TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 4, pp. 767–777, August 2018
DOI: 10.11650/tjm/171202

Pentavalent Arc-transitive Graphs of Order $2p^2q$

Hailin Liu, Bengong Lou* and Bo Ling

Abstract. In this paper, we complete a classification of pentavalent arc-transitive graphs of order $2p^2q$, where p and q are distinct odd primes. This result involves a subclass of pentavalent arc-transitive graphs of cube-free order.

1. Introduction

Throughout the paper, graphs considered are simple, connected, undirected and regular. For a graph Γ, we denote by $V\Gamma$, $E\Gamma$, $A\Gamma$ and $\mathrm{Aut}\,\Gamma$ the vertex set, edge set, arc set and full automorphism group of Γ, respectively. Γ is called *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if $G \leq \mathrm{Aut}\,\Gamma$ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$, respectively. In particular, when $G = \mathrm{Aut}\,\Gamma$ then Γ is called *vertex-transitive*, *edge-transitive* or *arc-transitive*, respectively. As we all know, Γ is G-arc-transitive for some $G \leq \mathrm{Aut}\,\Gamma$ if and only if G is vertex-transitive and the vertex stabilizer G_v of $v \in V\Gamma$ in G is transitive on the neighborhood $\Gamma(v)$ of v. Let Γ be a vertex-transitive graph, and let N be a subgroup of $\mathrm{Aut}\,\Gamma$. Denote by Γ_N the quotient graph induced by N with $V\Gamma_N = \{v^N \mid v \in V\Gamma\}$ and two orbits are adjacent in Γ_N if and only if that there is an edge in Γ between these two orbits. If Γ and Γ_N have the same valency, then Γ is called a normal cover of Γ_N.

Let G be a group, and $H \leq G$. Then we use G', $\mathrm{Aut}(G)$ and $C_G(H)$ to denote the derived group, automorphism group and the centralizer of H in G, respectively. Let M and N be two groups. Then we use $M : N$ and $M \times N$ to denote a semidirect product and direct product of M by N. For a positive integer n, we denote by D_{2n}, A_n, S_n, $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$ the dihedral group of order $2n$, the alternating group and the symmetric group of degree n, the cyclic group of order n and the ring of integers modulo n (and for the field of order n if n is a prime), and the multiplicative group of units of $\mathbb{Z}_n$ respectively.

A group G is called a *generalized dihedral group*, if there exists an abelian subgroup H and an involution τ such that $G = H : \langle \tau \rangle$ and $h^\tau = h^{-1}$ for each $h \in H$. This group is denoted by $\mathrm{Dih}(H)$.

Received January 12, 2017; Accepted December 13, 2017.
Communicated by Xuding Zhu.
2010 *Mathematics Subject Classification.* 20B25, 05C25.
Key words and phrases. arc-transitive graph, Cayley graph, cube-free order.
This work was partially supported by the NNSF of China (11231008, 11761079, 11701503).
*Corresponding author.

In the literature, the classification of arc-transitive graphs of small valency have been extensively studied, for examples [5, 9, 14, 20, 24]. In particular, arc-transitive graphs of square-free order have been studied for a long time, for instance [2, 12, 13]. More recently, arc-transitive graphs of cube-free order are studied in various special case, for examples [4, 16–18, 22], which will be a long-term project. In this paper, we study a subclass of pentavalent arc-transitive graphs of cube-free order, and give a classification of pentavalent arc-transitive graphs of order $2p^2q$ for distinct odd primes p and q. The special cases where $p = q$, $p = 2$, and $q = 2$ have been treated in [21], [8], and [10], respectively. The main result of this paper is the following theorem.

Theorem 1.1. *Let Γ be a pentavalent arc-transitive graph of order $2p^2q$, where p and q are distinct odd primes. Then either*

(1) *Γ is a Cayley graph on $\mathrm{Dih}(H)$, where $H \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q$ or $\mathbb{Z}_p^2 \times \mathbb{Z}_q$; or*

(2) *$(\Gamma, |V\Gamma|, \mathrm{Aut}\,\Gamma, (\mathrm{Aut}\,\Gamma)_v)$ lies in Table 1.1.*

Row	Γ	$2p^2q$	$\mathrm{Aut}\,\Gamma$	$(\mathrm{Aut}\,\Gamma)_v$	Remark
1	$\mathcal{C}_{126}$	126	S_9	$S_4 \times S_5$	Example 3.2(1)
2	$\mathcal{C}_{342}^1$	342	$\mathrm{PSL}(2,19)$	D_{10}	Example 3.2(2)
3	$\mathcal{C}_{342}^2$	342	$\mathrm{PGL}(2,19)$	D_{20}	Example 3.2(3)

Table 1.1

2. Preliminary results

In this section, we give some necessary preliminary results.

The following lemma determines the stabilizers of pentavalent arc-transitive graphs from [7, 23].

Lemma 2.1. *Let Γ be a pentavalent G-arc-transitive graph for some $G \le \mathrm{Aut}\,\Gamma$. Let $v \in V\Gamma$. If G_v is soluble, then $|G_v| \mid 80$. If G_v is insoluble, then $|G_v| \bigm| 2^9 \cdot 3^2 \cdot 5$. Furthermore, $G_v \cong \mathbb{Z}_5$, D_{10}, D_{20}, F_{20}, $F_{20} \times \mathbb{Z}_2$, A_5, S_5, $F_{20} \times \mathbb{Z}_4$, $A_4 \times A_5$, $(A_4 \times A_5) : \mathbb{Z}_2$, $S_4 \times S_5$, $\mathrm{ASL}(2,4)$, $\mathrm{AGL}(2,4)$, $A\Sigma L(2,4)$, $A\Gamma L(2,4)$ or $\mathbb{Z}_2^6 : \Gamma L(2,4)$.*

We now give a result that will be useful.

Lemma 2.2. *Let p and q be distinct odd primes, and let Γ be a connected pentavalent G-arc-transitive graph of order $2p^2q$, where $G \le \mathrm{Aut}\,\Gamma$. Let $N \lhd G$. If N is insoluble, then the following statements hold:*

(1) *N has at most two orbits on $V\Gamma$;*

(2) *For each $v \in V\Gamma$, $5 \bigm| |N_v^{\Gamma(v)}|$.*

Proof. (1) Suppose that N has at least three orbits on $V\Gamma$. Then, by [15, Theorem 9], N is semiregular on $V\Gamma$. Hence $|N| \bigm| 2p^2q$. Since a group of order $2p^2q$ is soluble, N is soluble, a contradiction.

(2) Let $v \in V\Gamma$. If $N_v = 1$, then N is a group with order divising $2p^2q$. It follows that N is soluble, which is a contradiction to our hypothesis. Thus $N_v \neq 1$. Since G is transitive on $V\Gamma$, $N_v^{\Gamma(v)} \neq 1$ by connectivity of Γ. Note that $G_v^{\Gamma(v)}$ acts primitively on $\Gamma(v)$ and $N_v^{\Gamma(v)} \trianglelefteq G_v^{\Gamma(v)}$, so $5 \bigm| |N_v^{\Gamma(v)}|$. $\square$

By checking order of nonabelian simple groups (see [3, pp. 303–304]), we have the following lemma.

Lemma 2.3. *Let p and q be distinct odd primes. Let T be a nonabelian simple group of order $|T| = 2^i \cdot 3^j \cdot 5 \cdot p^s \cdot q$, where $1 \leq i \leq 10$, $0 \leq j \leq 2$ and $0 \leq s \leq 2$. Then either T is in the following Table* 2.1*, or $T \cong \mathrm{PSL}(2,121)$ if $p \neq q > 5$ and $5p^2q \bigm| |T|$.*

T	$\lvert T\rvert$	T	$\lvert T\rvert$
A_5	$2^2 \cdot 3 \cdot 5$	A_6	$2^3 \cdot 3^2 \cdot 5$
$\mathrm{PSp}(4,3)$	$2^6 \cdot 3^4 \cdot 5$		
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
$\mathrm{PSL}(3,4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	$\mathrm{PSL}(3,5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$
$\mathrm{PSp}(4,4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	$\mathrm{PSp}(6,2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$
$\mathrm{PSU}(3,4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	$\mathrm{PSU}(3,5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	$\mathrm{PSL}(2,11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$
$\mathrm{PSL}(2,16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	$\mathrm{PSL}(2,19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$
$\mathrm{PSL}(2,25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	$\mathrm{PSL}(2,31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$
$\mathrm{PSL}(2,49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	$\mathrm{PSL}(2,81)$	$2^4 \cdot 3^4 \cdot 5 \cdot 41$
$\mathrm{Sz}(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$		

Table 2.1

A graph Γ is said a Cayley graph if there exists a group G and a subset $S \subset G$ with $1 \notin S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathrm{Cay}(G,S)$. As we all known, a graph Γ is a Cayley graph if and only if $\mathrm{Aut}\,\Gamma$ contains a subgroup which is regular on $V\Gamma$.

Lemma 2.4. *Let Γ be a connected and regular G-edge-transitive graph, where $G \le \mathrm{Aut}\,\Gamma$. Suppose that G contains an abelian normal subgroup H which acts semiregularly and has exactly two orbits on $V\Gamma$. Then Γ is a Cayley graph of the generalized dihedral $\mathrm{Dih}(H)$.*

Proof. Note that H is normal in G, and is semiregular and has exactly two orbits on $V\Gamma$, so $\Gamma_H \cong K_2$ by the connectivity of Γ. It follows that there exists a edge $\{\alpha,\beta\} \in E\Gamma$ such that $V\Gamma = \alpha^H \cup \beta^H$. We conclude that α^H is an independent set of Γ. Actually, if α^H is not an independent set of Γ, then there exist $h_1, h_2 \in H$ such that $\{\alpha^{h_1}, \alpha^{h_2}\} \in E\Gamma$. Since Γ is G-edge transitive, there exists $g \in G$ such that $\{\alpha^{h_1}, \alpha^{h_2}\}^g = \{\alpha,\beta\}$. Therefore $(\alpha^H)^g \cap \alpha^H \ne \emptyset$ and $(\alpha^H)^g \ne \alpha^H$, a contrary to the fact that α^H is a block of the action of G on $V\Gamma$. With the same reason, β^H is an independent set of Γ too. It follows that Γ is a bipartite graph with two parts α^H and β^H.

For any $h \in H$, define a map

$$\sigma\colon \alpha^h \mapsto \beta^{h^{-1}},\ \beta^h \mapsto \alpha^{h^{-1}}.$$

Clearly, σ is a permutation on $V\Gamma$ with order 2.

Since Γ is G-edge transitive, $E\Gamma = \{\alpha,\beta\}^G$. Let $g \in G$. Then there exist $h_1, h_2 \in H$ such that $\alpha^g = \alpha^{h_1}$ (or β^{h_2}) and $\beta^g = \beta^{h_2}$ (or α^{h_1}). Since H is abelian,

$$\{\alpha^g,\beta^g\}^\sigma = \{\alpha^{h_1},\beta^{h_2}\}^\sigma = \left\{\beta^{h_1^{-1}}, \alpha^{h_2^{-1}}\right\} = \left\{\alpha^{gh_1^{-1}h_2^{-1}}, \beta^{gh_2^{-1}h_1^{-1}}\right\} = \left\{\alpha^{gh_1^{-1}h_2^{-1}}, \beta^{gh_1^{-1}h_2^{-1}}\right\}$$

for each $\{\alpha^g,\beta^g\} \in E\Gamma$. Therefore, $\{\alpha^g,\beta^g\}^\sigma \in E\Gamma$, and so σ is an automorphism of Γ. Further, $(\alpha^{h'})^{\sigma h\sigma} = (\alpha^{h'^{-1}})^{h\sigma} = (\alpha^{h'^{-1}h})^\sigma = \alpha^{h^{-1}h'} = (\alpha^{h'})^{h^{-1}}$, and $(\beta^{h'})^{\sigma h\sigma} = (\beta^{h'})^{h^{-1}}$ for any $h, h' \in H$. Thus $\sigma^{-1}h\sigma = h^{-1}$ for any $h \in H$. So $\langle H, \alpha\rangle \cong \mathrm{Dih}(H)$. Since σ interchanges α^H and β^H, $\langle H,\sigma\rangle$ acts regularly on $V\Gamma$. Hence Γ is a Cayley graph on $\mathrm{Dih}(H)$. $\square$

3. Examples

In this section, we give some examples which are appearing in Theorem 1.1.

Example 3.1. (1) Let $H_1 = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_q$, and let $G_1 = \mathrm{Dih}(H_1) = \langle a,b,h \mid a^{p^2} = b^q = h^2 = [a,b] = 1, h^{-1}ah = a^{-1}, h^{-1}bh = b^{-1}\rangle$.

(1.1) Let $l = 1$ if $p = 5$, and let l be an element of order 5 in $\mathbb{Z}_p^*$ if $5 \mid (p-1)$. Define

$$\mathcal{CGD}^1_{2p^2q} = \mathrm{Cay}(G_1, S_1),$$

where $S_1 = \{h, ah, a^{l(l+1)^{-1}}b^{l^{-1}}h, a^l b^{(l+1)^{-1}}h, bh\}$. Note that $\alpha\colon h \mapsto ah, a \mapsto a^{l(l+1)^{-1}-1}b^{l^{-1}}, b \mapsto a^{-1}$ induces an automorphism of order 5 of G_1 permuting the elements in $\{h, ah, a^{l(l+1)^{-1}}b^{l^{-1}}h, a^l b^{(l+1)^{-1}}h, bh\}$ cyclicly, so $\mathrm{Aut}(G_1, S_1)$ is transitive on S_1. Hence $\mathcal{CGD}^1_{2p^2q}$ is an arc-transitive Cayley graphs of order $2p^2q$.

(1.2) For $5 \mid (p \pm 1)$, let λ be an element in $\mathbb{Z}_p^*$ such that $\lambda^2 = 5$. Define

$$\mathcal{CGD}^2_{2p^2q} = \mathrm{Cay}(G_1, S_2),$$

where $S_2 = \{h, ah, a^{2^{-1}(1+\lambda)}bh, ab^{2^{-1}(1+\lambda)}h, bh\}$. Note that $\beta\colon h \mapsto ah, a \mapsto a^{2^{-1}(1+\lambda)-1}b, b \mapsto a^{-1}$ induces an automorphism of G_1 permuting the elements in $\{h, ah, a^{2^{-1}(1+\lambda)}bh, ab^{2^{-1}(1+\lambda)}h, bh\}$ cyclicly, so $\mathrm{Aut}(G_1, S_2)$ is transitive on S_3. Hence $\mathcal{CGD}^2_{2p^2q}$ is an arc-transitive Cayley graphs of order $2p^2q$.

(2) Let $H_2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$, and let $G_2 = \mathrm{Dih}(H_2) = \langle a, b, c, h \mid a^p = b^p = c^q = h^2 = [a, b] = [a, c] = [b, c] = 1, h^{-1}ah = a^{-1}, h^{-1}bh = b^{-1}, h^{-1}ch = c^{-1}\rangle$. Let $l = 1$ if $p = 5$, and let l be an element of order 5 in $\mathbb{Z}_p^*$ if $5 \mid (p-1)$. Define

$$\mathcal{CGD}^3_{2p^2q} = \mathrm{Cay}(G_2, S_3),$$

where $S_3 = \{h, ah, a^{-l^2}b^{-l}c^{-l^{-1}}h, bh, ch\}$. Note that $\gamma\colon h \mapsto ah, a \mapsto ba^{-1}, b \mapsto a^{-l^2-1}b^{-l}c^{-l^{-1}}, c \mapsto a^{-1}$ induces an automorphism of G_2 permuting the elements in $\{h, ah, a^{-l^2}b^{-l}c^{-l^{-1}}h, bh, ch\}$ cyclicly, so $\mathrm{Aut}(G_2, S_3)$ is transitive on S_3. Hence $\mathcal{CGD}^3_{2p^2q}$ is an arc-transitive Cayley graphs of order $2p^2q$.

By using Magma program [1], we have the following example.

Example 3.2. (1) There exists a unique connected pentavalent graph of order 126 which admits A_9 as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}_{126}$, which satisfies the conditions in Row 1 of Table 1.1.

(2) There is a unique connected pentavalent graph of order 342 which admits $\mathrm{PSL}(2, 19)$ as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}^1_{342}$, which satisfies the conditions in Row 2 of Table 1.1.

(3) There is a unique connected pentavalent graph of order 342 which admits $\mathrm{PGL}(2, 19)$ as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}^2_{342}$, which satisfies the conditions in Row 3 of Table 1.1.

4. Proof of Theorem 1.1

Let Γ be a pentavalent arc-transitive graph of order $2p^2q$, where p and q are distinct odd primes. Let $A = \mathrm{Aut}\,\Gamma$. Then $|A_v| \bigm| 2^9 \cdot 3^2 \cdot 5$ for each $v \in V\Gamma$ by Lemma 2.1, and so $|A| \bigm| 2^{10} \cdot 3^2 \cdot 5 \cdot p^2 \cdot q$. Let N be a minimal normal subgroup of A.

We first consider the case where N is soluble.

Lemma 4.1. *If N is soluble, then part* (1) *of Theorem* 1.1 *holds.*

Proof. Since N is soluble, $N \cong \mathbb{Z}_r^d$ for some prime r and integer $d \geq 1$. Note that $|N|/|N_v| \mid |2p^2q|$, so N has at least 3 orbits on $V\Gamma$. It follows that N is semiregular and Γ is a normal cover of Γ_N by [15, Theorem 9]. Thus $|N| \mid 2p^2q$, and then $N \cong \mathbb{Z}_p$, $\mathbb{Z}_q$ or $\mathbb{Z}_p^2$. In what follows, we divide our proof into three cases:

Case 1. Assume that $N \cong \mathbb{Z}_p^2$. Then $\Gamma_N \cong K_6$, $K_{5,5}$ or $G(2q, 5)$ with $q \equiv 1 \pmod 5$ by [9, Proposition 2.7].

Suppose that $\Gamma_N \cong K_6$. Then $q = 3$ and $A/N \lesssim S_6$. Since $5 \cdot 6 \mid |A/N|$, $A/N \cong A_5$, S_5, A_6 or S_6. If $A/N \cong A_5$ or A_6, then $A = N.T$ is a central extension by [11]; further $A' \cong T$, $\mathbb{Z}_2.T$ or $\mathbb{Z}_3.T$, where $T = A_5$ or A_6. By Lemma 2.2, A' has at most two orbits on $V\Gamma$, and so $3 \cdot p^2 \mid |A'|$, which is impossible. If $A/N \cong S_5$ or S_6, then A/N contains a normal subgroup $M/N \cong A_5$ or A_6. Arguing as the above discussion, a contradiction occurs.

Suppose that $\Gamma_N \cong K_{5,5}$. Then $q = 5$ and $A/N \lesssim S_5 \wr S_2$. Let M/N be a minimal normal subgroup of A/N. If M/N is insoluble, then $M/N \cong A_5$ or A_5^2. Obviously, M/N has two orbits on $V\Gamma_N$ and $5 \mid |(M/N)_w|$ for any $w \in V\Gamma_N$, implying that $25 \mid |M/N|$. Thus, $M/N \cong A_5^2$. Let $B/N \trianglelefteq M/N$ such that $B/N \cong A_5$. Then B/N has two orbits on $V\Gamma_N$ and $5 \mid |(B/N)_w|$. Thus, $25 \mid |B/N|$, a contradiction. If M/N is soluble, then $M/N \cong \mathbb{Z}_5$ or $\mathbb{Z}_5^2$. Therefore $M_v \cong 1$ or $\mathbb{Z}_5$. It follows that $\Gamma \cong p^2K_{5,5}$, which contradicts the connectivity of Γ.

Thus $\Gamma_N \cong G(2q, 5)$. Assume that $q > 11$. Then $A/N = \mathrm{Aut}\,\Gamma_N := (Q : F) : \langle t\rangle \cong (\mathbb{Z}_q : \mathbb{Z}_5) : \mathbb{Z}_2$. Now Γ is a pentavalent 1-regular graph of order $2p^2q$. Since Q is characteristic in $Q : F$ and $Q : F \trianglelefteq \mathrm{Aut}\,\Gamma_N$, $Q \trianglelefteq \mathrm{Aut}\,\Gamma_N$. Thus A contains a normal subgroup H such that $H/N \cong \mathbb{Z}_q$, that is, $H = N.Q \cong \mathbb{Z}_p^2 : \mathbb{Z}_q$. If $p = 5$, then $H = N \times Q \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$ as $\mathrm{GL}(2, 5)$ has no cyclic subgroups of order more than 11. If $p \neq 5$, then $A = N.((Q : F) : \langle t\rangle) \cong \mathbb{Z}_p^2.((\mathbb{Z}_q : \mathbb{Z}_5) : \mathbb{Z}_2) = \mathbb{Z}_p^2 \times ((\mathbb{Z}_q : \mathbb{Z}_5) : \mathbb{Z}_2)$ by the groups structures of the $\mathrm{GL}(2, p)$. Thus $H = N.Q \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$. Now $H \lhd A$ is abelian, and has exactly two orbits on $V\Gamma$. So Γ is a Cayley graph on the generalized dihedral $\mathrm{Dih}(H)$ by Lemma 2.4. Assume that $q = 11$. Note that $A/N \leq \mathrm{Aut}\,\Gamma_N \cong \mathrm{PSL}(2, 11) : \mathbb{Z}_2$ is arc-transitive on Γ_N, and $\mathrm{PSL}(2, 11)$ has no subgroups of order 30, so $\mathrm{PSL}(2, 11)$ has exactly two orbits on $V\Gamma_N$. It concludes that $A/N = \mathrm{Aut}\,\Gamma_N \cong \mathrm{PSL}(2, 11) : \mathbb{Z}_2$. Let $B/N \lhd A/N$ such that $B/N \cong \mathrm{PSL}(2, 11)$. Then $B'N/N \lhd B/N \cong \mathrm{PSL}(2, 11)$. Thus $B'N/N = 1$ or B/N. If $B'N/N = 1$, then $B' \leq N$ is soluble, which is impossible as B is insoluble. If $B'N/N = B/N$, then $B = B'N = B' \times N$. Obviously, $B' \lhd A$ has exactly two orbits on $V\Gamma$. So $|B'| = p^2q$, implying that B' is soluble, a contradiction.

Case 2. Assume that $N \cong \mathbb{Z}_p$. Then $\Gamma_N \cong \mathcal{C}_{66}, \mathcal{C}_{114}, \mathcal{C}_{406}, \mathcal{C}_{3422}, \mathcal{C}_{3782}, \mathcal{C}_{574}, \mathcal{C}_{42}, \mathcal{C}_{170}$, or $\mathcal{CD}^l_{2pq}$ for some l satisfying $l^4 + l^3 + l^2 + l \equiv 0 \pmod{pq}$ by [9, Theorem 4.2].

Suppose that $\Gamma_N \cong \mathcal{C}_{66}$. Then $\{p, q\} = \{3, 11\}$ and $A/N \le \mathrm{Aut}\,\Gamma_N \cong \mathrm{PGL}(2, 11)$. Since $5 \cdot 66 \bigm| |A/N|$, $A/N \cong \mathrm{PSL}(2, 11).O$, where $O \le \mathbb{Z}_2$. Thus A/N contains a normal subgroup M/N isomorphic to $\mathrm{PSL}(2, 11)$. Then $M = N \times M' \cong \mathbb{Z}_p \times \mathrm{PSL}(2, 11)$ by [11]. Note that M' is a normal subgroup of A, so M' has at most two orbits on $V\Gamma$ by Lemma 2.2. Thus $|M'_v| = 2p^2q$ or p^2q. But $\mathrm{PSL}(2, 11)$ has no subgroups of these order, a contradiction. Similarly, we can exclude the cases where $\Gamma_N \cong \mathcal{C}_{406}, \mathcal{C}_{3422}, \mathcal{C}_{3782}$ and $\mathcal{C}_{574}$.

Suppose that $\Gamma_N \cong \mathcal{C}_{114}$. Then $\{p, q\} = \{3, 19\}$, and $A/N \le \mathrm{Aut}\,\Gamma_N \cong \mathrm{PGL}(2, 19)$. Thus A/N contains a normal subgroup $M/N \cong \mathrm{PSL}(2, 19)$, and so $M' \trianglelefteq A$ and $M' \cong \mathrm{PSL}(2, 19)$. It follows that M' has at most two orbits on $V\Gamma$ by Lemma 2.2. Obviously, we can exclude case where $(p, q) = (19, 3)$ by the same discussion above. If $(p, q) = (3, 19)$, then either $\Gamma \cong \mathcal{C}^1_{342}$ and $\mathrm{Aut}\,\Gamma \cong \mathrm{PSL}(2, 19)$ or $\Gamma \cong \mathcal{C}^2_{342}$ and $\mathrm{Aut}\,\Gamma \cong \mathrm{PGL}(2, 19)$. So $1 \le |\mathrm{Aut}\,\Gamma_N|/|\mathrm{Aut}\,\Gamma| \le 2$, which is impossible. Suppose that $\Gamma_N \cong \mathcal{C}_{170}$. Then $A/N \cong \mathrm{PSp}(4, 4).O$, where $O \le \mathbb{Z}_4$. Thus A/N contains a normal subgroup $M/N \cong \mathrm{PSp}(4, 4)$, and so $M' \trianglelefteq A$ and $M' \cong \mathrm{PSp}(4, 4)$. By Lemma 2.2, M' has at most two orbits on $V\Gamma$. So $p = 5$ and $q = 17$. It follows that $|M'_v| = 1152$ or 2304. On the one hand, the subgroups of M' with order 1152 or 2304 are all soluble by Magma [1]. On the other hand, A_v has no such normal subgroups that are isomorphic to M_v by Lemma 2.1, a contradiction. Similarly, we can exclude the case where $\Gamma_N \cong \mathcal{C}_{42}$.

Suppose that $\Gamma_N \cong \mathcal{CD}^l_{2pq}$. Then $A/N \le \mathrm{Aut}\,\Gamma_N \cong D_{2pq} : \mathbb{Z}_5$. Since $5 \cdot 2pq \bigm| |A/N|$, $A/N = \mathrm{Aut}(\Gamma_N) \cong D_{2pq} : \mathbb{Z}_5$. Note that D_{2pq} is regular on $V\Gamma_N$, so A has a normal regular subgroup $G \cong \mathbb{Z}_p.D_{2pq}$. Thus, by [19, Theprem 3.9], either $G \cong (\mathbb{Z}_p^2 \times \mathbb{Z}_q) : \mathbb{Z}_2$ or $(\mathbb{Z}_{p^2} \times \mathbb{Z}_q) : \mathbb{Z}_2$, that is, $G \cong \mathrm{Dih}(\mathbb{Z}_p^2 \times \mathbb{Z}_q)$ or $\mathrm{Dih}(\mathbb{Z}_{p^2} \times \mathbb{Z}_q)$. Hence Γ is a Cayley graph on $\mathrm{Dih}(H)$, where $H \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$ or $\mathbb{Z}_{p^2} \times \mathbb{Z}_q$.

Case 3. Assume that $N \cong \mathbb{Z}_q$. Then $\Gamma_N \cong \mathcal{CGD}^1_{2p^2}$ ($p = 5$ or $5 \mid (p-1)$), $\mathcal{CGD}^2_{2p^2}$ ($5 \mid (p \pm 1)$) or $\mathcal{CD}_{2p^2}$ ($5 \mid (p-1)$) by [6, Theorems 4.3 and 6.1].

Suppose that $\Gamma_N \cong \mathcal{CD}_{2p^2}$. Then $A/N = \mathrm{Aut}\,\mathcal{CD}_{p^2} \cong R(D_{2p^2}) : \mathbb{Z}_5$. Since $\mathbb{Z}_{p^2}$ is characteristic in $R(D_{2p^2})$ and $R(D_{2p^2}) \trianglelefteq \mathrm{Aut}\,\Gamma_N$, A/N has a normal subgroup isomorphic to $\mathbb{Z}_{p^2}$. Thus A has a normal subgroup H such that $H \cong \mathbb{Z}_q.\mathbb{Z}_{p^2}$. If $p = 5$, then $H \cong \mathbb{Z}_q \times \mathbb{Z}_{p^2}$. If $p \equiv 1 \pmod 5$, then $H \cong \mathbb{Z}_q \times \mathbb{Z}_{p^2}$ as $A \cong \mathbb{Z}_q \times (R(D_{2p^2}) : \mathbb{Z}_5)$. Thus H is abelian. Obviously, H has two orbits on $V\Gamma$. So Γ is a Cayley graph on $\mathrm{Dih}(H)$ by Lemma 2.4. Similarly, when $\Gamma_N \cong \mathcal{CGD}^2_{2p^2}, \mathcal{CGD}^1_{5^2}$ or $\mathcal{CGD}^1_{2p^2}$, then Γ is also a Cayley graph on $\mathrm{Dih}(H)$, where $H \cong \mathbb{Z}_p^2 \times \mathbb{Z}_q$. $\square$

Next we consider the case where N is insoluble.

Lemma 4.2. *If N is insoluble, then part* (2) *of Theorem* 1.1 *holds.*

Proof. Since N is insoluble, $N \cong T^d$ with T a nonabelian simple group and integer $d \geq 1$. By Lemma 2.2, N has at most two orbits on $V\Gamma$ and $5 \mid |N_v|$ for each $v \in V\Gamma$. Thus $5p^2q \mid |N|$. In the following, we process our analysis by several cases.

Case 1. Assume that $p \neq q > 5$. Then $5pq \mid |T|$. If $d \geq 2$, then $5^d p^d q^d \mid |N|$. But $|N| \mid |A| \mid 2^{10} \cdot 3^2 \cdot 5 \cdot p^2 \cdot q$, a contradiction. Hence $d = 1$ and $N \cong T$. By Lemma 2.3, $N \cong \mathrm{PSL}(2, 121)$ ($p = 11$, $q = 61$). Set $C := C_A(N)$. Since $C \cap N = 1$, $N \times C \leq A$, and so C is a $\{2, 3\}$-group. Therefore C is soluble, implying that $C = 1$ by the analysis of Lemma 4.1. Thus $A \leq \mathrm{Aut}(N)$. If N has two orbits on $V\Gamma$, then $|N_v| = |N|/(121 \cdot 61) = 120$. On the one hand, since $N \leq A \lesssim \mathrm{PSL}(2, 11^2).\mathbb{Z}_2^2$, $|A_v : N_v| = 2$ or 4. Thus A_v is insoluble because $|A_v| \nmid 80$, forcing that N_v is insoluble. On the other hand, N has no insoluble subgroups of order 120 by Magma [1], a contradiction. Hence N is transitive on $V\Gamma$. Further Γ is N-arc-transitive. But a computation by Magma [1] shows that no graph Γ appears.

Case 2. Assume that $(p, q) = (3, 5)$ or $(p, q) = (5, 3)$. Since there exists no graph of order 90 by [20] and 150 by [14], we can exclude this case.

Case 3. Assume that $p = 3$ and $q > 5$. Then $5 \cdot 3^2 \cdot q \mid |N| \mid |A| \mid 2^{10} \cdot 3^4 \cdot 5 \cdot q$. By Lemma 2.3, $N \cong M_{11}$, M_{12}, A_7, A_8, A_9, $\mathrm{PSL}(2, 19)$, $\mathrm{PSL}(2, 81)$, $\mathrm{PSL}(3, 4)$ or $\mathrm{PSp}(6, 2)$. Suppose that $N \cong M_{11}$. Then $q = 11$ and $|N_v| = 80$ or 40. But N has no subgroups of order 80 or 40 by [1], a contradiction. Similarly, we can exclude the cases where $N \cong M_{12}$ and A_8. Suppose that $N \cong \mathrm{PSL}(3, 4)$. Then $q = 7$ and $|N_v| = 320$ or 160. But N has no subgroups of order 320 by Magma [1]. Thus N is transitive on $V\Gamma$. It follows that N is arc-transitive on Γ. On the one hand, the subgroups of N with order 160 are soluble by Magma [1]. On the other hand, $N_v \lhd A_v$ is insoluble by Lemma 2.1, a contradiction. Suppose that $N \cong \mathrm{PSp}(6, 2)$. Then $q = 7$ and $|N_v| = 23040$ or 11520. For the former, since $N_v \lhd A_v$, $N_v = A_v \cong \mathbb{Z}_2^6 : \Gamma L(2, 4)$ by Lemma 2.1, which is insoluble. But all the subgroups of N with order 23040 are soluble by Magma [1], a contradiction. For the latter, since A_v has no such normal subgroups of order 11520 by Lemma 2.1, we can exclude this case. Similarly, we can exclude the cases where $N \cong \mathrm{PSL}(2, 81)$ and A_7. Suppose that $N \cong A_9$. Then $q = 7$ and $|N_v| = 40$ or 20. Since N has no subgroups of order 40 by Magma [1], N is transitive on $V\Gamma$. Thus N is arc-transitive on Γ. Hence $\Gamma \cong \mathcal{C}_{126}$ by Example 3.2. Suppose that $N \cong \mathrm{PSL}(2, 19)$. Then $q = 19$ and $\mathrm{PSL}(2, 19) \leq A \leq \mathrm{PGL}(2, 19)$. So $\Gamma \cong \mathcal{C}_{342}^1$ or $\mathcal{C}_{342}^2$ by Example 3.2.

Case 4. Assume that $p = 5$ and $q > 5$. Then $5^3 \cdot q \mid |N| \mid |A| \mid 2^{10} \cdot 3^2 \cdot 5^3 \cdot q$. By Lemma 2.3, $N \cong \mathrm{PSL}(3, 5)$ or $\mathrm{PSU}(3, 5)$. Suppose that $N \cong \mathrm{PSL}(3, 5)$. Then $q = 31$ and $|N_v| = 480$ or 240, which is impossible as A_v has no such normal subgroups which is isomorphic to N_v by Lemma 2.1 and Magma [1]. Similarly, we can also exclude the case where $N \cong \mathrm{PSU}(3, 5)$.

Case 5. Assume that $q = 3$ and $p > 5$. Then $3 \cdot 5 \cdot p^2 \mid |N| \mid |A| \mid 2^{10} \cdot 3^3 \cdot 5 \cdot p^2$.

It follows that $N \cong T$, which is impossible as there exists no nonabelian simple group satisfying the conditions by Lemma 2.3.

Case 6. Assume that $q = 5$ and $p > 5$. Then $5^2 \cdot p^2 \bigm| |N| \bigm| |A| \bigm| 2^{10} \cdot 3^2 \cdot 5^2 \cdot p^2$. It follows that $N \cong T^2$, and $T = \mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,16)$ or $\mathrm{PSL}(2,31)$ by Lemma 2.3. Assume that N is transitive on $V\Gamma$. Then N is arc-transitive on Γ. By Lemma 2.2, $5 \bigm| |T_v|$, and so $5^2 \bigm| |N_v|$, which is a contradiction as $|N_v| \bigm| 2^9 \cdot 3^2 \cdot 5$. Hence N has exactly two orbits on $V\Gamma$. Suppose that $T = \mathrm{PSL}(2,11)$. Then $p = 11$ and $|N_v| = |N|/(5p^2) = 720$. By Lemma 2.1, $A_v \cong A_4 \times A_5$, $(A_4 \times A_5) : \mathbb{Z}_2$ or $S_4 \times S_5$, and so $|A| = 2^5 \cdot 3^2 \cdot 5^2 \cdot 11^2$, $2^6 \cdot 3^2 \cdot 5^2 \cdot 11^2$ or $2^7 \cdot 3^2 \cdot 5^2 \cdot 11^2$. Thus $A \cong \mathrm{PSL}(2,11)^2.O$, where $O = \mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. But a calculation by Magma [1] shows no graph Γ in this case. Suppose that $T = \mathrm{PSL}(2,16)$. Then $p = 17$ and $|N_v| = |N|/(5p^2) = 11520$. But all of the subgroups with order 11520 of N are soluble by Magma [1], a contradiction. Suppose that $T = \mathrm{PSL}(2,31)$. Then $p = 31$ and $|N_v| = 46080$, which is not possible as $|A_v| \le 23040$ by Lemma 2.1. $\square$

Combining Lemmas 4.1 and 4.2, we complete the proof of Theorem 1.1.

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Hailin Liu
School of science, Jiangxi University of Science and Technology, Ganzhou 341000, P. R. China
and
School of Mathematics and Statistics, Yunnan University, Kunming 650091, P. R. China
E-mail address: `hailinliuqp@163.com`

Bengong Lou
School of Mathematics and Statistics, Yunnan University, Kunming 650091, P. R. China
E-mail address: `bengong188@163.com`

Bo Ling
School of Mathematics and Computer Science, Yunnan Minzu University, Kunming 650504, P. R. China
E-mail address: `bolinggxu@163.com`