TAIWANESE JOURNAL OF MATHEMATICS
Vol. 22, No. 3, pp. 677–693, June 2018
DOI: 10.11650/tjm/171104

Multiplication of Distributions and Travelling Wave Solutions for the Keyfitz-Kranzer System

Carlos Orlando R. Sarrico

Abstract. The present paper concerns the study of distributional travelling waves for the model problem $u_t + (u^2 - v)_x = 0$, $v_t + (u^3/3 - u)_x = 0$, also called the Keyfitz-Kranzer system. In the setting of a product of distributions, which is not defined by approximation processes, we are able to define a rigourous concept of a solution which extends the classical solution concept. As a consequence, we will establish necessary and sufficient conditions for the propagation of distributional profiles and explicit examples are given. A survey of the main ideas and formulas for multiplying distributions is also provided.

1. Introduction and contents

Let us consider the Keyfitz Kranzer system

$$u_t + (u^2 - v)_x = 0, \tag{1.1}$$

$$v_t + \left(\frac{1}{3}u^3 - u\right)_x = 0, \tag{1.2}$$

where $x \in \mathbb{R}$ is the space variable, $t \in \mathbb{R}$ is the time variable, and $u(x,t)$, $v(x,t)$ are the unknown real state variables.

Although this model has been invented with the purpose of satisfying certain mathematical properties, it is related with the isentropic gas dynamic system [11] and also with a model for a nonlinear elastic system [2]. It was also generalized and studied from several viewpoints [10, 12, 13, 28, 29]. In the present paper, travelling wave solutions with a distributional profile are studied; as far as we know this problem has not been addressed in the literature. The main result is a necessary and sufficient condition for the propagation of distributional wave profiles. This condition allows us to prove that continuous travelling wave solutions are necessarily constants. Therefore, if we want to seek for nonconstant travelling wave solutions we have to seek them among distributions that are not continuous functions. The main result makes also possible the identification of an interesting set

Received February 21, 2017; Accepted November 29, 2017.
Communicated by Cheng-Hsiung Hsu.
2010 *Mathematics Subject Classification.* 46F10, 35D99, 35L67.
Key words and phrases. products of distributions, conservation laws, travelling shock waves, travelling delta waves, travelling waves which are not measures.

of distributional profiles. Thus, we will see the possibility of propagation of wave profiles containing discontinuous functions, Dirac delta measures and also distributions that are not measures as, for example,

- $U = 2\sqrt{3}H$, and $V = 6H$ with speed $c = \sqrt{3}$;
- $U = H + \delta$, and $V = \frac{1}{2}\big(1 - \sqrt{11/3}\big)(H + \delta)$ with speed $c = \frac{1}{2}\big(1 + \sqrt{11/3}\big)$;
- $U = D\delta$ and $V = D\delta$ with speed $c = 2$;
- $U = D\delta$ and $V = 2D\delta$ with speed $c = 1$

(H stands for the Heaviside function, δ stands for the Dirac measure and D is the usual derivative operator in distributional sense).

It is usual to think that distributions cannot satisfy a nonlinear system like (1.1), (1.2) because a clear explanation of the sense in which they satisfy the system is still lacking. On the other hand, distributional solutions obtained by limit processes often depend on the chosen processes (in general asymptotic algorithms) and cannot be substituted into equations or systems owing to the well known difficulties of multiplying distributions. We will see that, in several cases, these difficulties can be overcome by defining a solution concept in the setting of a multiplication of distributions that gives a distribution as a result. To show the scope of these methods, let us recall some results we have obtained.

For the conservation law

$$u_t + [\phi(u)]_x = \psi(u),$$

where ϕ, ψ are entire functions taking real values on the real axis, we have established [21] necessary and sufficient conditions for the propagation of a travelling wave with a given distributional profile and we also have computed its speed. For C^1-wave profiles with one jump discontinuity, our methods easily lead to the well known Rankine-Hugoniot condition.

Conditions for the propagation of travelling waves with profiles $\beta + m\delta$ and $\beta + m\delta'$ (where β is a continuous function, $m \in \mathbb{R}$ and $m \neq 0$) were also obtained, as well as their speeds [22].

Gas dynamics phenomena, known as "infinitely narrow soliton solutions", discovered by Maslov and collaborators [5, 7, 14, 15], can be obtained directly in distributional form [19].

For a Riemann problem concerning the generalized pressureless gas dynamics system

$$u_t + [\phi(u)]_x = 0,$$
$$v_t + [\psi(u)v]_x = 0,$$

only assuming $\phi, \psi\colon \mathbb{R} \to \mathbb{R}$ continuous, we were able to show the formation of a delta shock wave solution [24]. In this case we arrived, more easily and in a much more general setting, to the same result of Danilov and Mitrovic [6], which have employed the weak asymptotic method, and also to the same result of Mitrovic et al. [16], which have used a different approach, based on a linearization process.

In the Brio system

$$u_t + \frac{1}{2}(u^2 + v^2)_x = 0,$$
$$v_t + (uv - v)_x = 0,$$

a simplified model for the study of plasmas, we got a delta shock wave as explicit solution for a Riemann problem [26]. This problem (suggested by Hayes and LeFlock in [8], p. 1558), was first studied by Kalish and Mitrovic [9] who also constructed a delta shock wave using an extension of the weak asymptotic method. Their solution coincides with our solution (in [9] p. 712 there is a mispring in formula (3.8); the correct $\alpha(t)$ has the opposite sign and in [26] p. 522, formula (17), $-k_0/c_0$ must replace $-k_0/b_0$).

Also for the Brio system we have subjected $u(x,t)$ and $v(x,t)$ to the initial conditions

$$u(x,0) = c_0\delta(x),$$
$$v(x,0) = h_0\delta(x)$$

with $c_0, h_0 \in \mathbb{R} \setminus \{0\}$. Under certain assumptions, we got, as solutions, travelling delta waves with speed $(c_0^2 + h_0^2)/(c_0^2 - h_0^2)$ and certain singular perturbations (which are not measures) propagating with speed 1 [25].

Regarding the interaction of singular waves, we have shown that delta waves under collision behave just as classical soliton collisions (as in the Korteweg-de Vries equation) in models ruled by a singular perturbation of Burgers conservative equation [20]. Also in a conservation law with singular flux, the interaction of a δ wave with a δ' wave was studied. Here, we were able to distinguish three distinct dynamics for that collision to which correspond phenomena of solitonic behavior, scattering, and merging [27].

In our framework, the product of two distributions is a distribution that depends on the choice of a certain function α encoding the indeterminacy inherent to such products. This indeterminacy generally is not avoidable and in many cases it also has a physical meaning; concerning this point let us mention [1,3,4,18]. Thus, the solutions of differential equations containing such products may depend (or not) of α. We call such solutions α-solutions. The possibility of their occurrence depends on the physical system: in certain cases we cannot previously know the behavior of the system, possibly due to physical features omitted in the formulation of the model with the goal of simplifying it. Thus, the mathematical indetermination sometimes observed may have this origin.

Let us now summarize the present paper's contents. In Section 2, we present the main ideas of our method for multiplying distributions. In Section 3, we define powers of certain distributions. In Section 4, we define the concept of α-solution for the system (1.1), (1.2); this concept is a consistent extension of the classical solution concept. In Section 5, we present necessary and sufficient conditions for the propagation of distributions as wave profiles and some examples are given.

2. The multiplication of distributions

2.1. A general product of distributions

Let $\mathcal{C}^\infty$ be the space of indefinitely differentiable real or complex-valued functions defined on $\mathbb{R}^N$, $N \in \{1, 2, 3, \ldots\}$, and $\mathcal{D}$ the subspace of C^∞ consisting of those functions with compact support. Let $\mathcal{D}'$ be the space of Schwartz distributions and $L(\mathcal{D})$ the space of continuous linear maps $\phi\colon \mathcal{D} \to \mathcal{D}$, where we suppose $\mathcal{D}$ endowed with the usual topology. We will sketch the main ideas of our distributional product (the reader can look at (2.4), (2.8), and (2.10) as definitions, if he prefers to skip this presentation). For proofs and other details concerning this product see [17].

First, we define a product $T\phi \in \mathcal{D}'$ for $T \in \mathcal{D}'$ and $\phi \in L(\mathcal{D})$ by

$$\langle T\phi, \xi\rangle = \langle T, \phi(\xi)\rangle$$

for all $\xi \in \mathcal{D}$; this makes $\mathcal{D}'$ a right $L(\mathcal{D})$-module. Next, we define an epimorphism $\widetilde{\zeta}\colon L(\mathcal{D}) \to \mathcal{D}'$, where the image of ϕ is the distribution $\widetilde{\zeta}(\phi)$ given by

$$\langle \widetilde{\zeta}(\phi), \xi\rangle = \int \phi(\xi)$$

for all $\xi \in \mathcal{D}$ (when the domain of the integral is not specified, we consider that it is extended all over $\mathbb{R}^N$); given $S \in \mathcal{D}'$, we say that ϕ is a representative operator of S if $\widetilde{\zeta}(\phi) = S$. For instance, if $\beta \in C^\infty$ is seen as a distribution, the operator $\phi_\beta \in L(\mathcal{D})$ defined by $\phi_\beta(\xi) = \beta\xi$, for all $\xi \in \mathcal{D}$, is a representative operator of β because, for all $\xi \in \mathcal{D}$, we have

$$\langle \widetilde{\zeta}(\phi_\beta), \xi\rangle = \int \phi_\beta(\xi) = \int \beta\xi = \langle \beta, \xi\rangle.$$

For this reason $\widetilde{\zeta}(\phi_\beta) = \beta$. If $T \in \mathcal{D}'$, we also have

$$\langle T\phi_\beta, \xi\rangle = \langle T, \phi_\beta(\xi)\rangle = \langle T, \beta\xi\rangle = \langle T\beta, \xi\rangle$$

for all $\xi \in \mathcal{D}$. Hence,

$$T\beta = T\phi_\beta.$$

Thus, given $T, S \in \mathcal{D}'$, we are tempted to define a natural product by setting $TS := T\phi$, where $\phi \in L(\mathcal{D})$ is a representative operator of S, i.e., ϕ is such that $\widetilde{\zeta}(\phi) = S$. Unfortunately, this product is not well defined, because TS depends on the representative $\phi \in L(\mathcal{D})$ of $S \in \mathcal{D}'$.

This difficulty can be overcome, if we fix $\alpha \in \mathcal{D}$ with $\int \alpha = 1$ and define $s_\alpha \colon L(\mathcal{D}) \to L(\mathcal{D})$ by

$$[(s_\alpha\phi)(\xi)](y) = \int \phi[(\tau_y\check{\alpha})\xi] \tag{2.1}$$

for all $\xi \in \mathcal{D}$ and all $y \in \mathbb{R}^N$, where $\tau_y\check{\alpha}$ is given by $(\tau_y\check{\alpha})(x) = \check{\alpha}(x-y) = \alpha(y-x)$ for all $x \in \mathbb{R}^N$. It can be proved that for each $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, $s_\alpha(\phi) \in L(\mathcal{D})$, s_α is linear, $s_\alpha \circ s_\alpha = s_\alpha$ (s_α is a projector of $L(\mathcal{D})$), $\ker s_\alpha = \ker \widetilde{\zeta}$, and $\widetilde{\zeta} \circ s_\alpha = \widetilde{\zeta}$.

Now, for each $\alpha \in \mathcal{D}$, we can define a general α-product $\underset{\alpha}{\odot}$ of $T \in \mathcal{D}'$ with $S \in \mathcal{D}'$ by setting

$$T \underset{\alpha}{\odot} S := T(s_\alpha\phi), \tag{2.2}$$

where $\phi \in L(\mathcal{D})$ is a representative operator of $S \in \mathcal{D}'$. This α-product is independent of the representative ϕ of S, because if ϕ, ψ are such that $\widetilde{\zeta}(\phi) = \widetilde{\zeta}(\psi) = S$, then $\phi - \psi \in \ker \widetilde{\zeta} = \ker s_\alpha$. Hence,

$$T(s_\alpha\phi) - T(s_\alpha\psi) = T[s_\alpha(\phi - \psi)] = 0.$$

Since ϕ in (2.2) satisfies $\widetilde{\zeta}(\phi) = S$, we have $\int \phi(\xi) = \langle S, \xi\rangle$ for all $\xi \in \mathcal{D}$, and by (2.1)

$$[(s_\alpha\phi)(\xi)](y) = \langle S, (\tau_y\check{\alpha})\xi\rangle = \langle S\xi, \tau_y\check{\alpha}\rangle = (S\xi * \alpha)(y)$$

for all $y \in \mathbb{R}^N$, which means that $(s_\alpha\phi)(\xi) = S\xi * \alpha$. Therefore, for all $\xi \in \mathcal{D}$,

$$\begin{aligned}\langle T \underset{\alpha}{\odot} S, \xi\rangle &= \langle T(s_\alpha\phi), \xi\rangle = \langle T, (s_\alpha\phi)(\xi)\rangle = \langle T, S\xi * \alpha\rangle \\ &= [T * (S\xi * \alpha)\check{}\,](0) = [(S\xi)\check{}\, * (T * \check{\alpha})](0) = \langle (T * \check{\alpha})S, \xi\rangle,\end{aligned}$$

and we obtain an easier formula for the general product (2.2),

$$T \underset{\alpha}{\odot} S = (T * \check{\alpha})S. \tag{2.3}$$

In general, this α-product is neither commutative nor associative but it is bilinear and satisfies the Leibniz rule written in the form

$$D_k(T \underset{\alpha}{\odot} S) = (D_kT) \underset{\alpha}{\odot} S + T \underset{\alpha}{\odot} (D_kS),$$

where D_k is the usual k-partial derivative operator in distributional sense ($k = 1, 2, \ldots, N$).

Recall that the usual Schwartz products of distributions are not associative and the commutative property is a convention inherent to the definition of such products (see the classical monograph of Schwartz [30] pp. 117, 118, and 121, where these products are defined). Unfortunately, the α-product (2.3), in general, is not consistent with the classical Schwartz products of distributions with functions.

2.2. How to get a product consistent with the Schwartz product of a distribution with a C^∞-function

In order to obtain the referred consistency, we are going to introduce some definitions and single out a certain subspace H_α of $L(\mathcal{D})$.

An operator $\phi \in L(\mathcal{D})$ is said to vanish on an open set $\Omega \subset \mathbb{R}^N$, if and only if $\phi(\xi) = 0$ for all $\xi \in \mathcal{D}$ with support contained in Ω. The support of an operator $\phi \in L(\mathcal{D})$ will be defined as the complement of the largest open set in which ϕ vanishes.

Let $\mathcal{N}$ be the set of operators $\phi \in L(\mathcal{D})$ whose support has Lebesgue measure zero, and $\rho(C^\infty)$ the set of operators $\phi \in L(\mathcal{D})$ defined by $\phi(\xi) = \beta\xi$ for all $\xi \in \mathcal{D}$, with $\beta \in C^\infty$. For each $\alpha \in \mathcal{D}$, with $\int \alpha = 1$, let us consider the space $H_\alpha = \rho(C^\infty) \oplus s_\alpha(\mathcal{N}) \subset L(\mathcal{D})$. It can be proved that $\zeta_\alpha := \widetilde{\zeta}|_{H_\alpha} \colon H_\alpha \to C^\infty \oplus \mathcal{D}'_\mu$ is an isomorphism ($\mathcal{D}'_\mu$ stands for the space of distributions whose support has Lebesgue measure zero). Therefore, if $T \in \mathcal{D}'$ and $S = \beta + f \in C^\infty \oplus \mathcal{D}'_\mu$, a new α-product, $\dot{\alpha}$, can be defined by $T \dot{\alpha} S := T\phi_\alpha$, where for each α, $\phi_\alpha = \zeta_\alpha^{-1}(S) \in H_\alpha$. Hence,

$$\begin{aligned} T \dot{\alpha} S &= T\zeta_\alpha^{-1}(S) = T[\zeta_\alpha^{-1}(\beta + f)] \\ &= T[\zeta_\alpha^{-1}(\beta) + \zeta_\alpha^{-1}(f)] = T\beta + T \underset{\alpha}{\odot} f = T\beta + (T * \check{\alpha})f, \end{aligned}$$

and putting α instead of $\check{\alpha}$ (to simplify), we get

$$T \dot{\alpha} S = T\beta + (T * \alpha)f. \tag{2.4}$$

Thus, the referred consistency is obtained when the C^∞-function is placed at the right-hand side: if $S \in C^\infty$, then $f = 0$, $S = \beta$, and $T \dot{\alpha} S = T\beta$.

2.3. How to obtain the consistency with all Schwartz products of D'^p-distributions with C^p-functions

The α-product (2.4) can be easily extended for $T \in \mathcal{D}'^p$ and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, \dots, \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $T\beta$ is the Schwartz product of a $\mathcal{D}'^p$-distribution with a C^p-function, and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. This extension is clearly consistent with all Schwartz products of $\mathcal{D}'^p$-distributions with C^p-functions, if the

C^p-functions are placed at the right-hand side. It also keeps the bilinearity and satisfies the Leibniz rule written in the form

$$D_k(T\dot{\alpha}S) = (D_kT)\dot{\alpha}S + T\dot{\alpha}(D_kS),$$

clearly under certain natural conditions; for $T \in \mathcal{D}'^p$, we must suppose $S \in C^{p+1} \oplus \mathcal{D}'_\mu$. Moreover, these products are invariant by translations, that is,

$$\tau_a(T\dot{\alpha}S) = (\tau_aT)\dot{\alpha}(\tau_aS),$$

where τ_a stands for the usual translation operator in distributional sense. These products are also invariant for the action of any group of linear transformations $h\colon \mathbb{R}^N \to \mathbb{R}^N$ with $|\det h| = 1$, that leave α invariant.

Thus, for each $\alpha \in \mathcal{D}$ with $\int \alpha = 1$, formula (2.4) allows us to evaluate the product of $T \in \mathcal{D}'^p$ with $S \in C^p \oplus \mathcal{D}'_\mu$; therefore, we have obtained a family of products, one for each α.

From now on, we always consider the dimension $N = 1$. For instance, if β is a continuous function we have for each α by applying (2.4),

$$\delta\dot{\alpha}\beta = \delta\dot{\alpha}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta,$$

$$\beta\dot{\alpha}\delta = \beta\dot{\alpha}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta,$$

$$\delta\dot{\alpha}\delta = \delta\dot{\alpha}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \tag{2.5}$$

$$H\dot{\alpha}\delta = (H * \alpha)\delta = \left[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\,d\tau\right]\delta = \left(\int_{-\infty}^{0} \alpha\right)\delta, \tag{2.6}$$

$$(D\delta)\dot{\alpha}(D\delta) = [(D\delta) * \alpha]D\delta = \alpha'(0)D\delta - \alpha''(0)\delta. \tag{2.7}$$

For each α, the support of the α-product (2.4) satisfies $\operatorname{supp}(T\dot{\alpha}S) \subset \operatorname{supp} S$, as for usual functions, but it may happen that $\operatorname{supp}(T\dot{\alpha}S) \not\subset \operatorname{supp} T$.

2.4. Other products we need in the present paper

It is also possible to multiply many other distributions preserving the consistency with all Schwartz products of distributions with functions. For instance, using the Leibniz formula to extend the α-products, it is possible to write

$$T\dot{\alpha}S = Tw + (T * \alpha)f \tag{2.8}$$

with $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ stands for the space of distributions $T \in \mathcal{D}'$ such that $DT \in \mathcal{D}'^0$ and Tw is the usual pointwise product of $T \in \mathcal{D}'^{-1}$ with $w \in L^1_{\text{loc}}$. Recall that, locally, T can be read as a function of bounded variation (see [23], Section 2 for details). For instance, since $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\text{loc}} \oplus \mathcal{D}'_\mu$, we have

$$H\dot{\alpha}H = HH + (H * \alpha)0 = H. \tag{2.9}$$

More generally, if $T \in \mathcal{D}'^{-1}$ and $S \in L^1_{\text{loc}}$, then $T_{\dot{\alpha}} S = TS$; actually, using (2.8) we can write

$$T_{\dot{\alpha}} S = T_{\dot{\alpha}}(S + 0) = TS + (T * \alpha)0 = TS.$$

Thus, in distributional sense, the α-products of functions that, locally, are of bounded variation coincide with the usual pointwise product of these functions considered as a distribution. We stress that in (2.4) or (2.8) the convolution $T * \alpha$ is not to be understood as an approximation of T. Those formulas are exact.

Another useful extension that will be applied is given by the formula

$$T_{\dot{\alpha}} S = D(Y_{\dot{\alpha}} S) - Y_{\dot{\alpha}}(DS) \tag{2.10}$$

for $T \in \mathcal{D}'^0 \cap \mathcal{D}'_\mu$ and $S, DS \in L^1_{\text{loc}} \oplus \mathcal{D}'_c$, where $\mathcal{D}'_c \subset \mathcal{D}'_\mu$ is the space of distributions whose support is at most countable, and $Y \in \mathcal{D}'^{-1}$ is such that $DY = T$ (the products $Y_{\dot{\alpha}} S$ and $Y_{\dot{\alpha}}(DS)$ are supposed to be computed by (2.4) or (2.8)). The value of $T_{\dot{\alpha}} S$ given by (2.10) is independent of the choice of $Y \in \mathcal{D}'^{-1}$ such that $DY = T$ (see [23] p. 1004 for the proof). For instance, by (2.10) and (2.6) we have, for any α,

$$\delta_{\dot{\alpha}} H = D(H_{\dot{\alpha}} H) - H_{\dot{\alpha}}(DH) = DH - H_{\dot{\alpha}} \delta = \delta - \left(\int_{-\infty}^{0} \alpha \right) \delta = \left(\int_{0}^{+\infty} \alpha \right) \delta \tag{2.11}$$

so that

$$H_{\dot{\alpha}} \delta + \delta_{\dot{\alpha}} H = \delta \tag{2.12}$$

for any α. The products (2.4), (2.8), and (2.10) are compatible, that is, if an α-product can be computed by two of them, the result is the same.

3. Powers of distributions

Let $M \subset \mathcal{D}'$ be a set of distributions such that, if $T_1, T_2 \in M$, then $T_1{}_{\dot{\alpha}} T_2$ is well defined and $T_1{}_{\dot{\alpha}} T_2 \in M$. For each $T \in M$ we define the α-power T^n_α by the recurrence relation

$$T^n_\alpha = (T^{n-1}_\alpha)_{\dot{\alpha}} T \quad \text{for } n \geq 1, \text{ with } T^0_\alpha = 1 \text{ for } T \neq 0; \tag{3.1}$$

naturally, if $0 \in M$, $0^n_\alpha = 0$ for all $n \geq 1$.

Since our distributional products are consistent with the Schwartz products of distributions with functions, when the functions are placed at the right-hand side, we have $\beta^n_\alpha = \beta^n$ for all $\beta \in C^0 \cap M$. Thus, this definition is consistent with the usual definition of powers of C^0-functions. Moreover, if M is such that $\tau_a T \in M$ for all $T \in M$ and all $a \in \mathbb{R}$, then we also have $(\tau_a T)^n_\alpha = \tau_a(T^n_\alpha)$.

Taking, for instance, $M = C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$ and supposing $T_1, T_2 \in M$, we have $T_1 = \beta_1 + f_1$, $T_2 = \beta_2 + f_2$ and by (2.4), we can write

$$\begin{aligned} T_1 \dot{\alpha} T_2 &= T_1\beta_2 + (T_1 * \alpha) f_2 = (\beta_1 + f_1)\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \\ &= \beta_1\beta_2 + f_1\beta_2 + [(\beta_1 + f_1) * \alpha] f_2 \in M. \end{aligned}$$

Therefore, we can define α-powers T^n_α of distributions $T \in C^p \oplus (\mathcal{D}'^p \cap \mathcal{D}'_\mu)$. For instance, we have $\delta^0_\alpha = 1$, $\delta^1_\alpha = \delta$, and for $n \geq 2$, $\delta^n_\alpha = \alpha(0)^{n-1}\delta$, as can be easily seen by induction applying (2.5).

Setting $M = \mathcal{D}'^{-1}$ and supposing $T_1, T_2 \in \mathcal{D}'^{-1}$, we have $T_1 \dot{\alpha} T_2 \in \mathcal{D}'^{-1}$. Thus, we can also define α-powers T^n_α of distributions $T \in \mathcal{D}'^{-1}$ by the recurrence relation (3.1) and clearly we get,

$$T^n_\alpha = T^n,$$

that is, in distributional sense the α-powers of functions that, locally, are of bounded variation, coincide with the usual powers of these functions when considered as distributions. The usual rule of the derivative of a power, in general, cannot be applied. In the sequel we will write, in all cases, T^n instead of T^n_α, supposing α fixed. For instance, we will write $\delta^3 = \alpha(0)^2\delta$ instead of $\delta^3_\alpha = \alpha(0)^2\delta$.

4. The α-solution concept

Let I be an interval of $\mathbb{R}$ with more that one point, and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\widetilde{u}\colon I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$, the notation $[\widetilde{u}(t)](x)$ is sometimes used for emphasizing that the distribution $\widetilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ depending on x.

Let $\Sigma(I)$ be the space of functions $u\colon \mathbb{R} \times I \to \mathbb{R}$ such that:

(a) for each $t \in I$, $u(x,t) \in L^1_{\text{loc}}(\mathbb{R})$;

(b) $\widetilde{u}\colon I \to \mathcal{D}'$, defined by $[\widetilde{u}(t)](x) = u(x,t)$ is in $\mathcal{F}(I)$.

The natural injection $u \mapsto \widetilde{u}$ from $\Sigma(I)$ into $\mathcal{F}(I)$ identifies any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R} \times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

Thus, identifying u with $\widetilde{u}$ and v with $\widetilde{v}$ the system (1.1), (1.2) can be read as follows:

$$\frac{d\widetilde{u}}{dt}(t) + D[\widetilde{u}(t)^2 - \widetilde{v}(t)] = 0, \tag{4.1}$$

$$\frac{d\widetilde{v}}{dt}(t) + D\left(\frac{1}{3}\widetilde{u}(t)^3 - \widetilde{u}(t)\right) = 0. \tag{4.2}$$

Definition 4.1. Given α, the pair $(\widetilde{u},\widetilde{v}) \in \mathcal{F}(I)\times\mathcal{F}(I)$ will be called an α- solution for the system (4.1), (4.2) on I, if $\widetilde{u}(t)^2$ and $\widetilde{u}(t)^3$ are well defined distributions, and if both equations are satisfied for all $t\in I$.

This definition sees the system (1.1), (1.2) as an evolution system and we have the following results:

Theorem 4.2. *If (u,v) is a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$ then, for any α, the pair $(\widetilde{u},\widetilde{v}) \in \mathcal{F}(I)\times\mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$, $[\widetilde{v}(t)](x) = v(x,t)$ is an α-solution of (4.1), (4.2) on I.*

Note that, by a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$, we mean a pair $(u(x,t), v(x,t))$ of C^1-functions that satisfies (1.1), (1.2) on $\mathbb{R}\times I$.

Theorem 4.3. *If $u, v\colon \mathbb{R}\times I\to\mathbb{R}$ are C^1-functions and, for a certain α, the pair $(\widetilde{u},\widetilde{v}) \in \mathcal{F}(I)\times\mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$, $[\widetilde{v}(t)](x) = v(x,t)$ is an α-solution of (4.1), (4.2) on I, then the pair $(u(x,t), v(x,t))$ is a classical solution of (1.1), (1.2) on $\mathbb{R}\times I$.*

For the proof, it is enough to observe that any C^1-functions $u(x,t)$ can be read as continuously differentiable function $\widetilde{u}\in\mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$ and to use the consistency of the α-products with the classical Schwartz products of distributions with functions.

Definition 4.4. Given α, any α-solution $(\widetilde{u},\widetilde{v})$ of (4.1), (4.2) on I, will be called an α-solution of the system (1.1), (1.2) on I.

As a consequence, an α-solution $(\widetilde{u},\widetilde{v})$ in this sense, read as a usual distributional solution (u,v), affords a consistent extension of the concept of a classical solution for the system (1.1), (1.2). Thus, and for short, we also call to (u,v) an α-solution of (1.1), (1.2).

5. Travelling waves for the Keyfitz Kranzer system

For the sake of simplicity we introduce the following definition:

Definition 5.1. Let $\gamma\colon\mathbb{R}\to\mathbb{R}$ be a C^1-function. Then, given α, the wave profiles $U, V\in\mathcal{D}'$ are said to α-propagate, according to (1.1), (1.2), with the movement $\gamma(t)$ (and speed $\gamma'(t)$) if the travelling waves $\widetilde{u}(t) = \tau_{\gamma(t)}U$ and $\widetilde{v}(t) = \tau_{\gamma(t)}V$ are α-solutions of (4.1), (4.2) on $\mathbb{R}$.

Theorem 5.2. *Given α, let $U, V\in\mathcal{D}'$ be such that U^2 and U^3 are well defined distributions. Then,*

- *if $DU\neq 0$, the wave profiles U, V α-propagate with the movement $\gamma(t)$ if and only if the following three conditions are satisfied:*

(a) $\gamma'(t) = c$ *is a constant function;*

(b) $cDU = DU^2 - DV$*;*

(c) $cDV = \frac{1}{3}DU^3 - DU$.

- *if* $DU = 0$, *the wave profiles* U, V α*-propagate if and only if* $DV = 0$*; in this case the movement* $\gamma(t)$ *is arbitrary.*

Remark 5.3. Recall that U^2 and U^3 may depend on α and also that, in general, we cannot apply the usual law of the derivative of a power.

Proof of Theorem 5.2. Suppose that the profiles U, V α-propagate with the movement $\gamma(t)$. Then, by Definition 5.1, $\widetilde{u}(t) = \tau_{\gamma(t)}U$ and $\widetilde{v}(t) = \tau_{\gamma(t)}V$ are α-solutions of (4.1), (4.2) on $\mathbb{R}$, which means that, for all $t \in \mathbb{R}$,

$$\tau_{\gamma(t)}DU[-\gamma'(t)] + D\left[\tau_{\gamma(t)}U^2 - \tau_{\gamma(t)}V\right] = 0,$$
$$\tau_{\gamma(t)}DV[-\gamma'(t)] + D\left[\frac{1}{3}\tau_{\gamma(t)}U^3 - \tau_{\gamma(t)}U\right] = 0.$$

These equations are respectively equivalent to

$$\tau_{\gamma(t)}\{DU[-\gamma'(t)] + DU^2 - DV\} = 0,$$
$$\tau_{\gamma(t)}\left\{DV[-\gamma'(t)] + \frac{1}{3}DU^3 - DU\right\} = 0,$$

and applying the operator $\tau_{-\gamma(t)}$ to both equalities we conclude that, for all t,

$$\gamma'(t)DU = DU^2 - DV, \tag{5.1}$$

$$\gamma'(t)DV = \frac{1}{3}DU^3 - DU. \tag{5.2}$$

Suppose $DU \neq 0$. Then, since the right-hand side of (5.1) is independent of t we conclude that $\gamma'(t) = c$ is a constant function and (b),(c) follow. Suppose $DU = 0$. Then U can be seen as an almost everywhere constant function and $DU^2 = DU^3 = 0$. From (5.1) $DV = 0$ follows and (5.1), (5.2) are clearly satisfied with $\gamma(t)$ arbitrarily chosen. The statement is proved. □

Now, let us suppose that $U, V \in C^1$. Then, using the consistency of the α-products with the Schwartz products of distributions with functions, (b) and (c) turn out to be

$$cU' = 2UU' - V',$$
$$cV' = U^2U' - U',$$

and this is the system we obtain when, for the system (1.1), (1.2), we seek for travelling wave solutions $u(x,t) = U(x - ct)$, $v(x,t) = V(x - ct)$. This shows that Definition (5.1) provides a consistent extension of the travelling wave classical concept for the system (1.1), (1.2).

Theorem 5.4. *Suppose that the wave profiles $U, V \in C^0$ α-propagate, according to* (1.1), (1.2)*, with the movement $\gamma(t)$. Then, U and V are constant functions.*

Proof. By assumption and Theorem 5.2, if $DU \neq 0$ we would have (a), (b) and (c). However this is impossible. Actually, from (b) we would have $DV = DU^2 - cDU$ and from (c) we would also have

$$D\left[\frac{1}{3}U^3 - cU^2 + (c^2-1)U\right] = 0.$$

Taking $F = \frac{1}{3}U^3 - cU^2 + (c^2-1)U$, we have $F \in C^0$ and $DF = 0$, which implies $F \in C^1$ and $F' = 0$ in the usual sense (see lemma of Du Bois-Reymond for instance, in [31] p. 162), that is, $F = k$ is a constant function. Thus, for all $x \in \mathbb{R}$ we have

$$\frac{1}{3}U^3(x) - cU^2(x) + (c^2-1)U(x) - k = 0.$$

This means that, for each x, $U(x)$ is a root of the polynomial $P(z) = \frac{1}{3}z^3 - cz^2 + (c^2-1)z - k$. Since this third degree polynomial does not vanish identically, its real roots are a nonempty set of isolated points. Thus, the continuous function U takes values on a non empty set of isolated points, that is, U is a constant function, which is a contradiction. Hence we always have $DU = 0$, and from Theorem 5.2, $DV = 0$ follows. Then $U, V \in C^1$ and $U' = V' = 0$. The theorem is proved. □

As a consequence, if we ask for nonconstant travelling waves of (1.1), (1.2), then we have to seek them among distributions which are not continuous functions. The following result provides an interesting particular class of travelling waves for the system (1.1), (1.2):

Theorem 5.5. *Given α, let $U, V \in \mathcal{D}'$ be such that U^2 and U^3 are well defined distributions and $U^2 = a + bU$ for certain $a, b \in \mathbb{R}$. Then, if $DU \neq 0$, the profiles $U, V \in \mathcal{D}'$ α-propagate with the movement $\gamma(t)$ if and only if the following four conditions are satisfied:*

(i) $b^2 - 4a \leq 12$;

(ii) $\gamma'(t) = c$ *is a constant function;*

(iii) $3c^2 - 3bc + a + b^2 - 3 = 0$;

(iv) $DV = (b-c)DU$.

Proof. By assumption we have $DU^2 = bDU$ and since $U^3 = (U^2)_{\dot\alpha} U = (a + bU)_{\dot\alpha} U = aU + bU^2$, we have

$$DU^3 = aDU + bDU^2 = (a + b^2)DU.$$

Now, suppose that the profiles U, V α-propagate with the movement $\gamma(t)$. Then, by Theorem 5.2, this is possible if and only if (a), (b) and (c) are satisfied. From (a) we have (ii) and (b),(c) turn out to be respectively

$$cDU = bDU - DV, \tag{5.3}$$

$$cDV = \frac{1}{3}(a + b^2)DU - DU. \tag{5.4}$$

Thus, (iv) follows from (5.3). From (5.4) we can write

$$(3c^2 - 3bc + a + b^2 - 3)DU = 0,$$

and, since $DU \neq 0$, (iii) follows. Also because c is a real number, (i) follows from (iii). The theorem is proved. □

We will apply Theorems 5.5 and 5.2 to some examples.

Example 5.6. Taking $U = r + (s - r)H$, with $s, r \in \mathbb{R}$ and $s \neq r$, it is easy to see that

$$U^2 = r^2 + (s^2 - r^2)H = a + bU$$

with $a = -sr$ and $b = s + r$. Since U^3 is also well defined, by Theorem 5.5 we conclude that, for any α, and any constant k, the α-propagation of the wave profiles

$$U = r + (s - r)H \quad \text{and} \quad V = (s - r)(s + r - c)H + (s + r - c)r + k,$$

becomes possible with constant speed c, if and only if

$$s^2 + 6sr + r^2 \leq 12,$$

and c satisfies the equation

$$3c^2 - 3(s + r)c + r^2 + rs + s^2 - 3 = 0.$$

A particular simple case can be obtained taking, for example, $r = 0$, $s = 2\sqrt{3}$ and $k = 0$: we conclude that, for any α, the wave profiles $U = 2\sqrt{3}H$ and $V = 6H$ α-propagate with speed $c = \sqrt{3}$. Exactly the same results can be obtained, within the classical setting, by applying the well known Rankine-Hugoniot conditions.

Example 5.7. Travelling waves for (1.1), (1.2) can also contain Dirac measures. Taking $U = H + \delta$, we have

$$U^2 = (H+\delta)\dot{_\alpha}(H+\delta) = H\dot{_\alpha}H + H\dot{_\alpha}\delta + \delta\dot{_\alpha}H + \delta\dot{_\alpha}\delta.$$

Using (2.5), (2.9) and (2.12) we get

$$U^2 = H + [1+\alpha(0)]\delta,$$

and U^3 is also clearly defined. Since $U^2 = a + bU$ if and only if $a = 0$, $b = 1$ and $\alpha(0) = 0$, by Theorem 5.5 we conclude that, for any α such that $\alpha(0) = 0$ and any constant k, the α-propagation of the wave profiles

$$U = H + \delta \quad \text{and} \quad V = (1-c)(H+\delta) + k \tag{5.5}$$

is possible with constant speed c if and only if $3c^2 - 3c - 2 = 0$.

In particular, taking $c = \frac{1}{2}\big(1 + \sqrt{11/3}\big)$ and $k = 0$, we conclude that, for any α such that $\alpha(0) = 0$ the α-propagation of wave profiles

$$U = H + \delta \quad \text{and} \quad V = \frac{1}{2}\left(1 - \sqrt{\frac{11}{3}}\right)(H+\delta)$$

is possible with speed $c = \frac{1}{2}\big(1 + \sqrt{11/3}\big)$.

However, if $\alpha(0) \neq 0$ it is not possible to apply Theorem 5.5. Hence, if in this setting we want a general solution, we must apply Theorem 5.2. Thus, using (2.11), we have

$$U^3 = U^2\,\dot{_\alpha}U = H + \left[1 + \alpha(0) + \alpha(0)\int_0^{+\infty}\alpha\right]\delta,$$

and conditions (b),(c) of Theorem 5.2 turn out to be respectively,

$$c(\delta + D\delta) = \delta + [1+\alpha(0)]D\delta - DV,$$
$$cDV = \frac{1}{3}\delta + \frac{1}{3}\left[1 + \alpha(0) + \alpha(0)\int_0^{+\infty}\alpha\right]D\delta - \delta - D\delta.$$

These conditions are satisfied if and only if

$$3c^2 - 3c - 2 = 0, \tag{5.6}$$

$$3c^2 - 3[1+\alpha(0)]c + \alpha(0)\left(1 + \int_0^{+\infty}\alpha\right) - 2 = 0. \tag{5.7}$$

Thus, we conclude that, in general, the α-propagation of the profiles (5.5) is possible if and only if c and α satisfy (5.6) and (5.7). Clearly, our first result can also be obtained as a particular case taking $\alpha(0) = 0$.

Example 5.8. Certain distributions which are not measures can also take the form of travelling waves for (1.1), (1.2). Taking $U = D\delta$, we have, applying (2.7), $U^2 = \alpha'(0)D\delta - \alpha''(0)\delta = a + bD\delta$ if and only if $\alpha''(0) = 0$, $a = 0$ and $b = \alpha'(0)$. Since U^3 is also clearly defined, by Theorem 5.5 we conclude that, for any α such that $\alpha''(0) = 0$ and $\alpha'(0)^2 \leq 12$, and any k, the α-propagation of the wave profiles

$$U = D\delta \quad \text{and} \quad V = [\alpha'(0) - c]D\delta + k$$

is possible with constant speed

$$c = \frac{1}{2}\alpha'(0) \pm \frac{1}{2\sqrt{3}}\sqrt{12 - \alpha'(0)^2}.$$

Two particular simple and interesting cases take place for $k = 0$, and any α such that $\alpha''(0) = 0$ and $\alpha'(0) = 3$:

- the wave profiles $U = D\delta$ and $V = D\delta$ α-propagate with speed $c = 2$;
- the wave profiles $U = D\delta$ and $V = 2D\delta$ α-propagate with speed $c = 1$.

Acknowledgments

The present research was supported by FCT, UID/MAT/04561/2013.

References

[1] A. Bressan and F. Rampazzo, *On differential systems with vector-valued impulsive controls*, Bull. Un. Mat. Ital. B (7) **2** (1988), no. 3, 641–656.

[2] J.-J. Cauret, J.-F. Colombeau, A. Y. LeRoux, *Discontinuous generalized solutions of nonlinear nonconservative hyperbolic equations*, J. Math. Anal. Appl. **139** (1989), no. 2, 552–573.

[3] J.-F. Colombeau and A. LeRoux, *Multiplications of distributions in elasticity and hydrodynamics*, J. Math. Phys. **29** (1988), no. 2, 315–319.

[4] G. Dal Maso, P. G. Lefloch and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl. (9) **74** (1995), no. 6, 483–548.

[5] V. G. Danilov, V. P. Maslov and V. M. Shelkovich, *Algebras of the singularities of singular solutions to first-order quasi-linear strictly hyperbolic systems*, Theoret. Mat. Phys. **114** (1998), no. 1, 1–42.

[6] V. G. Danilov and D. Mitrovic, *Delta shock wave formation in the case of triangular hyperbolic system of conservation laws*, J. Differerential Equations **245** (2008), no. 12, 3704–3734.

[7] Yu. V. Egorov, *On the theory of generalized functions*, Uspekhi Mat. Nauk **45** (1990), no. 5, 3–40.

[8] B. T. Hayes and P. G. LeFloch, *Measure solutions to a strictly hyperbolic system of conservation laws*, Nonlinearity **9** (1996), no. 6, 1547–1563.

[9] H. Kalisch and D. Mitrović, *Singular solutions of a fully nonlinear* 2×2 *system of conservation laws*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 3, 711–729.

[10] B. L. Keyfitz, *Conservation laws, delta-shocks and singular shocks*, in: *Nonlinear Theory of Generalized Functions* (Vienna, 1997), 99–111, Chapman & Hall/CRC Res. Notes Math **401**, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[11] ———, *Singular shocks: retrospective and prospective*, Confluentes Math. **3** (2011), no. 3, 445–470.

[12] B. L. Keyfitz and H. C. Kranzer, *The Riemann problem for a class of hyperbolic conservation laws exhibiting a parabolic degeneracy*, J. Differential Equations **47** (1983), no. 1, 35–65.

[13] ———, *Spaces of weighed measures for conservation laws with singular shock solutions*, J. Differential Equations **118** (1995), no. 2, 420–451.

[14] V. P. Maslov, *Non-standard characteristics in asymptotic problems*, Russian Math. Surveys **38** (1983), no. 6, 1–42.

[15] V. P. Maslov and G. A. Omel'yanov, *Asymptotic soliton-form solutions of equations with small dispersion*, Russian Math. Surveys **36** (1981), no. 3, 73–149.

[16] D. Mitrovic, V. Bojkovic and V. G. Danilov, *Linearization of the Riemann problem for a triangular system of conservation laws and delta shock wave formation process*, Math. Methods Appl. Sci. **33** (2010), no. 7, 904–921.

[17] C. O. R. Sarrico, *About a family of distributional products important in the applications*, Portugal. Math. **45** (1988), no. 3, 295–316.

[18] ———, *Distributional products and global solutions for nonconservative inviscid Burgers equation*, J. Math. Anal. Appl. **281** (2003), no. 2, 641–656.

[19] ______, *New solutions for the one-dimensional nonconservative inviscid Burgers equation*, J. Math. Anal. Appl. **317** (2006), no. 2, 496–509.

[20] ______, *Collision of delta-waves in a turbulent model studied via a distribution product*, Nonlinear Anal. **73** (2010), no. 9, 2868–2875.

[21] ______, *Products of distributions and singular travelling waves as solutions of advection-reaction equations*, Russ. J. Math. Phys. **19** (2012), no. 2, 244–255.

[22] ______, *Products of distributions, conservation laws and the propagation of δ'-shock waves*, Chin. Ann. Math. Ser. B **33** (2012), no. 3, 367–384.

[23] ______, *The multiplication of distributions and the Tsodyks model of synapses dynamics*, Int. J. Math. Anal. (Ruse) **6** (2012), no. 21-24, 999–1014.

[24] ______, *A distributional product approach to δ-shock wave solutions for a generalized pressureless gas dynamics system*, Int. J. Math. **25** (2014), no. 1, 1450007, 12 pp.

[25] ______, *The Brio system with initial conditions involving Dirac masses: a result afforded by a distributional product*, Chin. Ann. Math. Ser. B **35** (2014), no. 6, 941–954.

[26] ______, *The Riemann problem for the Brio system: a solution containing a Dirac mass obtained via a distributional product*, Russ. J. Math. Phys. **22** (2015), no. 4, 518–527.

[27] C. O. R. Sarrico and A. Paiva, *Products of distributions and collision of a δ-wave with a δ'-wave in a turbulent model*, J. Nonlinear Math. Phys. **22** (2015), no. 3, 381–394.

[28] D. G. Schaeffer, S. Schecter and M. Shearer, *Non-strictly hyperbolic conservation laws with a parabolic line*, J. Differential Equations **103** (1993), no. 1, 94–126.

[29] S. Schecter, *Existence of Dafermos profiles for singular shocks*, J. Differential Equations **205** (2004), no. 1, 185–210.

[30] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.

[31] M. Willem, *Analyse harmonique réele*, Hermann, Paris, 1995.

Carlos Orlando R. Sarrico
CMAFCIO, Faculdade de Ciências da Universidade de Lisboa, Campo Grande, 1749-016 Lisboa, Portugal
E-mail address: `corsarrico@gmail.com`