-1)$.

By (3.1), we have

$$\sum_{\substack{1\le i\le u\\1\le j\le v}} B_{i,j} = \sum_{\substack{1\le i\le u\\1\le j\le v}} \left(-\frac{s+2}{2}t + js + it\right) = \frac{uv}{2}(u-2v-1).$$

We proceed to prove

$$\sum_{\substack{1\le i\le u\\1\le j\le v}} |B_{i,j}| = \frac{v}{6}(4u^2v + u^2 + 2v - 3u + 2), \tag{4.2}$$

which means

$$\sum_{(i,j)\colon B_{i,j}<0} B_{i,j} = \frac{1}{2}\sum_{\substack{1\le i\le u\\1\le j\le v}} (B_{i,j} - |B_{i,j}|) = \frac{(u+1)v}{6}(u-v-2uv-1),$$

as expected.

If $s=2$, we have

$$\sum_{1\le j\le v} |B_{1,j}| = \sum_{1\le j\le v} (t-2j) = v^2,$$

which is the same as (4.2) with $u = s/2 = 1$. Thus we can assume $s \ge 4$ in the rest proof. Recall that the number of entries in the (s,t)-diagram is $(2u-1)v$ by (3.5), implying that the sum of the entries in the (s,t)-diagram is $(s^2-1)(t^2-1)/24 + \binom{(2u-1)v}{2}$, by Lemma 3.4 of [11]. Subtracting the sum of the entries in the $(u-1)\times v$-rectangular diagram with corners $st-s-t$ and $s/2+t = u+2v+1$ (see Figure 3.1), we derive

$$\begin{aligned}\sum_{(i,j)} |B_{i,j}| &= \frac{(s^2-1)(t^2-1)}{24} + \binom{(2u-1)v}{2}\\ &\quad - \left((u-1)v(u+2v+1) + (u-1)\sum_{i=0}^{v-1} is + v\sum_{j=0}^{u-2} jt\right)\\ &= \frac{(s^2-1)(t^2-1)}{24} + \binom{(2u-1)v}{2} - (u-1)v(u+2v+1)\\ &\quad - (u-1)s\binom{v}{2} - vt\binom{u-1}{2}\\ &= \frac{v}{6}(4u^2v + u^2 + 2v - 3u + 2),\end{aligned}$$

which completes the proof. □

Now we are ready to give the average size of an $(\overline{s},\overline{t})$-core for any coprime integer pair (s,t) with even s.

Proof of Theorem 1.4. Let $BC(s,t)$ denote the set of $(\overline{s},\overline{t})$-cores. Combining Theorem 3.2 and Lemma 4.1, we obtain

$$\sum_{\lambda\in BC(s,t)}|\lambda| = \sum_{P\in\mathcal{P}(B)}\left(\sum_{(i,j)\in\overline{P}}B_{i,j} - \frac{(u+1)v}{6}(u-v-2uv-1)\right). \tag{4.3}$$

By (3.1) and Lemma 2.6, we have

$$\begin{aligned}\sum_{P\in\mathcal{P}(B)}\sum_{(i,j)\in\overline{P}}B_{i,j} &= \sum_{P\in\mathcal{P}(B)}\sum_{(i,j)\in\overline{P}}\left(-\frac{s+2}{2}t+js+it\right)\\ &= -\frac{s+2}{2}t\sum_{\substack{1\le i\le u\\1\le j\le v}}f(i,j)+s\sum_{\substack{1\le i\le u\\1\le j\le v}}jf(i,j)+t\sum_{\substack{1\le i\le u\\1\le j\le v}}if(i,j)\\ &= -\frac{s+2}{2}t\binom{u+v}{v}\frac{uv}{2}+s\binom{v+2}{3}\binom{u+v}{v+1}+t\binom{u+2}{3}\binom{u+v}{u+1}\\ &= \left(-\frac{uv(s+2)t}{4}+\frac{uv}{6}(v+2)s+\frac{uv}{6}(u+2)t\right)\binom{u+v}{v}\\ &= \frac{uv}{6}(2u-2uv-2v-1)\binom{u+v}{v}.\end{aligned} \tag{4.4}$$

Applying (4.4) to (4.3), since $|\mathcal{P}(B)|=\binom{u+v}{u}$, we arrive at

$$\sum_{\lambda\in BC(s,t)}|\lambda| = \left(\frac{uv}{6}(2u-2uv-2v-1)-\frac{(u+1)v}{6}(u-v-2uv-1)\right)\binom{u+v}{u}. \tag{4.5}$$

Finally, replacing u by $s/2$ and v by $(t-1)/2$ in (4.5) and combining with Theorem 1.3 lead to the average size of an $(\overline{s},\overline{t})$-core:

$$\frac{\sum_{\lambda\in BC(s,t)}|\lambda|}{\binom{u+v}{u}} = \frac{(t-1)(s^2+st-3s+2t+2)}{48}.$$

The proof is completed. □

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