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Variable Anisotropic Hardy Spaces and Their Applications

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Abstract. Let $p(\cdot)\colon \mathbb{R}^n \to (0,\infty]$ be a variable exponent function satisfying the globally log-Hölder continuous condition and A a general expansive matrix on $\mathbb{R}^n$. In this article, the authors first introduce the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ associated with A, via the non-tangential grand maximal function, and then establish its radial or non-tangential maximal function characterizations. Moreover, the authors also obtain various equivalent characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$, respectively, by means of atoms, finite atoms, the Lusin area function, the Littlewood-Paley g-function or g_λ^*-function. As applications, the authors first establish a criterion on the boundedness of sublinear operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into a quasi-Banach space. Then, applying this criterion, the authors show that the maximal operators of the Bochner-Riesz and the Weierstrass means are bounded from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ and, as consequences, some almost everywhere and norm convergences of these Bochner-Riesz and Weierstrass means are also obtained. These results on the Bochner-Riesz and the Weierstrass means are new even in the isotropic case.

1. Introduction

The main purpose of this article is to introduce and to investigate the variable anisotropic Hardy space on $\mathbb{R}^n$. Due to the celebrated work [11–13] of Calderón and Torchinsky on parabolic Hardy spaces, there has been an increasing interest in extending classical function spaces from Euclidean spaces to some more general underlying spaces; see, for example, [28, 31, 32, 58, 59, 61, 65, 76]. Let A be a general expansive matrix on $\mathbb{R}^n$. Recall that the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ with $p \in (0,\infty)$ was first introduced by Bownik [6], which is a generalization of the parabolic Hardy space studied in [11]. Later on, Bownik et al. [7] further extended the anisotropic Hardy space to the weighted setting. For more progresses about this theory, we refer the reader to [25, 38, 44–48, 65, 66] and their references.

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On the other hand, as we all know, the variable function spaces have found their applications in fluid dynamics [1, 2, 57], image processing [14, 33, 64], partial differential equations and variational calculus [9, 23, 63] and harmonic analysis [4, 18, 21, 75]. Recall that, as a generalization of the classical Lebesgue space $L^p(\mathbb{R}^n)$, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, in which the constant exponent p is replaced by an exponent function $p(\cdot)\colon \mathbb{R}^n \to (0,\infty]$, can be traced back to the well-known article [56] of Orlicz and was also systematically studied by Musielak [51] and Nakano [53, 54]. Since then, a lot of interesting work on the theory of function spaces with variable exponents arose (see, for example, [17, 20, 26, 36]). In particular, the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with $p(\cdot)$ satisfying the so-called globally log-Hölder continuous condition was first introduced and investigated by Nakai and Sawano [52]. Then Sawano [60], Zhuo et al. [84] and Yang et al. [81] further completed the theory of this space. Independently, Cruz-Uribe and Wang [19], with some slightly weaker conditions on $p(\cdot)$ than those used in [52], also studied the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$. For more progresses about function spaces with variable exponents, we refer the reader to [5,22,34,35,55,69,73–75,82,83,85] and their references. In particular, Zhuo et al. [83] introduced the variable Hardy space $H^{p(\cdot)}(\mathcal{X})$ on an RD-space $\mathcal{X}$, with $p(\cdot) \in (n/(n+1),\infty)$, and established the real-variable theory of this space. Recall that a metric measure space of homogeneous type $\mathcal{X}$ is called an RD-space if it is a metric measure space of homogeneous type in the sense of Coifman and Weiss [15,16] and satisfies some reverse doubling property, which originates from Han et al. [32] (see also [31] and [80] for some equivalent characterizations). However, the real-variable theory of $H^{p(\cdot)}(\mathcal{X})$, with $p(\cdot) \in (0, n/(n+1)]$, is still unknown.

To give a complete real-variable theory of variable Hardy spaces in anisotropic setting, in this article, we first introduce the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ associated with some expansive matrix A, via the non-tangential grand maximal function, and then establish its radial or non-tangential maximal function characterizations. In addition, we also obtain various real-variable characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$, respectively, by means of atoms, finite atoms, the Lusin area function, the Littlewood-Paley g-function or g_λ^*-function. As applications, we first establish a criterion on the boundedness of sublinear operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into a quasi-Banach space. Then, applying this criterion, we further show that the maximal operators of the Bochner-Riesz and the Weierstrass means are bounded from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$. This implies some almost everywhere and norm convergences of these Bochner-Riesz and Weierstrass means. We point out that all results on the Bochner-Riesz and the Weierstrass means are new even in the isotropic case and the real-variable characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$ have proved important in [46] in the study on the real interpolation between $H_A^{p(\cdot)}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$.

To be precise, this article is organized as follows.

In Section 2, we first present some notation and notions used in this article, including variable Lebesgue spaces and some known facts on expansive matrixes from [6]. Then we introduce the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ via the non-tangential grand maximal function.

The aim of Section 3 is to establish the characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$ by means of the radial or the non-tangential maximal functions (see Theorem 3.10 below). To this end, via an auxiliary inequality (see Lemma 3.2 below), which originates from [3], and the boundedness of the Hardy-Littlewood maximal function as in (3.1) below on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 3.3 below) with $p(\cdot)$ satisfying the so-called globally log-Hölder continuous condition (see (2.5) and (2.6) below) and $1 < p_- \le p_+ < \infty$, where p_- and p_+ are as in (2.4) below, we first show that the $L^{p(\cdot)}(\mathbb{R}^n)$ quasi-norm of the tangential maximal function $T_\phi^{N(K,L)}(f)$ can be controlled by that of the non-tangential maximal function $M_\phi^{(K,L)}(f)$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$ (see Lemma 3.6 below), where K is the truncation level, L the decay level and $\mathcal{S}'(\mathbb{R}^n)$ denotes the set of all tempered distributions on $\mathbb{R}^n$. Using this, the monotone convergence property for increasing sequences on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 3.5 below) and the boundedness of the Hardy-Littlewood maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$ again, we then prove Theorem 3.10.

In Section 4, via borrowing some ideas from those used in the proofs of [6, p. 38, Theorem 6.4] and [52, Theorems 4.5 and 4.6] as well as [83, Theorem 4.3], we obtain the atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$. For this purpose, we first introduce the variable anisotropic atomic Hardy space $H_A^{p(\cdot),q,s}(\mathbb{R}^n)$ in Definition 4.2 below and then prove

$$H_A^{p(\cdot)}(\mathbb{R}^n) = H_A^{p(\cdot),q,s}(\mathbb{R}^n)$$

with equivalent quasi-norms (see Theorem 4.8 below). Indeed, we first present the density of the subset $L^q(\mathbb{R}^n) \cap H_A^{p(\cdot)}(\mathbb{R}^n)$ in $H_A^{p(\cdot)}(\mathbb{R}^n)$ for any $q \in [1,\infty] \cap (p_+,\infty]$ (see Lemma 4.7 below). By this density and the Calderón-Zygmund decomposition associated with non-tangential grand maximal functions on anisotropic $\mathbb{R}^n$ from [6, p. 9, Lemma 2.7] as well as an argument similar to that used in the proofs of [6, p. 38, Theorem 6.4] and [52, Theorem 4.5], we then prove that $H_A^{p(\cdot)}(\mathbb{R}^n)$ is continuously embedded into $H_A^{p(\cdot),\infty,s}(\mathbb{R}^n)$ and hence also into $H_A^{p(\cdot),q,s}(\mathbb{R}^n)$ due to the fact that each $(p(\cdot),\infty,s)$-atom is also a $(p(\cdot),q,s)$-atom for any $q \in (1,\infty)$. Conversely, as a special case of [83, Proposition 2.11], we first obtain that some estimates related to $L^{p(\cdot)}(\mathbb{R}^n)$ norms for some series of functions can be reduced into dealing with the $L^q(\mathbb{R}^n)$ norms of the corresponding functions (see Lemma 4.6 below), which plays a key role in the proof of Theorem 4.8 and is also of independent interest. Then, using this key lemma and the anisotropic Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator M_{HL} on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 4.4 below), we prove that $H_A^{p(\cdot),q,s}(\mathbb{R}^n) \subset H_A^{p(\cdot)}(\mathbb{R}^n)$ and the inclusion is continuous.

Section 5 is aimed to establish a finite atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 5.4 below). To be precise, via borrowing some ideas from those used in the proofs of [45, Theorem 5.7] and [44, Theorem 2.14], we prove that, for any given finite linear combination of (p, q, s)-atoms with $q \in (\max\{p_+, 1\}, \infty)$ (or continuous (p, ∞, s)-atoms), its quasi-norm in $H_A^{p(\cdot)}(\mathbb{R}^n)$ can be achieved via all its finite atomic decompositions. This extends [50, Theorem 3.1 and Remark 3.3] and [30, Theorem 5.6] to the present setting of variable anisotropic Hardy spaces.

In Section 6, by the anisotropic Calderón reproducing formula (see Lemma 6.6 below) and the way same as that used in the proof of Theorem 4.8, we first establish the Lusin area function characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ (see Theorem 6.1 below). Then, using this and an approach initiated by Ullrich [68] and further developed by Liang et al. [43] and Liu et al. [48], together with the anisotropic Fefferman-Stein vector-valued inequality of the Hardy-Littlewood maximal operator $M_{\rm HL}$ on $L^{p(\cdot)}(\mathbb{R}^n)$ (see Lemma 4.4 below), we establish the Littlewood-Paley g-function and g_λ^*-function characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$ (see Theorems 6.2 and 6.3 below). We point out that the aforementioned approach, via a key lemma (see Lemma 6.9 below) and an auxiliary function $g_{t,*}(f)$ (see (6.6) below), shows that the $L^{p(\cdot)}(\mathbb{R}^n)$ quasi-norm of the Lusin area function can be controlled by that of the Littlewood-Paley g-function.

As applications, in Section 7, we first establish a criterion on the boundedness of some sublinear operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into a quasi-Banach space (see Theorem 7.1 below). Then we recall a general summability method, namely, the so-called anisotropic θ-summation defined by a single function θ (see (7.2) and (7.3) below). Moreover, under some assumptions on θ, using the criterion established in Theorem 7.1, we show that the maximal operator of the θ-summability means is bounded from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ when p_- satisfies (7.5) below. As consequences, some almost everywhere and also $L^{p(\cdot)}(\mathbb{R}^n)$ norm convergences of the θ-means $\sigma_m^\theta f$ are presented. In addition, two special cases of the θ-summation are investigated, namely, the Bochner-Riesz and the Weierstrass summabilities.

Finally, we make some conventions on notation. We always let $\mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$ and $\vec{0}_n$ be the *origin* of $\mathbb{R}^n$. For any multi-index $\beta := (\beta_1, \ldots, \beta_n) \in (\mathbb{Z}_+)^n =: \mathbb{Z}_+^n$, let $|\beta| := \beta_1 + \cdots + \beta_n$. We denote by C a *positive constant* which is independent of the main parameters, but may vary from line to line. The notation $f \lesssim g$ means $f \le Cg$ and, if $f \lesssim g \lesssim f$, then we write $f \sim g$. For any $q \in [1, \infty]$, we denote by q' its *conjugate index*, namely, $1/q + 1/q' = 1$. In addition, for any set $E \subset \mathbb{R}^n$, we denote by $E^\complement$ the set $\mathbb{R}^n \setminus E$, by χ_E its *characteristic function*, by $|E|$ the *n-dimensional Lebesgue measure* of E and by $\sharp E$ the *cardinality* of E. For any $s \in \mathbb{R}$, we denote by $\lfloor s \rfloor$ the *largest integer not greater than s*.

2. Preliminaries

In this section, we first recall some notation and notions on dilations and variable Lebesgue spaces (see, for example, [6, 18, 21]). Then we introduce the variable anisotropic Hardy space via the non-tangential grand maximal function.

We begin with recalling the notion of expansive matrixes from [6].

Definition 2.1. A real $n \times n$ matrix A is called an *expansive matrix* (shortly, a *dilation*) if

$$\min_{\lambda \in \sigma(A)} |\lambda| > 1,$$

here and hereafter, $\sigma(A)$ denotes the *set of all eigenvalues of* A.

Let $b := |\det A|$. Then it follows, from [6, p. 6, (2.7)], that $b \in (1, \infty)$. From the fact that there exist an open ellipsoid Δ, with $|\Delta| = 1$, and $r \in (1, \infty)$ such that $\Delta \subset r\Delta \subset A\Delta$ (see [6, p. 5, Lemma 2.2]), we deduce that, for any $k \in \mathbb{Z}$, $B_k := A^k\Delta$ is open, $B_k \subset rB_k \subset B_{k+1}$ and $|B_k| = b^k$. For any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, an ellipsoid $x + B_k$ is called a *dilated ball*. In what follows, we always let $\mathfrak{B}$ be the set of all such dilated balls, namely,

$$\mathfrak{B} := \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\} \tag{2.1}$$

and

$$\tau := \inf\{\ell \in \mathbb{Z} : r^\ell \geq 2\}. \tag{2.2}$$

The following notion of the homogeneous quasi-norm is just [6, p. 6, Definition 2.3].

Definition 2.2. A *homogeneous quasi-norm*, associated with a dilation A, is a measurable mapping $\rho\colon \mathbb{R}^n \to [0, \infty)$ satisfying

(i) if $x \neq \vec{0}_n$, then $\rho(x) \in (0, \infty)$;

(ii) for each $x \in \mathbb{R}^n$, $\rho(Ax) = b\rho(x)$;

(iii) there exists an $H \in [1, \infty)$ such that, for any $x, y \in \mathbb{R}^n$, $\rho(x + y) \leq H[\rho(x) + \rho(y)]$.

For a given dilation A, by [6, p. 6, Lemma 2.4], we may use the *step homogeneous quasi-norm* ρ defined by setting, for any $x \in \mathbb{R}^n$,

$$\rho(x) := \sum_{j \in \mathbb{Z}} b^j \chi_{B_{j+1} \setminus B_j}(x) \quad \text{when } x \neq \vec{0}_n, \text{ or else} \quad \rho(\vec{0}_n) := 0 \tag{2.3}$$

for convenience.

For any measurable function $p(\cdot)\colon \mathbb{R}^n \to (0, \infty]$, let

$$p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}^n} p(x), \quad p_+ := \operatorname{ess\,sup}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad \underline{p} := \min\{p_-, 1\}. \tag{2.4}$$

Denote by $\mathcal{P}(\mathbb{R}^n)$ the *set of all measurable functions* $p(\cdot)$ satisfying $0 < p_- \le p_+ < \infty$.

For any $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, the *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\varrho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$, where, for any measurable function f, the *modular functional* $\varrho_{p(\cdot)}(f)$ and the *Luxemburg* (also called *Luxemburg-Nakano*) *quasi-norm* $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ of f are defined, respectively, as $\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$ and

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\{\lambda \in (0,\infty) : \varrho_{p(\cdot)}(f/\lambda) \le 1\}.$$

Let $C^{\log}(\mathbb{R}^n)$ be the set of all functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ satisfying the *globally log-Hölder continuous condition*, namely, there exist $C_{\log}(p)$, $C_\infty \in (0,\infty)$ and $p_\infty \in \mathbb{R}$ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/\rho(x-y))} \tag{2.5}$$

and

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + \rho(x))}. \tag{2.6}$$

Recall that a *Schwartz function* is a $C^\infty(\mathbb{R}^n)$ function φ satisfying, for any $m \in \mathbb{Z}_+$ and multi-index $\alpha \in \mathbb{Z}_+^n$,

$$\|\varphi\|_{\alpha,m} := \sup_{x\in\mathbb{R}^n} [\rho(x)]^m |\partial^\alpha \varphi(x)| < \infty.$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the set of all Schwartz functions, equipped with the topology determined by $\{\|\cdot\|_{\alpha,m}\}_{\alpha\in\mathbb{Z}_+^n, m\in\mathbb{Z}_+}$, and $\mathcal{S}'(\mathbb{R}^n)$ the *dual space* of $\mathcal{S}(\mathbb{R}^n)$, equipped with the weak-$*$ topology. For any $N \in \mathbb{Z}_+$, let

$$\mathcal{S}_N(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \|\varphi\|_{\alpha,\ell} \le 1, |\alpha| \le N, \ell \le N\},$$

equivalently,

$$\varphi \in \mathcal{S}_N(\mathbb{R}^n) \quad \Longleftrightarrow \quad \|\varphi\|_{\mathcal{S}_N(\mathbb{R}^n)} := \sup_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} \left[|\partial^\alpha \varphi(x)| \max\{1, [\rho(x)]^N\}\right] \le 1.$$

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{Z}$, let $\varphi_k(\cdot) := b^k \varphi(A^k \cdot)$.

Let $\lambda_-, \lambda_+ \in (0,\infty)$ be two *numbers* such that

$$1 < \lambda_- < \min\{|\lambda| : \lambda \in \sigma(A)\} \le \max\{|\lambda| : \lambda \in \sigma(A)\} < \lambda_+.$$

We should point out that, if A is diagonalizable over $\mathbb{C}$, then we may let $\lambda_- := \min\{|\lambda| : \lambda \in \sigma(A)\}$ and $\lambda_+ := \max\{|\lambda| : \lambda \in \sigma(A)\}$. Otherwise, we may choose them sufficiently close to these equalities in accordance with what we need in our arguments.

Definition 2.3. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *non-tangential maximal function* $M_\varphi(f)$ with respect to φ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\varphi(f)(x) := \sup_{y\in x+B_k, k\in\mathbb{Z}} |f * \varphi_k(y)|. \tag{2.7}$$

Moreover, for any given $N \in \mathbb{N}$, the *non-tangential grand maximal function* $M_N(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N(f)(x) := \sup_{\varphi\in\mathcal{S}_N(\mathbb{R}^n)} M_\varphi(f)(x).$$

We now introduce variable anisotropic Hardy spaces as follows.

Definition 2.4. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in \mathbb{N} \cap [\lfloor (1/\underline{p} - 1)\ln b/\ln\lambda_-\rfloor + 2, \infty)$, where $\underline{p}$ is as in (2.4). The *variable anisotropic Hardy space* $H_A^{p(\cdot)}(\mathbb{R}^n)$ is defined as

$$H_A^{p(\cdot)}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : M_N(f) \in L^{p(\cdot)}(\mathbb{R}^n)\}$$

and, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, let $\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} := \|M_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Remark 2.5. (i) The quasi-norm of $H_A^{p(\cdot)}(\mathbb{R}^n)$ in Definition 2.4 depends on N, however, by Theorem 3.10 below, we conclude that the space $H_A^{p(\cdot)}(\mathbb{R}^n)$ is independent of the choice of N as long as $N \in \mathbb{N}\cap[\lfloor (1/\underline{p}-1)\ln b/\ln\lambda_-\rfloor + 2, \infty)$. In addition, when $p(\cdot) \equiv p \in (0,\infty)$, the space $H_A^{p(\cdot)}(\mathbb{R}^n)$ becomes the anisotropic Hardy space $H_A^p(\mathbb{R}^n)$ from [6] and, when $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$, here and hereafter, $\mathrm{I}_{n\times n}$ denotes the $n \times n$ *unit matrix*, the space $H_A^{p(\cdot)}(\mathbb{R}^n)$ goes back to the variable Hardy space studied in [19,52].

(ii) Very recently, via the variable Lorentz spaces $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [24], where

$$p(\cdot),\ q(\cdot)\colon (0,\infty) \to (0,\infty)$$

are two measurable functions, Almeida et al. [3] introduced the variable anisotropic Hardy-Lorentz spaces $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ on $\mathbb{R}^n$. As was mentioned in [35, Remark 2.6] (see also [46, Remark 2.11(ii)]), the space $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in [24] never goes back to the space $L^{p(\cdot)}(\mathbb{R}^n)$, since the variable exponent $p(\cdot)$ in $\mathcal{L}^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is only defined on $(0,\infty)$ while not on $\mathbb{R}^n$. Thus, it is easy to see that the space $H_A^{p(\cdot)}(\mathbb{R}^n)$, in this article, is not covered by the space $H^{p(\cdot),q(\cdot)}(\mathbb{R}^n, A)$ in [3]. We should also point out that the space $H_A^{p(\cdot)}(\mathbb{R}^n)$, in this article, is also not covered by the variable anisotropic Hardy-Lorentz space $H_A^{p(\cdot),q}(\mathbb{R}^n)$ investigated in [44,46], since the exponent $q \in (0,\infty]$ in $H_A^{p(\cdot),q}(\mathbb{R}^n)$ is only a constant.

(iii) Recall that Li et al. [39–41] studied the anisotropic Musielak-Orlicz Hardy space $H_A^\varphi(\mathbb{R}^n)$ with a Musielak-Orlicz growth function $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$. Similarly to [74, Remark 2.8], we know that, if

$$\varphi(x,t) := t^{p(x)} \quad \text{for any } x \in \mathbb{R}^n \text{ and } t \in (0,\infty), \tag{2.8}$$

then $H_A^{\varphi}(\mathbb{R}^n) = H_A^{p(\cdot)}(\mathbb{R}^n)$. However, a general Musielak-Orlicz growth function φ satisfying all the assumptions in [39–41] may not have the form as in (2.8). On the other hand, as was pointed out in [74, Remark 2.14(iii)], it was proved in [77] that there exists a variable exponent function $p(\cdot)$ satisfying (2.5) and (2.6) which were required in this article, but $t^{p(\cdot)}$ is not a uniformly Muckenhoupt weight which was required in [39–41]. Thus, the anisotropic Musielak-Orlicz Hardy space $H_A^{\varphi}(\mathbb{R}^n)$ in [39–41] and the variable anisotropic Hardy space $H_A^{p(\cdot)}(\mathbb{R}^n)$ in this article can not cover each other.

3. Maximal function characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$

In this section, we characterize $H_A^{p(\cdot)}(\mathbb{R}^n)$ by means of the radial maximal function M_φ^0 (see Definition 3.9 below) or the non-tangential maximal function M_φ (see (2.7)). We begin with the following notions of some auxiliary maximal functions from [6].

Definition 3.1. Let $K \in \mathbb{Z}$, $L \in [0,\infty)$ and $N \in \mathbb{N}$. For any $\phi \in \mathcal{S}$, the *maximal functions* $M_\phi^{0(K,L)}(f)$, $M_\phi^{(K,L)}(f)$ and $T_\phi^{N(K,L)}(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ are, respectively, defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\phi^{0(K,L)}(f)(x) := \sup_{k\in\mathbb{Z}, k\le K} |(f*\phi_k)(x)| \left[\max\{1,\rho(A^{-K}x)\}\right]^{-L} (1+b^{-k-K})^{-L},$$
$$M_\phi^{(K,L)}(f)(x) := \sup_{k\in\mathbb{Z}, k\le K} \sup_{y\in x+B_k} |(f*\phi_k)(y)| \left[\max\{1,\rho(A^{-K}y)\}\right]^{-L} (1+b^{-k-K})^{-L}$$

and

$$T_\phi^{N(K,L)}(f)(x) := \sup_{k\in\mathbb{Z}, k\le K} \sup_{y\in\mathbb{R}^n} \frac{|(f*\phi_k)(y)|}{[\max\{1,\rho(A^{-k}(x-y))\}]^N} \frac{(1+b^{-k-K})^{-L}}{[\max\{1,\rho(A^{-K}y)\}]^L}.$$

Moreover, the *maximal functions* $M_N^{0(K,L)}(f)$ and $M_N^{(K,L)}(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ are, respectively, defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N^{0(K,L)}(f)(x) := \sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)} M_\phi^{0(K,L)}(f)(x)$$

and

$$M_N^{(K,L)}(f)(x) := \sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)} M_\phi^{(K,L)}(f)(x).$$

Let $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be the collection of all locally integrable functions on $\mathbb{R}^n$. Recall that the *Hardy-Littlewood maximal operator* $M_{\mathrm{HL}}(f)$ of $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{y\in x+B_k} \frac{1}{|B_k|}\int_{y+B_k} |f(z)|\,dz = \sup_{x\in B\in\mathfrak{B}} \frac{1}{|B|}\int_B |f(z)|\,dz, \tag{3.1}$$

where $\mathfrak{B}$ is as in (2.1).

The following Lemmas 3.2 through 3.5 are just [3, Lemma 2.3], [46, Lemma 3.3], [74, Remark 2.1(i)] and [18, Corollary 2.64], respectively.

Lemma 3.2. *Let $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $r \in (0,\infty)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. Then there exists a positive constant C, independent of K, N, L, r and ϕ, such that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$\left[T_\phi^{N(K,L)}(f)(x)\right]^r \le C M_{\rm HL}\left(\left[M_\phi^{(K,L)}(f)\right]^r\right)(x),$$

where $M_{\rm HL}$ is as in (3.1).

Lemma 3.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$.*

(i) *If $1 \le p_- \le p_+ < \infty$, then, for any given $s \in [1,\infty)$ and any $f \in L^{sp(\cdot)}(\mathbb{R}^n)$,*

$$\sup_{\lambda\in(0,\infty)} \left\|\lambda\chi_{\{x\in\mathbb{R}^n: M_{\rm HL}(f)(x)>\lambda\}}\right\|_{L^{sp(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{sp(\cdot)}(\mathbb{R}^n)},$$

where C is a positive constant independent of f;

(ii) *If $1 < p_- \le p_+ < \infty$, then, for any given $s \in [1,\infty)$ and any $f \in L^{sp(\cdot)}(\mathbb{R}^n)$,*

$$\|M_{\rm HL}(f)\|_{L^{sp(\cdot)}(\mathbb{R}^n)} \le \widetilde{C}\|f\|_{L^{sp(\cdot)}(\mathbb{R}^n)},$$

where $\widetilde{C}$ is a positive constant independent of f.

Lemma 3.4. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then, for any $s \in (0,\infty)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$,*

$$\||f|^s\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{L^{sp(\cdot)}(\mathbb{R}^n)}.$$

In addition, for any $\lambda \in \mathbb{C}$ and $f, g \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda|\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ and

$$\|f+g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (2.4).

Lemma 3.5. *Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\{f_k\}_{k\in\mathbb{N}} \subset L^{p(\cdot)}(\mathbb{R}^n)$ be any sequence of non-negative functions satisfying that f_k, as $k \to \infty$, increases pointwisely almost everywhere to some $f \in L^{p(\cdot)}(\mathbb{R}^n)$ in $\mathbb{R}^n$. Then*

$$\|f - f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \to 0 \quad \text{as } k \to \infty.$$

From Lemmas 3.2, 3.3 and 3.4, we easily deduce the following conclusion, the details being omitted.

Lemma 3.6. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then there exists a positive constant C such that, for any $K \in \mathbb{Z}$, $N, L \in \mathbb{N}$, $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\left\| T_\phi^{N(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

We also need the following two technical lemmas, which are just [6, p. 45, Lemma 7.5 and p. 46, Lemma 7.6], respectively.

Lemma 3.7. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Then, for any given $N \in \mathbb{N}$ and $L \in [0, \infty)$, there exist an $I \in \mathbb{N}$ and a positive constant $C_{(N,L)}$, depending on N and L, such that, for any $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M_I^{0(K,L)}(f)(x) \leq C_{(N,L)} T_\phi^{N(K,L)}(f)(x).$$

Lemma 3.8. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Then, for any given $M \in (0, \infty)$ and $K \in \mathbb{Z}_+$, there exist $L \in (0, \infty)$ and a positive constant $C_{(K,M)}$, depending on K and M, such that, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,*

$$M_\phi^{(K,L)}(f)(x) \leq C_{(K,\,M)}[\max\{1, \rho(x)\}]^{-M}. \tag{3.2}$$

Definition 3.9. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The *radial maximal function* $M_\phi^0(f)$ of f with respect to ϕ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_\phi^0(f)(x) := \sup_{k\in\mathbb{Z}} |f * \phi_k(x)|.$$

Moreover, for any given $N \in \mathbb{N}$, the *radial grand maximal function* $M_N^0(f)$ of $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$M_N^0(f)(x) := \sup_{\phi\in\mathcal{S}_N(\mathbb{R}^n)} M_\phi^0(f)(x).$$

Now, it is a position to state the main result of this section.

Theorem 3.10. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\,dx \neq 0$. Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$, the following statements are mutually equivalent:*

(i) $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$;

(ii) $M_\phi(f) \in L^{p(\cdot)}(\mathbb{R}^n)$;

(iii) $M_\phi^0(f) \in L^{p(\cdot)}(\mathbb{R}^n)$.

Moreover, there exist two positive constants C_1 and C_2, independent of f, such that

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \leq C_1 \|M_\phi^0(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_1 \|M_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C_2 \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. Due to the obvious facts that (i) implies (ii) and that (ii) implies (iii), we next only need to show that (ii) implies (i) and that (iii) implies (ii).

We first prove that (ii) implies (i). Indeed, notice that Lemma 3.7 with $N \in \mathbb{N}$ and $L = 0$ implies that there exists an $I \in \mathbb{N}$ such that, for any $K \in \mathbb{Z}_+$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, it holds true that $M_I^{0(K,0)}(f)(x) \lesssim T_\phi^{N(K,0)}(f)(x)$. Thus, by Lemma 3.6, we conclude that, for any $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\left\| M_I^{0(K,0)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| M_\phi^{(K,0)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{3.3}$$

Letting $K \to \infty$ and applying Lemma 3.5 to (3.3), we obtain

$$\|M_I^0(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|M_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which, combined with [6, p. 17, Proposition 3.10], implies that, if (ii) holds true, then (i) also holds true.

Next we prove that (iii) implies (ii). To this end, let $M_\phi^0(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Then, for any $M \in (1/p_-, \infty)$ and $K \in \mathbb{Z}_+$, by Lemma 3.8, we know that there exists some $L \in (0,\infty)$ such that (3.2) holds true and hence $M_\phi^{(K,L)}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Indeed, by Lemma 3.4, we have

$$\begin{aligned}
\left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} &\le \left\| M_\phi^{(K,L)}(f)\chi_{B_1} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} + \sum_{k\in\mathbb{N}} \left\| M_\phi^{(K,L)}(f)\chi_{B_{k+1}\setminus B_k} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \|\chi_{B_1}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} + \sum_{k\in\mathbb{N}} b^{-k\underline{p}M} \|\chi_{B_{k+1}\setminus B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \sum_{k\in\mathbb{Z}_+} b^{-k\underline{p}M} b^{(k+1)\underline{p}/p_-} < \infty.
\end{aligned}$$

Thus, $M_\phi^{(K,L)}(f) \in L^{p(\cdot)}(\mathbb{R}^n)$.

On the other hand, from Lemmas 3.7 and 3.6, we deduce that, for any given $L \in (0,\infty)$, there exist some $I \in \mathbb{N}$ and a positive constant C_3 such that, for any $K \in \mathbb{Z}_+$ and $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\left\| M_I^{0(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_3 \left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

For any fixed $K \in \mathbb{Z}_+$, let

$$E_K := \left\{ x \in \mathbb{R}^n : M_I^{0(K,L)}(f)(x) \le C_4 M_\phi^{(K,L)}(f)(x) \right\}$$

with $C_4 := 2C_3$. Then, since

$$\left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(E_K^\complement)} \le C_4^{-1} \left\| M_I^{0(K,L)}(f) \right\|_{L^{p(\cdot)}(E_K^\complement)} \le \frac{C_3}{C_4} \left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

it follows that

$$\left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| M_\phi^{(K,L)}(f) \right\|_{L^{p(\cdot)}(E_K)} \tag{3.4}$$

holds true.

For any given $L\in(0,\infty)$, by a proof similar to that of [45, (4.17)], we find that, for any $t\in(0,p_-)$, $K\in\mathbb{Z}_+$, $f\in\mathcal{S}'(\mathbb{R}^n)$ and $x\in E_K$,

$$\left[M_\phi^{(K,L)}(f)(x)\right]^t\lesssim M_{\rm HL}\left(\left[M_\phi^{0(K,L)}(f)\right]^t\right)(x), \tag{3.5}$$

which, together with (3.4), Lemma 3.4, (3.5) and Lemma 3.3(ii), further implies that, for any $K\in\mathbb{Z}_+$ and $f\in\mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned}\left\|M_\phi^{(K,L)}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^t&\lesssim\left\|M_\phi^{(K,L)}(f)\right\|_{L^{p(\cdot)}(E_K)}^t\sim\left\|\left[M_\phi^{(K,L)}(f)\right]^t\right\|_{L^{p(\cdot)/t}(E_K)}\\&\lesssim\left\|M_{\rm HL}\left(\left[M_\phi^{0(K,L)}(f)\right]^t\right)\right\|_{L^{p(\cdot)/t}(\mathbb{R}^n)}\\&\lesssim\left\|\left[M_\phi^{0(K,L)}(f)\right]^t\right\|_{L^{p(\cdot)/t}(\mathbb{R}^n)}\sim\left\|M_\phi^{0(K,L)}(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^t.\end{aligned} \tag{3.6}$$

Letting $K\to\infty$ in (3.6), by Lemma 3.5, we conclude that

$$\|M_\phi(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\|M_\phi^0(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which shows that (iii) implies (ii) and hence completes the proof of Theorem 3.10. □

4. Atomic characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$

In this section, we establish the atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$. We begin with recalling the definition of anisotropic $(p(\cdot),q,s)$-atoms from [46].

Definition 4.1. Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $q\in(1,\infty]$ and

$$s\in\left[\left\lfloor\left(\frac{1}{p_-}-1\right)\frac{\ln b}{\ln\lambda_-}\right\rfloor,\infty\right)\cap\mathbb{Z}_+. \tag{4.1}$$

An *anisotropic* $(p(\cdot),q,s)$-*atom* is a measurable function a on $\mathbb{R}^n$ satisfying

(i) $\operatorname{supp}a\subset B$, where $B\in\mathfrak{B}$ and $\mathfrak{B}$ is as in (2.1);

(ii) $\|a\|_{L^q(\mathbb{R}^n)}\le|B|^{1/q}/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$;

(iii) $\int_{\mathbb{R}^n}a(x)x^\gamma\,dx=0$ for any $\gamma\in\mathbb{Z}_+^n$ with $|\gamma|\le s$.

In what follows, we call an anisotropic $(p(\cdot),q,s)$-atom simply by a $(p(\cdot),q,s)$-atom. Now, using $(p(\cdot),q,s)$-atoms, we introduce the variable anisotropic atomic Hardy space $H_A^{p(\cdot),q,s}(\mathbb{R}^n)$ as follows.

Definition 4.2. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (1,\infty]$, s be as in (4.1) and A a dilation. The *variable anisotropic atomic Hardy space* $H_A^{p(\cdot),q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot),q,s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H_A^{p(\cdot),q,s}(\mathbb{R}^n)$, let

$$\|f\|_{H_A^{p(\cdot),q,s}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where the infimum is taken over all decompositions of f as above.

To establish the atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$, we need several technical lemmas as follows. First, by a proof similar to that of [6, p. 21, Theorem 4.5], we easily obtain the following property of $H_A^{p(\cdot)}(\mathbb{R}^n)$, the details being omitted.

Lemma 4.3. *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *and* $N \in \mathbb{N} \cap [\lfloor (1/\underline{p} - 1)\ln b/\ln \lambda_- \rfloor + 2, \infty)$, *where* $\underline{p}$ *is as in* (2.4). *Then* $H_A^{p(\cdot)}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ *and the inclusion is continuous.*

The following Lemmas 4.4 and 4.5 are just [46, Lemma 4.3] and [6, p. 9, Lemma 2.7], respectively.

Lemma 4.4. *Let* $r \in (1,\infty]$. *Assume that* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *satisfies* $1 < p_- \le p_+ < \infty$. *Then there exists a positive constant* C *such that, for any sequence* $\{f_k\}_{k\in\mathbb{N}}$ *of measurable functions,*

$$\left\| \left\{ \sum_{k\in\mathbb{N}} [M_{\mathrm{HL}}(f_k)]^r \right\}^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{k\in\mathbb{N}} |f_k|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with the usual modification made when $r = \infty$, *where* M_{HL} *denotes the Hardy-Littlewood maximal operator as in* (3.1).

Lemma 4.5. *Let* $\Omega \subset \mathbb{R}^n$ *be an open set with* $|\Omega| < \infty$. *Then, for any* $m \in \mathbb{Z}_+$, *there exist a sequence of points,* $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$, *and a sequence of integers,* $\{\ell_k\}_{k\in\mathbb{N}}$, *such that*

(i) $\Omega = \bigcup_{k\in\mathbb{N}} (x_k + B_{\ell_k})$;

(ii) $\{x_k + B_{\ell_k - \tau}\}_{k\in\mathbb{N}}$ *are pairwise disjoint, where* τ *is as in* (2.2);

(iii) *for each* $k \in \mathbb{N}$, $(x_k + B_{\ell_k + m}) \cap \Omega^{\complement} = \emptyset$, *but* $(x_k + B_{\ell_k + m + 1}) \cap \Omega^{\complement} \ne \emptyset$;

(iv) *for any* $i, j \in \mathbb{N}$, $(x_i + B_{\ell_i+m-2\tau}) \cap (x_j + B_{\ell_j+m-2\tau}) \neq \emptyset$ *implies* $|\ell_i - \ell_j| \leq \tau$*;*

(v) *there exists a positive constant* R *such that, for any* $i \in \mathbb{N}$,

$$\sharp\{j \in \mathbb{N} : (x_i + B_{\ell_i+m-2\tau}) \cap (x_j + B_{\ell_j+m-2\tau}) \neq \emptyset\} \leq R.$$

Observe that $(\mathbb{R}^n, \rho, dx)$ is an RD-space (see [32, 80]). From this and [83, Proposition 2.11 and Lemma 4.8], we deduce the following Lemmas 4.6 and 4.7, which play an important role in this section and are also of independent interest, the details being omitted.

Lemma 4.6. *Let* $r(\cdot) \in C^{\log}(\mathbb{R}^n)$ *and* $q \in [1, \infty] \cap (r_+, \infty]$ *with* r_+ *as in* (2.4)*. Assume that* $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$*,* $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ *and* $\{a_i\}_{i\in\mathbb{N}} \subset L^q(\mathbb{R}^n)$ *satisfy, for any* $i \in \mathbb{N}$*,* $\operatorname{supp} a_i \subset B^{(i)}$*,*

$$\|a_i\|_{L^q(\mathbb{R}^n)} \leq \frac{|B^{(i)}|^{1/q}}{\|\chi_{B^{(i)}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}$$

and

$$\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} < \infty.$$

Then

$$\left\|\left[\sum_{i\in\mathbb{N}}|\lambda_i a_i|^{\underline{r}}\right]^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)} \leq C\left\|\left\{\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{r(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{r}}\right\}^{1/\underline{r}}\right\|_{L^{r(\cdot)}(\mathbb{R}^n)},$$

where C *is a positive constant independent of* λ_i*,* $B^{(i)}$ *and* a_i*.*

Lemma 4.7. *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *and* $q \in [1, \infty] \cap (p_+, \infty]$ *with* p_+ *as in* (2.4)*. Then* $H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ *is dense in* $H_A^{p(\cdot)}(\mathbb{R}^n)$*.*

Now we state the main result of this section as follows.

Theorem 4.8. *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$*,* $q \in (\max\{p_+, 1\}, \infty]$ *with* p_+ *as in* (2.4)*,* s *be as in* (4.1) *and* $N \in \mathbb{N} \cap [\lfloor(1/\underline{p} - 1)\ln b/\ln\lambda_-\rfloor + 2, \infty)$ *with* $\underline{p}$ *as in* (2.4)*. Then* $H_A^{p(\cdot)}(\mathbb{R}^n) = H_A^{p(\cdot),q,s}(\mathbb{R}^n)$ *with equivalent quasi-norms.*

Proof. First, we show that

$$H_A^{p(\cdot),q,s}(\mathbb{R}^n) \subset H_A^{p(\cdot)}(\mathbb{R}^n). \tag{4.2}$$

To this end, for any $f \in H_A^{p(\cdot),q,s}(\mathbb{R}^n)$, by Definition 2.2, we know that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), q, s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\|f\|_{H_A^{p(\cdot),q,s}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then, by [46, (4.8)], it is easy to see that, for any $N \in \mathbb{N} \cap [\lfloor (1/\underline{p}-1)\ln b/\ln\lambda_- \rfloor + 2, \infty)$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} M_N(f)(x) &\le \sum_{i\in\mathbb{N}} |\lambda_i| M_N(a_i)(x)\chi_{A^\tau B^{(i)}}(x) + \sum_{i\in\mathbb{N}} |\lambda_i| M_N(a_i)(x)\chi_{(A^\tau B^{(i)})^\complement}(x) \\ &\lesssim \left\{ \sum_{i\in\mathbb{N}} [|\lambda_i| M_N(a_i)(x)\chi_{A^\tau B^{(i)}}(x)]^{\underline{p}} \right\}^{1/\underline{p}} \\ &\quad + \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_{\mathrm{HL}}(\chi_{B^{(i)}})(x)]^\beta =: \mathrm{I}_1 + \mathrm{I}_2, \end{aligned} \tag{4.3}$$

where $\underline{p}$ is as in (2.4),

$$\beta := \left(\frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\underline{p}} \tag{4.4}$$

and M_{HL} denotes the Hardy-Littlewood maximal operator as in (3.1).

For the term I_1, by the boundedness of M_N on $L^r(\mathbb{R}^n)$ with $r \in (1,\infty]$ (see [45, Remark 2.10]), Lemma 4.6 and [46, Remark 4.4(i)], we conclude that

$$\|\mathrm{I}_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot),q,s}(\mathbb{R}^n)}. \tag{4.5}$$

To deal with I_2, by Lemmas 3.4 and 4.4, we find that

$$\begin{aligned} \|\mathrm{I}_2\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\sim \left\| \left\{ \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} [M_{\mathrm{HL}}(\chi_{B^{(i)}})(x)]^\beta \right\}^{1/\beta} \right\|_{L^{\beta p(\cdot)}}^\beta \\ &\lesssim \left\| \sum_{i\in\mathbb{N}} \frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\sim \|f\|_{H_A^{p(\cdot),q,s}(\mathbb{R}^n)}. \end{aligned}$$

This, combined with (4.3) and (4.5), implies that

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \sim \|M_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot),q,s}(\mathbb{R}^n)}.$$

Thus, (4.2) holds true.

We now prove that $H_A^{p(\cdot)}(\mathbb{R}^n) \subset H_A^{p(\cdot),q,s}(\mathbb{R}^n)$. To this end, it suffices to show that

$$H_A^{p(\cdot)}(\mathbb{R}^n) \subset H_A^{p(\cdot),\infty,s}(\mathbb{R}^n), \tag{4.6}$$

due to the fact that each $(p(\cdot),\infty,s)$-atom is also a $(p(\cdot),q,s)$-atom and hence $H_A^{p(\cdot),\infty,s}(\mathbb{R}^n) \subset H_A^{p(\cdot),q,s}(\mathbb{R}^n)$.

Next we prove (4.6) by two steps.

Step 1. In this step, we show that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $q \in (\max\{p_+,1\},\infty]$,

$$\|f\|_{H_A^{p(\cdot),\infty,s}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \tag{4.7}$$

holds true.

To prove (4.7), we borrow some ideas from those used in the proofs of [6, p. 38, Theorem 6.4] and [52, Theorem 4.5]. For any $k\in\mathbb{Z}$, $N\in\mathbb{N}\cap[\lfloor(1/\underline{p}-1)\ln b/\ln\lambda_-\rfloor+2,\infty)$ and $f\in H_A^{p(\cdot)}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, let

$$\Omega_k := \{x\in\mathbb{R}^n : M_N(f)(x) > 2^k\}. \tag{4.8}$$

Then, by Lemma 4.5 with $m = 6\tau$, we know that there exist a sequence $\{x_i^k\}_{i\in\mathbb{N}}\subset\Omega_k$ and a sequence $\{\ell_i^k\}_{i\in\mathbb{N}}\subset\mathbb{Z}$ such that

$$\Omega_k = \bigcup_{i\in\mathbb{N}} (x_i^k + B_{\ell_i^k}); \tag{4.9}$$

$$(x_i^k + B_{\ell_i^k-\tau})\cap(x_j^k + B_{\ell_j^k-\tau}) = \emptyset \quad \text{for any } i,j\in\mathbb{N} \text{ with } i\neq j;$$

$$(x_i^k + B_{\ell_i^k+6\tau})\cap\Omega_k^{\complement} = \emptyset, \quad (x_i^k + B_{\ell_i^k+6\tau+1})\cap\Omega_k^{\complement} \neq \emptyset \quad \text{for any } i\in\mathbb{N};$$

$$(x_i^k + B_{\ell_i^k+4\tau})\cap(x_j^k + B_{\ell_j^k+4\tau}) \neq \emptyset \quad \text{implies} \quad |\ell_i^k-\ell_j^k|\le\tau;$$

$$\sharp\left\{j\in\mathbb{N} : (x_i^k + B_{\ell_i^k+4\tau})\cap(x_j^k + B_{\ell_j^k+4\tau}) \neq \emptyset\right\} \le R \quad \text{for any } i\in\mathbb{N}, \tag{4.10}$$

where τ and R are same as in Lemma 4.5.

Let $\eta\in\mathcal{S}(\mathbb{R}^n)$ satisfy that $\operatorname{supp}\eta\subset B_\tau$, $0\le\eta\le1$ and $\eta\equiv1$ on B_0. For any $i\in\mathbb{N}$, $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, let $\eta_i^k(x) := \eta(A^{-\ell_i^k}(x-x_i^k))$ and

$$\theta_i^k(x) := \frac{\eta_i^k(x)}{\sum_{j\in\mathbb{N}}\eta_j^k(x)}. \tag{4.11}$$

Then it is easy to see that $\{\theta_i^k\}_{i\in\mathbb{N}}$ forms a smooth partition of unity of Ω_k. For any $r\in\mathbb{Z}_+$, denote by $\mathbb{P}_r(\mathbb{R}^n)$ the linear space of all polynomials on $\mathbb{R}^n$ with degree not greater than r. By an argument similar to that used in [45, p. 1679], we conclude that there exists a unique polynomial $P_i^k\in\mathbb{P}_r(\mathbb{R}^n)$ such that, for any $Q\in\mathbb{P}_r(\mathbb{R}^n)$,

$$\frac{1}{\int_{\mathbb{R}^n}\theta_i^k(x)\,dx}\langle f, Q\theta_i^k\rangle = \frac{1}{\int_{\mathbb{R}^n}\theta_i^k(x)\,dx}\langle P_i^k, Q\theta_i^k\rangle = \frac{1}{\int_{\mathbb{R}^n}\theta_i^k(x)\,dx}\int_{\mathbb{R}^n} P_i^k(x)Q(x)\theta_i^k(x)\,dx.$$

For each $i\in\mathbb{N}$ and $k\in\mathbb{Z}$, let $b_i^k := [f-P_i^k]\theta_i^k$ and

$$g^{(k)} := f - \sum_{i\in\mathbb{N}} b_i^k = f - \sum_{i\in\mathbb{N}}[f-P_i^k]\theta_i^k = f\chi_{\Omega_k^{\complement}} + \sum_{i\in\mathbb{N}} P_i^k\theta_i^k. \tag{4.12}$$

From this and an argument same as that used in [45, p. 1679], we deduce that $\|g^{(k)}\|_{L^\infty(\mathbb{R}^n)} \lesssim 2^k$ and $\|g^{(k)}\|_{L^\infty(\mathbb{R}^n)} \to 0$ as $k \to -\infty$.

Fix some $N \in \mathbb{N} \cap [\lfloor (1/\underline{p}-1)\ln b/\ln \lambda_-\rfloor + 2, \infty)$ large enough such that $(\ln \lambda_-/\ln b)Np_- \in (1,\infty)$. Notice that $f \in L^q(\mathbb{R}^n) = H_A^q(\mathbb{R}^n)$ with $q \in (\max\{p_+,1\},\infty]$ (see [6, p. 17]), where $H_A^q(\mathbb{R}^n)$ denotes the anisotropic Hardy space from [6]. Then, repeating the proof of [6, p. 31, Lemma 5.7] with some slight modifications, we find that, for any $k \in \mathbb{Z}$, $\{\sum_{i=1}^m b_i^k\}_{m\in\mathbb{N}}$ converges in $H_A^q(\mathbb{R}^n)$ and hence converges in $\mathcal{S}'(\mathbb{R}^n)$. By this, a proof similar to those of [6, p. 27, Lemma 5.4 and p. 28, Lemma 5.6], (4.9), (4.10), (2.3) and (3.1), we conclude that, for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} M_N\left(\sum_{i\in\mathbb{N}} b_i^k\right)(x) &\le \sum_{i\in\mathbb{N}} M_N(b_i^k)(x)\chi_{x_i^k+B_{\ell_i^k+2\tau}}(x) + \sum_{i\in\mathbb{N}} M_N(b_i^k)(x)\chi_{(x_i^k+B_{\ell_i^k+2\tau})^\complement}(x) \\ &\lesssim M_N(f)(x)\chi_{\Omega_k}(x) + 2^k\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}_+}(\lambda_-)^{-jN}\chi_{x_i^k+B_{\ell_i^k+2\tau+j+1}\setminus B_{\ell_i^k+2\tau+j}}(x) \\ &\lesssim M_N(f)(x)\chi_{\Omega_k}(x) \\ &\quad + 2^k\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{Z}_+} b^{N\ell_i^k\frac{\ln\lambda_-}{\ln b}} b^{-(\ell_i^k+2\tau+j)N\frac{\ln\lambda_-}{\ln b}}\chi_{x_i^k+B_{\ell_i^k+2\tau+j+1}\setminus B_{\ell_i^k+2\tau+j}}(x) \\ &\lesssim M_N(f)(x)\chi_{\Omega_k}(x) + 2^k\sum_{i\in\mathbb{N}}\frac{|x_i^k+B_{\ell_i^k}|^{N\ln\lambda_-/\ln b}}{[\rho(x-x_i^k)]^{N\ln\lambda_-/\ln b}} \\ &\lesssim M_N(f)(x)\chi_{\Omega_k}(x) + 2^k\sum_{i\in\mathbb{N}}\left[M_{\mathrm{HL}}(\chi_{x_i^k+B_{\ell_i^k}})(x)\right]^{N\ln\lambda_-/\ln b}. \end{aligned}$$

Thus, from the fact that $(\ln \lambda_-/\ln b)Np_- \in (1,\infty)$, Lemma 4.4, (4.9), (4.10) again and the definition of Ω_k, it follows that, for any $k \in \mathbb{Z}$,

$$\begin{aligned} &\left\|M_N\left(\sum_{i\in\mathbb{N}} b_i^k\right)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \|M_N(f)(x)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \left\|2^k\sum_{i\in\mathbb{N}}\left[M_{\mathrm{HL}}(\chi_{x_i^k+B_{\ell_i^k}})(x)\right]^{N\ln\lambda_-/\ln b}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\lesssim \|M_N(f)(x)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|2^k\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|M_N(f)(x)\chi_{\Omega_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned}$$

This, together with (4.12), further implies that, as $k \to \infty$,

$$\left\|f-g^{(k)}\right\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} = \left\|\sum_{i\in\mathbb{N}} b_i^k\right\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \sim \left\|M_N\left(\sum_{i\in\mathbb{N}} b_i^k\right)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \to 0.$$

By this, the fact that $\|g^{(k)}\|_{L^\infty(\mathbb{R}^n)} \to 0$, as $k \to -\infty$, and Lemma 4.3, we have

$$f = \sum_{k\in\mathbb{Z}}\left[g^{(k+1)} - g^{(k)}\right] \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

On the other hand, for any $k\in\mathbb{Z}$, $i\in\mathbb{N}$ and $P\in\mathbb{P}_r(\mathbb{R}^n)$ with $r\in\mathbb{Z}_+$, let

$$\|P\|_{i,k}:=\left[\frac{1}{\int_{\mathbb{R}^n}\theta_i^k(x)\,dx}\int_{\mathbb{R}^n}|P(x)|^2\theta_i^k(x)\,dx\right]^{1/2}, \tag{4.13}$$

where θ_i^k is as in (4.11). Then, by an argument same as that used in [6, p. 38] (see also [45, pp. 1680–1681]), we find that

$$f=\sum_{k\in\mathbb{Z}}\left[g^{(k+1)}-g^{(k)}\right]=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[b_i^k-\sum_{j\in\mathbb{N}}(b_j^{k+1}\theta_i^k-P_{i,j}^{k+1}\theta_j^{k+1})\right]=:\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}h_i^k \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where, for any $i,j\in\mathbb{N}$ and $k\in\mathbb{Z}$, $P_{i,j}^{k+1}$ is the orthogonal projection of $[f-P_j^{k+1}]\theta_i^k$ on $\mathbb{P}_r(\mathbb{R}^n)$ with respect to the norm defined by (4.13) and h_i^k is a multiple of a $(p(\cdot),\infty,s)$-atom satisfying

$$\int_{\mathbb{R}^n}h_i^k(x)Q(x)\,dx=0 \quad \text{for any } Q\in\mathbb{P}_r(\mathbb{R}^n), \tag{4.14}$$

$$\operatorname{supp} h_i^k\subset(x_i^k+B_{\ell_i^k+4\tau}) \tag{4.15}$$

and

$$\left\|h_i^k\right\|_{L^\infty(\mathbb{R}^n)}\le\widetilde{C}2^k, \tag{4.16}$$

where $\widetilde{C}$ is a positive constant independent of k and i.

Now, for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, let

$$\lambda_i^k:=\widetilde{C}2^k\left\|\chi_{x_i^k+B_{\ell_i^k+4\tau}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \quad \text{and} \quad a_i^k:=\left[\lambda_i^k\right]^{-1}h_i^k, \tag{4.17}$$

where $\widetilde{C}$ is as in (4.16). Then, by (4.14), (4.15) and (4.16), we easily know that, for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, a_i^k is a $(p(\cdot),\infty,s)$-atom. Moreover, we have

$$f=\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\lambda_i^k a_i^k \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

In addition, from (4.17), (4.9), (4.10) and the definition of Ω_k, we further deduce that

$$\begin{aligned}
&\left\|\left\{\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left[\frac{|\lambda_i^k|\chi_{x_i^k+B_{\ell_i^k+4\tau}}}{\|\chi_{x_i^k+B_{\ell_i^k+4\tau}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\sim\left\|\left[\sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}}\left(2^k\chi_{x_i^k+B_{\ell_i^k+4\tau}}\right)^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\left\|\left[\sum_{k\in\mathbb{Z}}(2^k\chi_{\Omega_k})^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\sim\left\|\left[\sum_{k\in\mathbb{Z}}(2^k\chi_{\Omega_k\setminus\Omega_{k+1}})^{\underline{p}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\left\|M_N(f)\left[\sum_{k\in\mathbb{Z}}\chi_{\Omega_k\setminus\Omega_{k+1}}\right]^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\
&\lesssim\|M_N(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}\sim\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

This implies that (4.7) holds true.

Step 2. In this step, we prove that (4.7) also holds true for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$.

To this end, let $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 4.7, we know that there exists a sequence $\{f_j\}_{j\in\mathbb{N}} \subset H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $q \in (\max\{p_+, 1\}, \infty]$ such that $f = \sum_{j\in\mathbb{N}} f_j$ in $H_A^{p(\cdot)}(\mathbb{R}^n)$ and, for any $j \in \mathbb{N}$,

$$\|f_j\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \le 2^{2-j}\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

Notice that, for any $j \in \mathbb{N}$, by the conclusion obtained in Step 1, we find that f_j has an atomic decomposition, namely,

$$f_j = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^{j,k} a_i^{j,k} \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where $\{\lambda_i^{j,k}\}_{k\in\mathbb{Z},i\in\mathbb{N}}$ and $\{a_i^{j,k}\}_{k\in\mathbb{Z},i\in\mathbb{N}}$ are constructed as in (4.17). Thus, $\{a_i^{j,k}\}_{k\in\mathbb{Z},i\in\mathbb{N}}$ are $(p(\cdot), \infty, s)$-atoms and hence we have

$$f = \sum_{j\in\mathbb{N}} \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^{j,k} a_i^{j,k} \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\|f\|_{H_A^{p(\cdot),\infty,s}(\mathbb{R}^n)} \le \left[\sum_{j\in\mathbb{N}} \|f_j\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}^{\underline{p}}\right]^{1/\underline{p}} \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)},$$

which implies that (4.7) holds true for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ and hence completes the proof of Theorem 4.8. □

5. Finite atomic characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$

In this section, we establish finite atomic characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$. We begin with introducing the notion of variable anisotropic finite atomic Hardy spaces $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ as follows.

Definition 5.1. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (1, \infty]$, s be as in (4.1) and A a dilation. The *variable anisotropic finite atomic Hardy space* $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that there exist $I \in \mathbb{N}$, $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathbb{C}$ and a finite sequence of $(p(\cdot), q, s)$-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in[1,I]\cap\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i=1}^{I} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n).$$

Moreover, for any $f \in H^{p(\cdot),q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{p(\cdot),q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} := \inf \left\| \left\{ \sum_{i=1}^{I} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

where $\underline{p}$ is as in (2.4) and the infimum is taken over all decompositions of f as above.

The following conclusion is from Theorem 4.8 and its proof, which is used in the proof of Theorem 5.4 below.

Lemma 5.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (1,\infty]$, s be as in (4.1) and τ as in (2.2). Then, for any $f \in H^{p(\cdot)}_A(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, there exist $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \mathbb{C}$, dilated balls $\{x_i^k + B_{\ell_i^k}\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \mathfrak{B}$ and $(p(\cdot),\infty,s)$-atoms $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$ such that*

$$f = \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \lambda_i^k a_i^k,$$

where the series converge both almost everywhere and in $\mathcal{S}'(\mathbb{R}^n)$,

$$\text{(5.1)} \qquad \operatorname{supp} a_i^k \subset x_i^k + B_{\ell_i^k+4\tau}, \quad \Omega_k = \bigcup_{j\in\mathbb{N}} (x_j^k + B_{\ell_j^k+4\tau}) \quad \text{for any } k \in \mathbb{Z} \text{ and } i \in \mathbb{N}$$

with Ω_k as in (4.8),

$$\text{(5.2)} \qquad (x_i^k + B_{\ell_i^k-\tau}) \cap (x_j^k + B_{\ell_j^k-\tau}) = \emptyset \quad \text{for any } k \in \mathbb{Z} \text{ and } i,j \in \mathbb{N} \text{ with } i \neq j,$$

and

$$\text{(5.3)} \qquad \sharp \left\{ j \in \mathbb{N} : (x_i^k + B_{\ell_i^k+4\tau}) \cap (x_j^k + B_{\ell_j^k+4\tau}) \neq \emptyset \right\} \le R \quad \text{for any } i \in \mathbb{N}$$

with R being a positive constant independent of k and f. Moreover, there exists a positive constant C, independent of f, such that, for any $k \in \mathbb{Z}$, $i \in \mathbb{N}$ and almost every $x \in \mathbb{R}^n$,

$$\text{(5.4)} \qquad \left| \lambda_i^k a_i^k(x) \right| \le C 2^k$$

and

$$\text{(5.5)} \qquad \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i^k| \chi_{x_i^k + B_{\ell_i^k+4\tau}}}{\|\chi_{x_i^k + B_{\ell_i^k+4\tau}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \|f\|_{H^{p(\cdot)}_A(\mathbb{R}^n)}$$

with $\underline{p}$ as in (2.4).

Remark 5.3. For any $i \in \mathbb{N}$, $k \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$, let θ_i^k and $\mathbb{P}_r(\mathbb{R}^n)$ be the same as those used in the proof of Theorem 4.8. For any $f \in H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ with $q \in (1, \infty]$, by an argument same as that used in the proof of Theorem 4.8, we also conclude that there exists a unique polynomial $P_i^k \in \mathbb{P}_r(\mathbb{R}^n)$ such that, for any $Q \in \mathbb{P}_r(\mathbb{R}^n)$,

$$\langle f, Q\theta_i^k \rangle = \langle P_i^k, Q\theta_i^k \rangle = \int_{\mathbb{R}^n} P_i^k(x)Q(x)\theta_i^k(x)\,dx.$$

In addition, for any $i, j \in \mathbb{N}$ and $k \in \mathbb{Z}$, we let the polynomial $P_{i,j}^{k+1}$ be the orthogonal projection of $(f - P_j^{k+1})\theta_i^k$ on $\mathbb{P}_r(\mathbb{R}^n)$ with respect to the norm defined by (4.13), namely, $P_{i,j}^{k+1}$ is the unique element of $\mathbb{P}_r(\mathbb{R}^n)$ such that, for any $Q \in \mathbb{P}_r(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \left[f(x) - P_j^{k+1}(x)\right] \theta_i^k(x)Q(x)\theta_j^{k+1}(x)\,dx = \int_{\mathbb{R}^n} P_{i,j}^{k+1}(x)Q(x)\theta_j^{k+1}(x)\,dx$$

and, for any $i \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$\lambda_i^k a_i^k = (f - P_i^k)\theta_i^k - \sum_{j\in\mathbb{N}} \left[(f - P_j^{k+1})\theta_i^k - P_{i,j}^{k+1}\right] \theta_j^{k+1}.$$

We always denote by $C(\mathbb{R}^n)$ the *set of all continuous functions.* Then we obtain the following finite atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$, which extends [50, Theorem 3.1 and Remark 3.3] to the present setting of variable anisotropic Hardy spaces.

Theorem 5.4. *Let* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty]$ *with* p_+ *as in* (2.4) *and* s *be as in* (4.1).

(i) *If* $q \in (\max\{p_+, 1\}, \infty)$, *then* $\|\cdot\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}$ *are equivalent quasi-norms on* $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$;

(ii) $\|\cdot\|_{H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n)}$ *and* $\|\cdot\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}$ *are equivalent quasi-norms on* $H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$.

Proof. Assume that $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty]$ and s is as in (4.1). Then it follows, from Theorem 4.8, that $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n) \subset H_A^{p(\cdot)}(\mathbb{R}^n)$ and, for any $f \in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$, $\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}$. Thus, to prove Theorem 5.4, we only need to show that, for any $f \in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ when $q \in (\max\{p_+, 1\}, \infty)$ and, for any $f \in \left[H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)\right]$ when $q = \infty$,

$$\|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

We prove this by three steps.

Step 1. Let $q \in (\max\{p_+, 1\}, \infty]$. Then, without loss of generality, we may assume that $f \in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ and $\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} = 1$. Clearly, there exists some $K \in \mathbb{Z}$ such that $\operatorname{supp} f \subset B_K$ because f has compact support, where B_K is as in Section 2. In the remainder of this section, we always let $N := \lfloor (1/\underline{p}-1)\ln b/\ln\lambda_-\rfloor + 2$ and, for any $k \in \mathbb{Z}$, let

$$\Omega_k := \left\{x \in \mathbb{R}^n : M_N(f)(x) > 2^k\right\}.$$

Since $f \in H_A^{p(\cdot)}(\mathbb{R}^n) \cap L^{\widetilde{q}}(\mathbb{R}^n)$, where $\widetilde{q} := q$ when $q \in (\max\{p_+, 1\}, \infty)$ and $\widetilde{q} := 2$ when $q = \infty$, it follows, from Lemma 5.2, that there exist $\{\lambda_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), \infty, s)$-atoms, $\{a_i^k\}_{k\in\mathbb{Z},i\in\mathbb{N}}$, such that

$$f = \sum_{k\in\mathbb{Z}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k \tag{5.6}$$

holds true both almost everywhere and in $\mathcal{S}'(\mathbb{R}^n)$ and, in addition, (5.1) through (5.5) also hold true.

By this and an argument similar to that used in Step 2 of the proof of [45, Theorem 5.7] (see also [44, (4.9)]), we conclude that there exists a positive constant C_5 such that, for any $x \in (B_{K+4\tau})^{\complement}$,

$$M_N(f)(x) \le C_5 \|\chi_{B_K}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{5.7}$$

Then, for any $k \in (\widetilde{k}, \infty] \cap \mathbb{Z}$, we have

$$\Omega_k \subset B_{K+4\tau}, \tag{5.8}$$

where τ is as in (2.2) and

$$\widetilde{k} := \sup\left\{k \in \mathbb{Z} : 2^k < C_5\|\chi_{B_K}\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}\right\} \tag{5.9}$$

with C_5 as in (5.7). Using this $\widetilde{k}$, we rewrite (5.6) as

$$f = \sum_{k=-\infty}^{\widetilde{k}}\sum_{i\in\mathbb{N}} \lambda_i^k a_i^k + \sum_{k=\widetilde{k}+1}^{\infty}\sum_{i\in\mathbb{N}} \cdots =: h + \ell,$$

where the series converge both almost everywhere and in $\mathcal{S}'(\mathbb{R}^n)$. From this and an argument same as that used in Step 2 of the proof of [44, Theorem 2.14], we further deduce that there exists a positive constant C_6, independent of f, such that h/C_6 is a $(p(\cdot), q, s)$-atom for any $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty]$ and s being as in (4.1).

Step 2. This step is aimed to prove (i). For this purpose, for any $k_0 \in (\widetilde{k}, \infty) \cap \mathbb{Z}$ and $k \in [\widetilde{k}+1, k_0] \cap \mathbb{Z}$ with $\widetilde{k}$ as in (5.9), let

$$I_{(k_0,k)} := \{i \in \mathbb{N} : |i| + |k| \le k_0\} \quad \text{and} \quad \ell_{(k_0)} := \sum_{k=\widetilde{k}+1}^{k_0}\sum_{i\in I_{(k_0,k)}} \lambda_i^k a_i^k.$$

On the other hand, for any $q\in(\max\{p_+,1\},\infty)$, by an argument similar to that used in Step 4 of the proof of [45, Theorem 5.7], we know that $\ell\in L^q(\mathbb{R}^n)$. This further implies that, for any given $\epsilon\in(0,1)$, there exists a $k_0\in[\widetilde{k}+1,\infty)\cap\mathbb{Z}$ large enough, depending on ϵ, such that $[\ell-\ell_{(k_0)}]/\epsilon$ is a $(p(\cdot),q,s)$-atom and hence $f=h+\ell_{(k_0)}+[\ell-\ell_{(k_0)}]$ is a finite linear combination of $(p(\cdot),q,s)$-atoms. From this, Step 1 and (5.5), we deduce that

$$\|f\|_{H^{p(\cdot),q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)}\lesssim C_6+\left\|\left\{\sum_{k=\widetilde{k}+1}^{k_0}\sum_{i\in I_{(k_0,k)}}\left[\frac{|\lambda_i^k|\chi_{x_i^k+B_{\ell_i^k+4\tau}}}{\|\chi_{x_i^k+B_{\ell_i^k+4\tau}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\epsilon\lesssim 1,$$

which completes the proof of (i).

Step 3. In this step, we prove (ii). To this end, assume that $f\in H^{p(\cdot),\infty,s}_{A,\mathrm{fin}}(\mathbb{R}^n)\cap C(\mathbb{R}^n)$. Thus, by (4.10), we know that, for any $k\in\mathbb{Z}$ and $i\in\mathbb{N}$, a_i^k is continuous. In addition, from the fact that there exists a positive constant $C_{(n,N)}$, depending only on n and N, such that, for any $x\in\mathbb{R}^n$,

$$M_N(f)(x)\le C_{(n,N)}\|f\|_{L^\infty(\mathbb{R}^n)},$$

it follows that the level set Ω_k is empty for any k such that $2^k\ge C_{(n,N)}\|f\|_{L^\infty(\mathbb{R}^n)}$. Let

$$\widehat{k}:=\sup\left\{k\in\mathbb{Z}:2^k<C_{(n,N)}\|f\|_{L^\infty(\mathbb{R}^n)}\right\}.$$

Then we conclude that the index k in the sum defining ℓ runs only over $k\in\{\widetilde{k}+1,\dots,\widehat{k}\}$.

Notice that f is uniformly continuous. Then, for any $\epsilon\in(0,\infty)$, we can choose a $\delta\in(0,\infty)$ such that $|f(x)-f(y)|<\epsilon$ whenever $\rho(x-y)<\delta$. Furthermore, for this ϵ, define

$$\ell_1^\epsilon:=\sum_{k=\widetilde{k}+1}^{\widehat{k}}\sum_{i\in E_1^{(k,\delta)}}\lambda_i^k a_i^k\quad\text{and}\quad\ell_2^\epsilon:=\sum_{k=\widetilde{k}+1}^{\widehat{k}}\sum_{i\in E_2^{(k,\delta)}}\lambda_i^k a_i^k,$$

where, for any $k\in\{\widetilde{k}+1,\dots,\widehat{k}\}$,

$$E_1^{(k,\delta)}:=\left\{i\in\mathbb{N}:b^{\ell_i^k+\tau}\ge\delta\right\}\quad\text{and}\quad E_2^{(k,\delta)}:=\left\{i\in\mathbb{N}:b^{\ell_i^k+\tau}<\delta\right\}.$$

Clearly, it follows, from (5.2) and (5.8), that, for any fixed $k\in\{\widetilde{k}+1,\dots,\widehat{k}\}$, $E_1^{(k,\delta)}$ is a finite set and hence ℓ_1^ϵ is a finite linear combination of continuous $(p(\cdot),\infty,s)$-atoms. Then, by (4.6), we have

$$\left\|\left\{\sum_{k=\widetilde{k}+1}^{\widehat{k}}\sum_{i\in E_1^{(k,\delta)}}\left[\frac{|\lambda_i^k|\chi_{x_i^k+B_{\ell_i^k}}}{\|\chi_{x_i^k+B_{\ell_i^k}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}. \tag{5.10}$$

Moreover, by an argument same as that used in Step 4 of the proof of [44, Theorem 2.14], we conclude that there exists an $\epsilon \in (0,\infty)$ small enough such that $f = h+\ell_1^\epsilon+\ell_2^\epsilon$ gives the desired finite atomic decomposition of f. Then, from (5.10) and the fact that h/C_6 is a $(p(\cdot),\infty,s)$-atom, we further deduce that

$$\|f\|_{H^{p(\cdot),\infty,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} \lesssim \|h\|_{H^{p(\cdot),\infty,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} + \|\ell_1^\epsilon\|_{H^{p(\cdot),\infty,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} + \|\ell_2^\epsilon\|_{H^{p(\cdot),\infty,s}_{A,\mathrm{fin}}(\mathbb{R}^n)} \lesssim 1.$$

This finishes the proof of (ii) and hence of Theorem 5.4. □

6. Littlewood-Paley function characterizations of $H_A^{p(\cdot)}(\mathbb{R}^n)$

In this section, we characterize $H_A^{p(\cdot)}(\mathbb{R}^n)$ by means of the Lusin area function, the Littlewood-Paley g-function and the Littlewood-Paley g_λ^*-function, respectively. We begin with recalling the following notion of Littlewood-Paley functions (see, for example, [40,45]).

Assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function such that, for any $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \le s$, where $s \in \mathbb{N} \cap [\lfloor (1/p_- - 1)\ln b/\ln \lambda_- \rfloor, \infty)$ with p_- as in (2.4),

$$\int_{\mathbb{R}^n} \phi(x)x^\alpha\, dx = 0 \tag{6.1}$$

and, for any $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,

$$\sum_{k\in\mathbb{Z}} \left|\widehat{\phi}((A^*)^k\xi)\right|^2 = 1, \tag{6.2}$$

here and hereafter, A^* denotes the adjoint matrix of A and $\widehat{\phi}$ the *Fourier transform* of ϕ, namely, for any $\xi \in \mathbb{R}^n$,

$$\widehat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x)e^{-2\pi\imath x\cdot\xi}\, dx, \tag{6.3}$$

where $\imath := \sqrt{-1}$ and, for any $x := (x_1,\ldots,x_n)$, $\xi := (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$, $x\cdot\xi := \sum_{i=1}^n x_i\xi_i$. Then, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\lambda \in (0,\infty)$, the *anisotropic Lusin area function* $S(f)$, the *Littlewood-Paley g-function* $g(f)$ and the *Littlewood-Paley g_λ^*-function* $g_\lambda^*(f)$ are defined, respectively, by setting, for any $x \in \mathbb{R}^n$,

$$S(f)(x) := \left[\sum_{k\in\mathbb{Z}} b^{-k}\int_{x+B_k} |f*\phi_{-k}(y)|^2\, dy\right]^{1/2}, \quad g(f)(x) := \left[\sum_{k\in\mathbb{Z}} |f*\phi_k(x)|^2\right]^{1/2}$$

and

$$g_\lambda^*(f)(x) := \left\{\sum_{k\in\mathbb{Z}} b^{-k}\int_{\mathbb{R}^n}\left[\frac{b^k}{b^k+\rho(x-y)}\right]^\lambda |f*\phi_{-k}(y)|^2\, dy\right\}^{1/2}.$$

Recall also that $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to *vanish weakly at infinity* if, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to -\infty$. In what follows, we always denote by $\mathcal{S}'_0(\mathbb{R}^n)$ the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ vanishing weakly at infinity.

Then the main results of this section are stated as follows.

Theorem 6.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \le C\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Theorem 6.2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \le C\|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Moreover, by Theorems 6.1 and 6.2 and an argument similar to that used in the proof of [46, Theorem 2.10], we easily obtain the following result, the details being omitted.

Theorem 6.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $\lambda \in (1 + 2/\min\{p_-, 2\}, \infty)$ with p_- as in (2.4). Then $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'_0(\mathbb{R}^n)$ and $g_\lambda^*(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,*

$$C^{-1}\left\|g_\lambda^*(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \le C\left\|g_\lambda^*(f)\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 6.4. We should point out that, in [42, Theorem 4.8], via the g_λ^*-function, Liang et al. characterized the Musielak-Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$ with $\varphi\colon \mathbb{R}^n \times [0,\infty) \to [0,\infty)$ being a Musielak-Orlicz growth function (see [42, Definition 2.3]). As was mentioned in [42, p. 428], the range of λ in [42, Theorem 4.8] coincides with the best known one of the g_λ^*-function characterization, namely, $\lambda \in (2/p, \infty)$ with $p \in (0, 1]$, of the classical Hardy space $H^p(\mathbb{R}^n)$. However, it is still unclear whether or not the g_λ^*-function, when $\lambda \in (2/\min\{p_-, 2\}, 1 + 2/\min\{p_-, 2\}]$, can characterize $H_A^{p(\cdot)}(\mathbb{R}^n)$, since the method used in the proof of Theorem 6.3 does not work in this case, while the method used in [42, Theorem 4.8] strongly depends on the properties of uniformly Muckenhoupt weights, which are not satisfied by $t^{p(\cdot)}$ with $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ (see [74, Remark 2.14(iii)]).

To prove Theorem 6.1, we need several technical lemmas. First, by a proof similar to that of [74, Lemma 6.5] with some slight modifications, we obtain the following conclusion, the details being omitted.

Lemma 6.5. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then $H_A^{p(\cdot)}(\mathbb{R}^n) \subset \mathcal{S}'_0(\mathbb{R}^n)$.*

The following Calderón reproducing formula is just [8, Proposition 2.14].

Lemma 6.6. *Let $s \in \mathbb{Z}_+$ and A be a dilation. Then there exist $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that*

(i) $\operatorname{supp}\varphi \subset B_0$, $\int_{\mathbb{R}^n} x^\gamma \varphi(x)\,dx = 0$ *for any* $\gamma \in \mathbb{Z}_+^n$ *with* $|\gamma| \le s$, $\widehat{\varphi}(\xi) \ge C$ *for any* $\xi \in \{x \in \mathbb{R}^n : m \le \rho(x) \le t\}$, *where* $0 < m < t < 1$ *and* $C \in (0,\infty)$ *are constants;*

(ii) $\operatorname{supp}\widehat{\psi}$ *is compact and bounded away from the origin;*

(iii) *for any* $\xi \in \mathbb{R}^n \setminus \{\vec{0}_n\}$, $\sum_{k\in\mathbb{Z}} \widehat{\psi}((A^*)^k\xi)\widehat{\varphi}((A^*)^k\xi) = 1$, *where* A^* *denotes the adjoint matrix of* A.

Moreover, for any $f \in \mathcal{S}_0'(\mathbb{R}^n)$, $f = \sum_{k\in\mathbb{Z}} f * \psi_k * \varphi_k$ *in* $\mathcal{S}'(\mathbb{R}^n)$.

The following lemma is just [8, Lemma 2.3].

Lemma 6.7. *Let A be a dilation. Then there exists a set*

$$\mathcal{Q} := \left\{Q_\alpha^k \subset \mathbb{R}^n : k \in \mathbb{Z}, \alpha \in E_k\right\}$$

of open subsets, where E_k is certain index set, such that

(i) *for each* $k \in \mathbb{Z}$, $\left|\mathbb{R}^n \setminus \bigcup_\alpha Q_\alpha^k\right| = 0$ *and, when* $\alpha \ne \beta$, $Q_\alpha^k \cap Q_\beta^k = \emptyset$;

(ii) *for any* α, β, k, ℓ *with* $\ell \ge k$, *either* $Q_\alpha^k \cap Q_\beta^\ell = \emptyset$ *or* $Q_\alpha^\ell \subset Q_\beta^k$;

(iii) *for each* (ℓ,β) *and each* $k < \ell$, *there exists a unique* α *such that* $Q_\beta^\ell \subset Q_\alpha^k$;

(iv) *there exist some* $v \in \mathbb{Z}\setminus\mathbb{Z}_+$ *and* $u \in \mathbb{N}$ *such that, for any* Q_α^k *with* $k \in \mathbb{Z}$ *and* $\alpha \in E_k$, *there exists* $x_{Q_\alpha^k} \in Q_\alpha^k$ *such that, for any* $x \in Q_\alpha^k$,

$$x_{Q_\alpha^k} + B_{vk-u} \subset Q_\alpha^k \subset x + B_{vk+u}.$$

In what follows, we call $\mathcal{Q} := \{Q_\alpha^k\}_{k\in\mathbb{Z},\alpha\in E_k}$ from Lemma 6.7 *dyadic cubes* and k the *level*, denoted by $\ell(Q_\alpha^k)$, of the dyadic cube Q_α^k for any $k \in \mathbb{Z}$ and $\alpha \in E_k$.

Remark 6.8. In the definition of $(p(\cdot),q,s)$-atoms (see Definition 4.1), if we replace dilated balls $\mathfrak{B}$ by dyadic cubes, then it follows, from Lemma 6.7, that the corresponding variable anisotropic atomic Hardy space coincides with the original one (see Definition 4.2) in the sense of equivalent quasi-norms.

Now we prove Theorem 6.1.

Proof of Theorem 6.1. We first show the sufficiency of this theorem. For this purpose, let $f \in \mathcal{S}_0'(\mathbb{R}^n)$ with $S(f) \in L^{p(\cdot)}(\mathbb{R}^n)$. Then we need to prove that $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ and

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{6.4}$$

To this end, for any $k \in \mathbb{Z}$, let $\Omega_k := \{x \in \mathbb{R}^n : S(f)(x) > 2^k\}$ and

$$\mathcal{Q}_k := \left\{ Q \in \mathcal{Q} : |Q \cap \Omega_k| > \frac{|Q|}{2} \text{ and } |Q \cap \Omega_{k+1}| \le \frac{|Q|}{2} \right\}.$$

It is easy to see that, for any $Q \in \mathcal{Q}$, there exists a unique $k \in \mathbb{Z}$ such that $Q \in \mathcal{Q}_k$. Denote by $\{Q_i^k\}_i$ the collection of all *maximal dyadic cubes* in $\mathcal{Q}_k$, namely, there exists no $Q \in \mathcal{Q}_k$ such that $Q_i^k \subsetneq Q$ for any i.

Then, by Lemmas 6.6 and 6.7 and an argument similar to that used in the proof of the sufficiency of [46, Theorem 5.2], we conclude that

$$f = \sum_{k\in\mathbb{Z}} \sum_i \lambda_i^k a_i^k \quad \text{in } \mathcal{S}'(\mathbb{R}^n),$$

where, for any $k \in \mathbb{Z}$ and i, $\lambda_i^k \sim 2^k \|\chi_{B_i^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with the equivalent positive constants independent of k and i and a_i^k is a $(p(\cdot), q, s)$-atom satisfying, for any $q \in (\max\{p_+, 1\}, \infty)$, $k \in \mathbb{Z}$, i and $\gamma \in \mathbb{Z}_+^n$ as in Definition 4.1,

$$\operatorname{supp} a_i^k \subset B_i^k =: x_{Q_i^k} + B_{v[\ell(Q_i^k)-1]+u+3\tau} \quad \text{with } v \text{ and } u \text{ as in Lemma 6.7(iv)},$$

$$\left\|a_i^k\right\|_{L^q(\mathbb{R}^n)} \le \left\|\chi_{B_i^k}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} |B_i^k|^{1/q} \quad \text{and} \quad \int_{\mathbb{R}^n} a_i^k(x) x^\gamma \, dx = 0.$$

From this, Theorem 4.8, the mutual disjointness of $\{Q_i^k\}_{k\in\mathbb{Z},i}$, Lemma 6.7(iv), the fact that $\left|Q_i^k \cap \Omega_k\right| \ge \left|Q_i^k\right|/2$ and [46, Lemma 5.4], we further deduce that

$$\begin{aligned}
\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} &\sim \left\| \left\{ \sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left[\frac{\lambda_i^k \chi_{B_i^k}}{\|\chi_{B_i^k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \left\| \left[\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left(2^k \chi_{B_i^k}\right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left[\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left(2^k \chi_{Q_i^k}\right)^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \left\| \left[\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left(2^k \chi_{Q_i^k}\right)^{\underline{p}} \right]^{1/2} \right\|_{L^{2p(\cdot)/\underline{p}}(\mathbb{R}^n)}^{2/\underline{p}} \lesssim \left\| \left[\sum_{k\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \left(2^k \chi_{Q_i^k \cap \Omega_k}\right)^{\underline{p}} \right]^{1/2} \right\|_{L^{2p(\cdot)/\underline{p}}(\mathbb{R}^n)}^{2/\underline{p}} \\
&\lesssim \left\| \left[\sum_{k\in\mathbb{Z}} (2^k \chi_{\Omega_k})^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left[\sum_{k\in\mathbb{Z}} (2^k \chi_{\Omega_k \setminus \Omega_{k+1}})^{\underline{p}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| S(f) \left[\sum_{k\in\mathbb{Z}} \chi_{\Omega_k \setminus \Omega_{k+1}} \right]^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

which implies that $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ and (6.4) holds true. This finishes the proof of the sufficiency of Theorem 6.1.

Next we show the necessity of this theorem. To this end, let $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 6.5, we know that $f \in \mathcal{S}_0'(\mathbb{R}^n)$. On the other hand, it follows, from Theorem 4.8, that there exist $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and a sequence of $(p(\cdot), q, s)$-atoms, $\{a_i\}_{i\in\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in\mathbb{N}} \subset \mathfrak{B}$ such that

$$f = \sum_{i\in\mathbb{N}} \lambda_i a_i \quad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Then, by [46, (5.10)], we find that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} S(f)(x) &\le \sum_{i\in\mathbb{N}} |\lambda_i| S(a_i)(x)\chi_{A^w B^{(i)}}(x) + \sum_{i\in\mathbb{N}} |\lambda_i| S(a_i)(x)\chi_{(A^w B^{(i)})^\complement}(x) \\ &\lesssim \left\{ \sum_{i\in\mathbb{N}} [|\lambda_i| S(a_i)(x)\chi_{A^w B^{(i)}}(x)]^{\underline{p}} \right\}^{1/\underline{p}} \\ &\quad + \sum_{i\in\mathbb{N}} \frac{|\lambda_i|}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \left[M_{\mathrm{HL}}(\chi_{B^{(i)}})(x) \right]^\beta, \end{aligned} \tag{6.5}$$

where $w := u - v + 2\tau$ with u and v as in Lemma 6.7, $\underline{p}$ is as in (2.4),

$$\beta := \left(\frac{\ln b}{\ln \lambda_-} + s + 1 \right) \frac{\ln \lambda_-}{\ln b} > \frac{1}{\underline{p}}$$

and M_{HL} denotes the Hardy-Littlewood maximal operator as in (3.1).

By (6.5) and an argument same as that used in the proof of Theorem 4.8, we further conclude that

$$\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)},$$

which completes the proof of the necessity and hence of Theorem 6.1. □

Recall that, for any given dilation A, $\phi \in \mathcal{S}(\mathbb{R}^n)$, $t \in (0, \infty)$, $j \in \mathbb{Z}$ and any $f \in \mathcal{S}'(\mathbb{R}^n)$, the *anisotropic Peetre maximal function* $(\phi_j^* f)_t$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$(\phi_j^* f)_t(x) := \operatorname*{ess\,sup}_{y\in\mathbb{R}^n} \frac{|(\phi_j * f)(x+y)|}{[1 + b^j \rho(y)]^t}$$

and the *g-function associated with* $(\phi_j^* f)_t$ is defined by setting, for any $x \in \mathbb{R}^n$,

$$g_{t,*}(f)(x) := \left\{ \sum_{j\in\mathbb{Z}} \left[(\phi_j^* f)_t(x) \right]^2 \right\}^{1/2}, \tag{6.6}$$

where, for any $j \in \mathbb{Z}$, $\phi_j(\cdot) := b^j \phi(A^j \cdot)$.

To prove Theorem 6.2, we need the following estimate, which is just [44, Lemma 3.6] and originates from [68].

Lemma 6.9. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying* (6.1) *and* (6.2)*. Then, for any given $N_0 \in \mathbb{N}$ and $r \in (0,\infty)$, there exists a positive constant $C_{(N_0,r)}$, which may depends on N_0 and r, such that, for any $t \in (0,N_0)$, $\ell \in \mathbb{Z}$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, it holds true that*

$$[(\phi_\ell^* f)_t(x)]^r \le C_{(N_0,r)} \sum_{k\in\mathbb{Z}_+} b^{-kN_0r} b^{k+\ell} \int_{\mathbb{R}^n} \frac{|(\phi_{k+\ell} * f)(y)|^r}{[1+b^\ell \rho(x-y)]^{tr}}\,dy.$$

We now prove Theorem 6.2.

Proof of Theorem 6.2. First, let $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then, by Lemma 6.5, we know that $f \in \mathcal{S}_0'(\mathbb{R}^n)$. In addition, repeating the proof of the necessity of Theorem 6.1 with some slight modifications, we easily find that $g(f) \in L^{p(\cdot)}(\mathbb{R}^n)$ and $\|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}$. Thus, to prove Theorem 6.2, by Theorem 6.1, we only need to show that, for any $f \in \mathcal{S}_0'(\mathbb{R}^n)$ with $g(f) \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\|S(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{6.7}$$

holds true. Notice that, for any $f \in \mathcal{S}_0'(\mathbb{R}^n)$, $t \in (0,\infty)$ and almost every $x \in \mathbb{R}^n$, $S(f)(x) \lesssim g_{t,*}(f)(x)$. Thus, to show (6.7), it suffices to prove that

$$\|g_{t,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \lesssim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{6.8}$$

holds true for any $f \in \mathcal{S}_0'(\mathbb{R}^n)$ and some $t \in (1/\min\{p_-,2\},\infty)$.

Now we show (6.8). To this end, assume that $\phi \in \mathcal{S}(\mathbb{R}^n)$ is a radial function and satisfies (6.1) and (6.2). Obviously, $t \in (1/\min\{p_-,2\},\infty)$ implies that there exists $r \in (0,\min\{p_-,2\})$ such that $t \in (1/r,\infty)$. Fix $N_0 \in (1/r,\infty)$. By this, Lemma 6.9 and the Minkowski inequality, we know that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned} g_{t,*}(f)(x) &= \left\{\sum_{k\in\mathbb{Z}} [(\phi_k^* f)_t(x)]^2\right\}^{1/2} \\ &\lesssim \left[\sum_{k\in\mathbb{Z}} \left\{\sum_{j\in\mathbb{Z}_+} b^{-jN_0r} b^{j+k} \int_{\mathbb{R}^n} \frac{|(\phi_{j+k} * f)(y)|^r}{[1+b^k\rho(x-y)]^{tr}}\,dy\right\}^{2/r}\right]^{1/2} \\ &\lesssim \left\{\sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)} \left[\sum_{k\in\mathbb{Z}} b^{2k/r} \left\{\int_{\mathbb{R}^n} \frac{|(\phi_{j+k} * f)(y)|^r}{[1+b^k\rho(x-y)]^{tr}}\,dy\right\}^{2/r}\right]^{r/2}\right\}^{1/r}, \end{aligned}$$

which, together with Lemma 3.4, implies that

$$\begin{aligned} &\|g_{t,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{rp} \\ &\lesssim \left\|\sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)} \left[\sum_{k\in\mathbb{Z}} b^{2k/r} \left\{\int_{\mathbb{R}^n} \frac{|(\phi_{j+k} * f)(y)|^r}{[1+b^k\rho(\cdot-y)]^{tr}}\,dy\right\}^{2/r}\right]^{r/2}\right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{p} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)\underline{p}} \left\| \left[\sum_{k\in\mathbb{Z}} b^{2k/r} \left\{ \int_{\mathbb{R}^n} \frac{|(\phi_{j+k}*f)(y)|^r}{[1+b^k\rho(\cdot-y)]^{tr}}\,dy \right\}^{2/r} \right]^{r/2} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)\underline{p}} \left\| \left\{ \sum_{k\in\mathbb{Z}} b^{2k/r} \left[\sum_{i\in\mathbb{N}} b^{-itr} \int_{\rho(\cdot-y)\sim b^{i-k}} |(\phi_{j+k}*f)(y)|^r\,dy \right]^{2/r} \right\}^{r/2} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{\underline{p}},
\end{aligned}$$

where $\rho(\cdot-y)\sim b^{i-k}$ means that $\{x\in\mathbb{R}^n:\rho(x-y)<b^{-k}\}$ when $i=0$, or $\{x\in\mathbb{R}^n: b^{i-k-1}\le\rho(x-y)<b^{i-k}\}$ when $i\in\mathbb{N}$. Then, by the Minkowski inequality again and Lemma 4.4, we further conclude that

$$\begin{aligned}
&\|g_{t,*}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{r\underline{p}} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)\underline{p}} \left\| \sum_{i\in\mathbb{N}} b^{-itr} \left\{ \sum_{k\in\mathbb{Z}} b^{2k/r} \left[\int_{\rho(\cdot-y)\sim b^{i-k}} |(\phi_{j+k}*f)(y)|^r\,dy \right]^{2/r} \right\}^{r/2} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)\underline{p}} \left\| \sum_{i\in\mathbb{N}} b^{(1-tr)i} \left\{ \sum_{k\in\mathbb{Z}} [M_{\mathrm{HL}}(|\phi_{j+k}*f|^r)]^{2/r} \right\}^{r/2} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{\underline{p}} \\
&\lesssim \sum_{j\in\mathbb{Z}_+} b^{-j(N_0r-1)\underline{p}} \sum_{i\in\mathbb{N}} b^{(1-tr)i\underline{p}} \left\| \left(\sum_{k\in\mathbb{Z}} |\phi_{j+k}*f|^2 \right)^{r/2} \right\|_{L^{p(\cdot)/r}(\mathbb{R}^n)}^{\underline{p}} \sim \|g(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{r\underline{p}}.
\end{aligned}$$

This implies that (6.8) holds true and hence finishes the proof of Theorem 6.2. □

7. Some applications

As applications, in this section, we first establish a criterion on the boundedness of sublinear operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into a quasi-Banach space. Applying this criterion, we then give some applications for the anisotropic summability of Fourier transforms introduced in [44].

Recall that a *quasi-Banach space* $\mathcal{B}$ is a complete vector space equipped with a quasi-norm $\|\cdot\|_{\mathcal{B}}$, which satisfies

(i) $\|f\|_{\mathcal{B}}=0$ if and only if f is the zero element of $\mathcal{B}$;

(ii) there exists a positive constant $L\in[1,\infty)$ such that, for any $f,g\in\mathcal{B}$,

$$\|f+g\|_{\mathcal{B}}\le L(\|f\|_{\mathcal{B}}+\|g\|_{\mathcal{B}}).$$

Clearly, a quasi-Banach space $\mathcal{B}$ becomes a Banach space when $L=1$. In addition, for any given $\gamma\in(0,1]$, a quasi-Banach space $\mathcal{B}_\gamma$ with quasi-norm $\|\cdot\|_{\mathcal{B}_\gamma}$ is called a γ*-quasi-Banach space* if there exists a constant $\kappa\in[1,\infty)$ such that, for any $m\in\mathbb{N}$ and $\{f_i\}_{i=1}^m\subset\mathcal{B}_\gamma$, it holds true that $\|\sum_{i=1}^m f_i\|_{\mathcal{B}_\gamma}^\gamma\le\kappa\sum_{i=1}^m\|f_i\|_{\mathcal{B}_\gamma}^\gamma$ (see [37, 78, 79]).

Let $\mathcal{B}_\gamma$ be a γ-quasi-Banach space with $\gamma \in (0,1]$ and $\mathcal{Y}$ a linear space. An operator T from $\mathcal{Y}$ to $\mathcal{B}_\gamma$ is said to be $\mathcal{B}_\gamma$-*sublinear* if there exists a positive constant C such that, for any $m \in \mathbb{N}$, $\{\lambda\}_{i=1}^m \subset \mathbb{C}$ and $\{f_i\}_{i=1}^m \subset \mathcal{Y}$,

$$\left\| T\left(\sum_{i=1}^m \lambda_i f_i\right)\right\|_{\mathcal{B}_\gamma}^\gamma \leq C\sum_{i=1}^m |\lambda_i|^\gamma \|T(f_i)\|_{\mathcal{B}_\gamma}^\gamma$$

and $\|T(f)-T(g)\|_{\mathcal{B}_\gamma} \leq C\|T(f-g)\|_{\mathcal{B}_\gamma}$ (see [37, 78, 79]). Obviously, if T is linear, then T is $\mathcal{B}_\gamma$-sublinear for any $\gamma \in (0,1]$.

As an application of the finite atomic characterization of $H_A^{p(\cdot)}(\mathbb{R}^n)$ obtained in Section 5 (see Theorem 5.4), we establish the following criterion for the boundedness of sublinear operators from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into a quasi-Banach space $\mathcal{B}_\gamma$.

Theorem 7.1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (\max\{p_+, 1\}, \infty]$, $\gamma \in (0,1]$, s be as in* (4.1) *and $\mathcal{B}_\gamma$ a γ-quasi-Banach space. If one of the following statements holds true:*

(i) *$q \in (\max\{p_+, 1\}, \infty)$ and $T\colon H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator satisfying that there exists a positive constant C_7 such that, for any $f \in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$,*

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C_7 \|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}; \tag{7.1}$$

(ii) *$T\colon H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n) \to \mathcal{B}_\gamma$ is a $\mathcal{B}_\gamma$-sublinear operator satisfying that there exists a positive constant C_8 such that, for any $f \in H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$,*

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C_8 \|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),\infty,s}(\mathbb{R}^n)},$$

then T uniquely extends to a bounded $\mathcal{B}_\gamma$-sublinear operator from $H_A^{p(\cdot)}(\mathbb{R}^n)$ into $\mathcal{B}_\gamma$. Moreover, there exists a positive constant C_9 such that, for any $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$,

$$\|T(f)\|_{\mathcal{B}_\gamma} \leq C_9 \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

From Theorem 7.1, we easily deduce the following conclusion, which extends the corresponding results of Meda et al. [50, Corollary 3.4] and Grafakos et al. [30, Theorem 5.9] as well as Ky [37, Theorem 3.5] to the present setting, the details being omitted.

Corollary 7.2. *Assume that $p(\cdot)$, q, γ, s and $\mathcal{B}_\gamma$ are as in Theorem* 7.1*. If one of the following statements holds true:*

(i) *$q \in (\max\{p_+, 1\}, \infty)$ and T is a $\mathcal{B}_\gamma$-sublinear operator from $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ to $\mathcal{B}_\gamma$ satisfying that*

$$\sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any } (p(\cdot), r, s)\text{-atom}\} < \infty;$$

(ii) *T is a $\mathcal{B}_\gamma$-sublinear operator defined on continuous (p,∞,s)-atoms satisfying that*

$$\sup\{\|T(a)\|_{\mathcal{B}_\gamma} : a \text{ is any continuous } (p(\cdot),\infty,s)\text{-atom}\} < \infty,$$

then T has a unique bounded $\mathcal{B}_\gamma$-sublinear extension $\widetilde{T}$ from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $\mathcal{B}_\gamma$.

To prove Theorem 7.1, we need the following density of $H_A^{p(\cdot)}(\mathbb{R}^n)$, which can be easily obtained by Lemma 4.7 and a proof similar to that of [45, Lemma 5.2], the details being omitted.

Lemma 7.3. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then $H_A^{p(\cdot)}(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$ is dense in $H_A^{p(\cdot)}(\mathbb{R}^n)$, here and hereafter, $C_c^\infty(\mathbb{R}^n)$ denotes the set of all infinite differentiable functions with compact supports.*

We now prove Theorem 7.1.

Proof of Theorem 7.1. We first prove (i). To this end, assume that $q \in (\max\{p_+,1\},\infty)$ and $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$. Then it follows, from the obvious density of $H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ in $H_A^{p(\cdot)}(\mathbb{R}^n)$, that there exists a Cauchy sequence $\{f_k\}_{k\in\mathbb{N}} \subset H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$ such that

$$\lim_{k\to\infty} \|f_k - f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} = 0.$$

By this, (7.1) and Theorem 5.4(i), we find that, as $k,\ell \to \infty$,

$$\|T(f_k) - T(f_\ell)\|_{\mathcal{B}_\gamma} \lesssim \|T(f_k - f_\ell)\|_{\mathcal{B}_\gamma} \lesssim \|f_k - f_\ell\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)} \sim \|f_k - f_\ell\|_{H_A^{p(\cdot)}(\mathbb{R}^n)} \to 0,$$

which implies that $\{T(f_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{B}_\gamma$. Thus, there exists some $h \in \mathcal{B}_\gamma$ such that $h = \lim_{k\to\infty} T(f_k)$ in $\mathcal{B}_\gamma$ by the completeness of $\mathcal{B}_\gamma$. Then let $T(f) := h$. By this, (7.1) and Theorem 5.4(i) again, we further conclude that

$$\begin{aligned}\|T(f)\|_{\mathcal{B}_\gamma}^\gamma &\lesssim \limsup_{k\to\infty} \left[\|T(f) - T(f_k)\|_{\mathcal{B}_\gamma}^\gamma + \|T(f_k)\|_{\mathcal{B}_\gamma}^\gamma\right] \lesssim \limsup_{k\to\infty} \|T(f_k)\|_{\mathcal{B}_\gamma}^\gamma \\ &\lesssim \limsup_{k\to\infty} \|f_k\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}^\gamma \sim \lim_{k\to\infty} \|f_k\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}^\gamma \sim \|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}^\gamma,\end{aligned}$$

which completes the proof of (i).

We now prove (ii). Indeed, by Lemma 7.3 and an argument similar to that used in the proof of [45, Theorem 6.13(ii)], it is easy to see that (ii) holds true. This finishes the proof of (ii) and hence of Theorem 7.1. □

Next, we investigate the anisotropic summability of Fourier transforms. Recall that the classical θ-summation was considered in a great number of articles and monographs; see, for example, Butzer and Nessel [10], Grafakos [29], Trigub and Belinsky [67] and Feichtinger and Weisz [27, 70–72] and the references therein.

Let $f \in L^p(\mathbb{R}^n)$ for some $p \in [1,2]$. Then the Fourier inversion formula, namely, for any $x \in \mathbb{R}^n$,

$$f(x) := \int_{\mathbb{R}^n} \widehat{f}(t) e^{2\pi \imath x \cdot t}\, dt$$

holds true if $\widehat{f} \in L^1(\mathbb{R}^n)$, where $\widehat{f}$ denotes the Fourier transform of f as in (6.3). This motivates the following definition of summability. We always assume that

$$\theta \in C_0(\mathbb{R}), \quad \theta(|\cdot|) \in L^1(\mathbb{R}^n), \quad \theta(0) = 1 \quad \text{and} \quad \theta \text{ is even}, \tag{7.2}$$

where $C_0(\mathbb{R})$ is the *set of all continuous functions f on $\mathbb{R}$ satisfying that* $\lim_{|x|\to\infty} |f(x)| = 0$. Let A^* be the transposed matrix of A. The *m-th anisotropic θ-mean* of the function $f \in L^p(\mathbb{R}^n)$, with $p \in [1,2]$, is defined by setting, for any $m \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := \int_{\mathbb{R}^n} \theta(|(A^*)^{-m} u|) \widehat{f}(u) e^{2\pi \imath x \cdot u}\, du. \tag{7.3}$$

Let $\theta_0(x) := \theta(|x|)$ for any $x \in \mathbb{R}^n$ and assume that

$$\widehat{\theta_0} \in L^1(\mathbb{R}^n). \tag{7.4}$$

It was proved in [44] that, for any $m \in \mathbb{Z}$, $f \in L^1(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we can rewrite $\sigma_m^\theta f$ as

$$\sigma_m^\theta f(x) = b^m \int_{\mathbb{R}^n} f(t) \widehat{\theta_0}(A^m(x-t))\, dt.$$

Moreover, we can extend the definition of the anisotropic θ-means to any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $p_- \in [1,\infty)$ by setting, for any $x \in \mathbb{R}^n$,

$$\sigma_m^\theta f(x) := b^m \int_{\mathbb{R}^n} f(x-t) \widehat{\theta_0}(A^m t)\, dt.$$

Then we define the *maximal θ-operator* σ_*^θ by setting, for any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ with $p_- \in [1,\infty)$,

$$\sigma_*^\theta f := \sup_{m \in \mathbb{Z}} \left| \sigma_m^\theta f \right|.$$

As an application of Theorem 7.1, we obtain the following boundedness of the maximal θ-operator from $H_A^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$.

Theorem 7.4. *Let θ and θ_0 be, respectively, as in* (7.2) *and* (7.4) *satisfying that there exists a positive constant $\beta \in (1,\infty)$ such that, for any $\alpha \in \mathbb{Z}_+^n$ and $x \in \mathbb{R}^n \setminus \{\vec{0}_n\}$,*

$$|\partial^\alpha \widehat{\theta_0}(x)| \le C_{(\alpha,\beta)} |x|^{-\beta},$$

where $C_{(\alpha,\beta)}$ is a positive constant independent of x. If $p(\cdot) \in C^{\log}(\mathbb{R}^n)$,

$$\beta \in \left(\frac{\ln b}{\ln \lambda_-}, \infty \right) \quad \textit{and} \quad p_- \in \left(\frac{\ln b}{\beta \ln \lambda_-}, \infty \right), \tag{7.5}$$

then there exists a positive constant $C_{(p_-,p_+)}$, with p_- and p_+ as in (2.4)*, such that, for any $f\in H_A^{p(\cdot)}(\mathbb{R}^n)$,*

$$\left\|\sigma_*^\theta f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\le C_{(p_-,p_+)}\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

Proof. By Theorem 7.1(i), to show Theorem 7.4, it suffices to prove that, for any $f\in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$,

$$\left\|\sigma_*^\theta f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\lesssim\|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)} \tag{7.6}$$

with s being as in (4.1) large enough and $q\in(\max\{p_+,1\},\infty)$ to be chosen later, where p_+ is as in (2.4).

To this end, assume now $f\in H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)$. Then it follows, from Definition 5.1, that there exist $I\in\mathbb{N}$, $\{\lambda_i\}_{i\in[1,I]\cap\mathbb{N}}\subset\mathbb{C}$ and a finite sequence of $(p(\cdot),q,s)$-atoms, $\{a_i\}_{i\in[1,I]\cap\mathbb{N}}$, supported, respectively, on $\{B^{(i)}\}_{i\in[1,I]\cap\mathbb{N}}\subset\mathfrak{B}$ such that $f=\sum_{i=1}^I\lambda_ia_i$ in $\mathcal{S}'(\mathbb{R}^n)$ and

$$\|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}\sim\left\|\left\{\sum_{i=1}^I\left[\frac{|\lambda_i|\chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\right]^{\underline{p}}\right\}^{1/\underline{p}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{7.7}$$

where $\underline{p}$ is as in (2.4). It is easy to see that

$$\begin{aligned}&\left\|\sigma_*^\theta f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim\left\|\sum_{i=1}^I|\lambda_i|\sigma_*^\theta(a_i)\chi_{A^\tau B^{(i)}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\left\|\sum_{i=1}^I|\lambda_i|\sigma_*^\theta(a_i)\chi_{(A^\tau B^{(i)})^\complement}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &=:\mathrm{I}_1+\mathrm{I}_2.\end{aligned} \tag{7.8}$$

We first deal with I_1. For this purpose, choose $g\in L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)$ with $\|g\|_{L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)}\le 1$ such that

$$\left\|\sum_{i=1}^I|\lambda_i|^{\underline{p}}\left[\sigma_*^\theta(a_i)\right]^{\underline{p}}\chi_{A^\tau B^{(i)}}\right\|_{L^{p(\cdot)/\underline{p}}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\sum_{i=1}^I|\lambda_i|^{\underline{p}}\left[\sigma_*^\theta(a_i)(x)\right]^{\underline{p}}\chi_{A^\tau B^{(i)}}(x)g(x)\,dx.$$

Then, by the Hölder inequality, we know that, for any $u\in(1,\infty)$ satisfying that $p_+<u\underline{p}<q$, it holds true that

$$\begin{aligned}(\mathrm{I}_1)^{\underline{p}}&\lesssim\left\|\sum_{i=1}^I|\lambda_i|^{\underline{p}}\left[\sigma_*^\theta(a_i)\right]^{\underline{p}}\chi_{A^\tau B^{(i)}}\right\|_{L^{p(\cdot)/\underline{p}}(\mathbb{R}^n)}\\ &\sim\int_{\mathbb{R}^n}\sum_{i=1}^I|\lambda_i|^{\underline{p}}\left[\sigma_*^\theta(a_i)(x)\right]^{\underline{p}}\chi_{A^\tau B^{(i)}}(x)g(x)\,dx\end{aligned}$$

$$\lesssim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\| \left[\sigma_*^\theta(a_i)\right]^{\underline{p}} \chi_{A^\tau B^{(i)}} \right\|_{L^u(\mathbb{R}^n)} \|\chi_{A^\tau B^{(i)}} g\|_{L^{u'}(\mathbb{R}^n)}$$
$$\lesssim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \left\| \sigma_*^\theta(a_i) \right\|_{L^q(\mathbb{R}^n)}^{\underline{p}} \|\chi_{A^\tau B^{(i)}}\|_{L^{q/(q-u\underline{p})}(\mathbb{R}^n)}^{1/u} \|\chi_{A^\tau B^{(i)}} g\|_{L^{u'}(\mathbb{R}^n)}.$$

From this, the boundedness of σ_*^θ on $L^r(\mathbb{R}^n)$ with $r \in (1,\infty)$, Definition 4.1 and the Hölder inequality again, we further deduce that

$$(\mathrm{I}_1)^{\underline{p}} \lesssim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \left|A^\tau B^{(i)}\right|^{\underline{p}/q} \left|A^\tau B^{(i)}\right|^{(q-u\underline{p})/(qu)} \|\chi_{A^\tau B^{(i)}} g\|_{L^{u'}(\mathbb{R}^n)}$$
$$\sim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \left|A^\tau B^{(i)}\right|^{1/u} \|\chi_{A^\tau B^{(i)}} g\|_{L^{u'}(\mathbb{R}^n)}$$
$$\sim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \left|A^\tau B^{(i)}\right| \left[\frac{1}{|A^\tau B^{(i)}|} \int_{A^\tau B^{(i)}} [g(x)]^{u'}\,dx\right]^{1/u'}$$
$$\lesssim \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \int_{\mathbb{R}^n} \chi_{A^\tau B^{(i)}}(x) \left[M_{\mathrm{HL}}(g^{u'})(x)\right]^{1/u'}\,dx$$
$$\lesssim \left\| \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \chi_{A^\tau B^{(i)}} \right\|_{L^{p(\cdot)/\underline{p}}(\mathbb{R}^n)} \left\| \left[M_{\mathrm{HL}}(g^{u'})\right]^{1/u'} \right\|_{L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)}.$$

On the other hand, it follows, from $p_+/\underline{p} \in (0,u)$, that $(p(\cdot)/\underline{p})' \in (u',\infty]$. By this, Lemmas 3.3(ii) and 3.4, the fact that $\|g\|_{L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)} \le 1$, [46, Remark 4.4(i)] and (7.7), we conclude that

$$\mathrm{I}_1 \lesssim \left\| \sum_{i=1}^{I} |\lambda_i|^{\underline{p}} \|\chi_{A^\tau B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\underline{p}} \chi_{B^{(i)}} \right\|_{L^{p(\cdot)/\underline{p}}(\mathbb{R}^n)}^{1/\underline{p}} \|g\|_{L^{(p(\cdot)/\underline{p})'}(\mathbb{R}^n)}^{1/\underline{p}}$$
$$\lesssim \left\| \left\{ \sum_{i=1}^{I} \left[\frac{|\lambda_i| \chi_{A^\tau B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \tag{7.9}$$
$$\sim \left\| \left\{ \sum_{i=1}^{I} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$
$$\sim \|f\|_{H_{A,\mathrm{fin}}^{p(\cdot),q,s}(\mathbb{R}^n)}.$$

For I_2, by an argument similar to that used in the proof of [44, (5.10)], we easily find that, for any $i \in [1,I] \cap \mathbb{N}$ and $x \in (A^\tau B^{(i)})^\complement$,

$$\sigma_*^\theta(a_i)(x) \lesssim \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}(\chi_{B^{(i)}})(x)\right]^{\beta \ln \lambda_- / \ln b} \tag{7.10}$$

with β as in (4.4). Then, from (7.5), (7.10), Lemmas 3.4 and 4.4, it follows that

$$\begin{aligned}
\mathrm{I}_2 &\lesssim \left\| \sum_{i=1}^{I} |\lambda_i| \, \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \left[M_{\mathrm{HL}}(\chi_{B^{(i)}})\right]^{\beta \ln \lambda_- / \ln b} \chi_{(A^\tau B^{(i)})^\complement} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \left\| \sum_{i=1}^{I} \left[|\lambda_i|^{\ln b/(\beta \ln \lambda_-)} \, \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\ln b/(\beta \ln \lambda_-)} M_{\mathrm{HL}}(\chi_{B^{(i)}}) \right]^{\beta \ln \lambda_- / \ln b} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\
&\sim \left\| \left\{ \sum_{i=1}^{I} \left[|\lambda_i|^{\ln b/(\beta \ln \lambda_-)} \, \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-\ln b/(\beta \ln \lambda_-)} M_{\mathrm{HL}}(\chi_{B^{(i)}}) \right]^{\beta \ln \lambda_- / \ln b} \right\}^{\frac{\ln b}{\beta \ln \lambda_-}} \right\|_{L^{p(\cdot)\beta \frac{\ln \lambda_-}{\ln b}}(\mathbb{R}^n)}^{\beta \frac{\ln \lambda_-}{\ln b}} \\
&\lesssim \left\| \left[\sum_{i=1}^{I} |\lambda_i| \, \|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{-1} \chi_{B^{(i)}} \right]^{\ln b/(\beta \ln \lambda_-)} \right\|_{L^{p(\cdot)\beta \ln \lambda_- / \ln b}(\mathbb{R}^n)}^{\beta \ln \lambda_- / \ln b} \\
&\lesssim \left\| \left\{ \sum_{i=1}^{I} \left[\frac{|\lambda_i| \chi_{B^{(i)}}}{\|\chi_{B^{(i)}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \sim \|f\|_{H^{p(\cdot),q,s}_{A,\mathrm{fin}}(\mathbb{R}^n)},
\end{aligned}$$

which, combined with (7.8) and (7.9), further implies that (7.6) holds true. This finishes the proof of Theorem 7.4. □

Remark 7.5. If $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1, \infty)$, then $\ln b / \ln \lambda_- = n$ and Theorem 7.4 goes back to the classical result with $\beta \in (n, \infty)$ and $p \in (n/\beta, \infty)$ (see Weisz [72]). The classical result was proved in a special case, namely, for the Bochner-Riesz means, in Stein et al. [62] and Lu [49]. For the same case, a counterexample was also given in [62] to show that the same conclusion is not true for $p \in (0, n/\beta]$.

The following Corollaries 7.6 and 7.7 can be deduced from Theorem 7.4 and an argument same as that used in the proofs of [44, Corollaries 2.19 and 2.20], respectively, the details being omitted.

Corollary 7.6. *With the same assumptions as in Theorem* 7.4, *if* $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$, *then* $\sigma_m^\theta f$ *converges almost everywhere as well as in the* $L^{p(\cdot)}(\mathbb{R}^n)$*-norm as* $m \to \infty$.

Corollary 7.7. *With the same assumptions as in Theorem* 7.4, *if* $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$ *and there exists a subset* $I \subset \mathbb{R}^n$ *such that the restriction* $f|_I \in L^{r(\cdot)}(I)$ *with* $r_- \in [1, \infty)$, *then*

$$\lim_{m\to\infty} \sigma_m^\theta f(x) = f(x) \quad \textit{for almost every } x \in I \textit{ as well as in the } L^{p(\cdot)}(I) \textit{ quasi-norm.}$$

Notice that, if $p_- \in (1, \infty)$, then $H_A^{p(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ with equivalent quasi-norms (see [83, Corollary 4.20]). Thus, Corollary 7.7 further implies the following result, the details being omitted.

Corollary 7.8. *Besides the same assumptions as in Theorem* 7.4*, suppose that* $p_- \in (1,\infty)$ *and* $f \in L^{p(\cdot)}(\mathbb{R}^n)$*. Then*

$$\lim_{m\to\infty} \sigma_m^\theta f(x) = f(x) \quad \textit{for almost every } x \in \mathbb{R}^n \textit{ as well as in the } L^{p(\cdot)}(\mathbb{R}^n) \textit{ quasi-norm.}$$

Remark 7.9. Corollary 7.8 for the Bochner-Riesz means in the classical case (namely, when $p(\cdot) =$ a constant $\in (0,\infty)$ and $A := d\,\mathrm{I}_{n\times n}$ for some $d \in \mathbb{R}$ with $|d| \in (1,\infty)$) can be found in Stein et al. [62] as well as Lu [49] and Weisz [72].

As special cases, we next consider two summability methods. For any $\alpha \in (0,\infty)$ and $\gamma \in \mathbb{N}$, the *Bochner-Riesz summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := \begin{cases} (1-|t|^\gamma)^\alpha & \text{when } |t| \in (1,\infty), \\ 0 & \text{when } |t| \in [0,1]. \end{cases} \tag{7.11}$$

The following conclusion follows from [44, Lemma 2.24] and Theorem 7.4, the details being omitted.

Theorem 7.10. *Let* θ_0 *be as in* (7.11) *and* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$*. If*

$$\alpha \in \left(\max\left\{\frac{n-1}{2}, \frac{\ln b}{\ln\lambda_-} - \frac{n+1}{2}\right\}, \infty\right) \quad \textit{and} \quad p_- \in \left(\frac{\ln b}{\ln\lambda_-(n/2+\alpha+1/2)}, \infty\right),$$

then there exists a positive constant $C_{(p_-,p_+)}$*, with* p_- *and* p_+ *as in* (2.4)*, such that, for any* $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$*,*

$$\left\|\sigma_*^\theta f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{(p_-,p_+)}\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

The *Weierstrass summation* is defined by setting, for any $t \in \mathbb{R}^n$,

$$\theta_0(t) := e^{-|t|^2/2}. \tag{7.12}$$

It is known that $\widehat{\theta_0}(x) = e^{-|x|^2/2}$ for any $x \in \mathbb{R}^n$. Then the following result follows from [44, Lemma 2.27] and Theorem 7.4, the details being omitted.

Theorem 7.11. *Let* θ_0 *be as in* (7.12)*. If* $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ *and* $p_- \in (0,\infty)$*, then there exists a positive constant* $C_{(p_-,p_+)}$*, with* p_- *and* p_+ *as in* (2.4)*, such that, for any* $f \in H_A^{p(\cdot)}(\mathbb{R}^n)$*,*

$$\left\|\sigma_*^\theta f\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C_{(p_-,p_+)}\|f\|_{H_A^{p(\cdot)}(\mathbb{R}^n)}.$$

Remark 7.12. Let θ_0 be as in (7.11) or (7.12). Then the corresponding conclusions in Corollaries 7.6 through 7.8 hold true as well, the details being omitted.

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. **164** (2002), no. 3, 213–259.

[2] ______, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. **584** (2005), 117–148.

[3] V. Almeida, J. J. Betancor and L. Rodríguez-Mesa, *Anisotropic Hardy-Lorentz spaces with variable exponents*, Canad. J. Math. **69** (2017), no. 6, 1219–1273.

[4] A. Almeida, P. Harjulehto, P. Höstä and T. Lukkari, *Riesz and Wolff potentials and elliptic equations in variable exponent weak Lebesgue spaces*, Ann. Mat. Pura Appl. (4) **194** (2015), no. 2, 405–424.

[5] A. Almeida and P. Hästö, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258** (2010), no. 5, 1628–1655.

[6] M. Bownik, *Anisotropic Hardy spaces and wavelets*, Mem. Amer. Math. Soc. **164** (2003), no. 781, vi+122 pp.

[7] M. Bownik, B. Li, D. Yang and Y. Zhou, *Weighted anisotropic Hardy spaces and their applications in boundedness of sublinear operators*, Indiana Univ. Math. J. **57** (2008), no. 7, 3065–3100.

[8] ______, *Weighted anisotropic product Hardy spaces and boundedness of sublinear operators*, Math. Nachr. **283** (2010), no. 3, 392–442.

[9] D. Breit, L. Diening and S. Schwarzacher, *Finite element approximation of the $p(\cdot)$-Laplacian*, SIAM J. Numer. Anal. **53** (2015), no. 1, 551–572.

[10] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation, Volume 1: One-dimensional theory*, Pure and Applied Mathematics **40**, Academic Press, New York-London, 1971.

[11] A.-P. Calderón, *An atomic decomposition of distributions in parabolic H^p spaces*, Advances in Math. **25** (1977), no. 3, 216–225.

[12] A.-P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*, Advances in Math. **16** (1975), 1–64.

[13] ______, *Parabolic maximal functions associated with a distribution II*, Advances in Math. **24** (1977), no. 2, 101–171.

[14] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), no. 4, 1383–1406.

[15] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes*, (French) Étude de certaines intégrales singulières, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-New York, 1971.

[16] ______, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), no. 4, 569–645.

[17] D. Cruz-Uribe, *The Hardy-Littlewood maximal operator on variable-L^p spaces*, in *Seminar of Mathematical Analysis* (Malaga/Seville, 2002/2003), 147–156, Colecc. Abierta **64**, Univ. Sevilla Secr. Publ., Seville, 2003.

[18] D. V. Cruz-Uribe and A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, Heidelberg, 2013.

[19] D. Cruz-Uribe and L.-A. D. Wang, *Variable Hardy spaces*, Indiana Univ. Math. J. **63** (2014), no. 2, 447–493.

[20] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl. **7** (2004), no. 2, 245–253.

[21] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics **2017**, Springer, Heidelberg, 2011.

[22] L. Diening, P. Hästö and S. Roudenko, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. **256** (2009), no. 6, 1731–1768.

[23] L. Diening and S. Schwarzacher, *Global gradient estimates for the $p(\cdot)$-Laplacian*, Nonlinear Anal. **106** (2014), 70–85.

[24] L. Ephremidze, V. Kokilashvili and S. Samko, *Fractional, maximal and singular operators in variable exponent Lorentz spaces*, Fract. Calc. Appl. Anal. **11** (2008), no. 4, 407–420.

[25] X. Fan, J. He, B. Li and D. Yang, *Real-variable characterizations of anisotropic product Musielak-Orlicz Hardy spaces*, Sci. China Math. **60** (2017), no. 11, 2093–2154.

[26] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263** (2001), no. 2, 424–446.

[27] H. G. Feichtinger and F. Weisz, *Wiener amalgams and pointwise summability of Fourier transforms and Fourier series*, Math. Proc. Cambridge Philos. Soc. **140** (2006), no. 3, 509–536.

[28] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes **28**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[29] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Upper Saddle River, NJ, 2004.

[30] L. Grafakos, L. Liu and D. Yang, *Maximal function characterizations of Hardy spaces on RD-spaces and their applications*, Sci. China Ser. A **51** (2008), no. 12, 2253–2284.

[31] Y. Han, D. Müller and D. Yang, *Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type*, Math. Nachr. **279** (2006), no. 13-14, 1505–1537.

[32] ———, *A theory of Besov and Triebel-Lizorkin spaces on metric measure spaces modeled on Carnot-Carathéodory spaces*, Abstr. Appl. Anal. **2008**, Art. ID 893409, 250 pp.

[33] P. Harjulehto, P. Hästö, V. Latvala and O. Toivanen, *Critical variable exponent functionals in image restoration*, Appl. Math. Lett. **26** (2013), no. 1, 56–60.

[34] Y. Jiao, Y. Zuo, D. Zhou and L. Wu, *Variable Hardy-Lorentz spaces* $H^{p(\cdot),q}(\mathbb{R}^n)$, Submitted.

[35] H. Kempka and J. Vybíral, *Lorentz spaces with variable exponents*, Math. Nachr. **287** (2014), no. 8-9, 938–954.

[36] O. Kováčik and J. Rákosník, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. **41** (116) (1991), no. 4, 592–618.

[37] L. D. Ky, *New Hardy spaces of Musielak-Orlicz type and boundedness of sublinear operators*, Integral Equations Operator Theory **78** (2014), no. 1, 115–150.

[38] B. Li, M. Bownik and D. Yang, *Littlewood-Paley characterization and duality of weighted anisotropic product Hardy spaces*, J. Funct. Anal. **266** (2014), no. 5, 2611–2661.

[39] B. D. Li, X. Y. Fan, Z. W. Fu and D. C. Yang, *Molecular characterization of anisotropic Musielak-Orlicz Hardy spaces and their applications*, Acta Math. Sin. (Engl. Ser.) **32** (2016), no. 11, 1391–1414.

[40] B. Li, X. Fan and D. Yang, *Littlewood-Paley characterizations of anisotropic Hardy spaces of Musielak-Orlicz type*, Taiwanese J. Math. **19** (2015), no. 1, 279–314.

[41] B. Li, D. Yang and W. Yuan, *Anisotropic Hardy spaces of Musielak-Orlicz type with applications to boundedness of sublinear operators*, The Scientific World Journal **2014** (2014), Article ID 306214, 19 pages.

[42] Y. Liang, J. Huang and D. Yang, *New real-variable characterizations of Musielak-Orlicz Hardy spaces*, J. Math. Anal. Appl. **395** (2012), no. 1, 413–428.

[43] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, *New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets*, J. Fourier Anal. Appl. **18** (2012), no. 5, 1067–1111.

[44] J. Liu, F. Weisz, D. Yang and W. Yuan, *Littlewood-Paley and finite atomic characterizations of anisotropic variable Hardy-Lorentz spaces and their applications*, Submitted.

[45] J. Liu, D. Yang and W. Yuan, *Anisotropic Hardy-Lorentz spaces and their applications*, Sci. China Math. **59** (2016), no. 9, 1669–1720.

[46] ———, *Anisotropic variable Hardy-Lorentz spaces and their real interpolation*, J. Math. Anal. Appl. **456** (2017), no. 1, 356–393.

[47] ———, *Littlewood-Paley characterizations of anisotropic Hardy-Lorentz spaces*, Acta Math. Sci. Ser. B Engl. Ed. (to appear) or arXiv:1601.05242.

[48] ———, *Littlewood-Paley characterizations of weighted anisotropic Triebel-Lizorkin spaces via averages on balls*, Submitted.

[49] S. Z. Lu, *Four Lectures on Real H^p Spaces*, World Scientific, River Edge, NJ, 1995.

[50] S. Meda, P. Sjögren and M. Vallarino, *On the H^1-L^1 boundedness of operators*, Proc. Amer. Math. Soc. **136** (2008), no. 8, 2921–2931.

[51] J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics **1034**, Springer-Verlag, Berlin, 1983.

[52] E. Nakai and Y. Sawano, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. **262** (2012), no. 9, 3665–3748.

[53] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Maruzen, Tokyo, 1950.

[54] ———, *Topology of Linear Topological Spaces*, Maruzen, Tokyo, 1951.

[55] T. Noi and Y. Sawano, *Complex interpolation of Besov spaces and Triebel-Lizorkin spaces with variable exponents*, J. Math. Anal. Appl. **387** (2012), no. 2, 676–690.

[56] W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), no. 1, 200–211.

[57] M. Růžička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Lecture Notes in Mathematics **1748**, Springer-Verlag, Berlin, 2000.

[58] S. Sato, *Estimates for singular integrals on homogeneous groups*, J. Math. Anal. Appl. **400** (2013), no. 2, 311–330.

[59] ______, *Characterization of parabolic Hardy spaces by Littlewood-Paley functions*, arXiv:1607.03645.

[60] Y. Sawano, *Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators*, Integral Equations Operator Theory **77** (2013), no. 1, 123–148.

[61] H.-J. Schmeisser and H. Triebel, *Topics in Fourier Analysis and Function Spaces*, A Wiley-Interscience Publication, John Wiley & Sons, Chichester, 1987.

[62] E. M. Stein, M. H. Taibleson and G. Weiss, *Weak type estimates for maximal operators on certain H^p classes*, Proceedings of the Seminar on Harmonic Analysis (Pisa, 1980), Rend. Circ. Mat. Palermo (2) 1981, suppl. 1, 81–97.

[63] L. Tang, *$L^{p(\cdot),\lambda(\cdot)}$ regularity for fully nonlinear elliptic equations*, Nonlinear Anal. **149** (2017), 117–129.

[64] J. Tiirola, *Image decompositions using spaces of variable smoothness and integrability*, SIAM J. Imaging Sci. **7** (2014), no. 3, 1558–1587.

[65] H. Triebel, *Theory of Function Spaces III*, Monographs in Mathematics **100**, Birkhäuser Verlag, Basel, 2006.

[66] ______, *Tempered Homogeneous Function Spaces*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2015.

[67] R. M. Trigub and E. S. Bellinsky, *Fourier Analysis and Approximation of Functions*, Kluwer Academic Publishers, Dordrecht, 2004.

[68] T. Ullrich, *Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpretations as coorbits*, J. Funct. Spaces Appl. **2012**, Art. ID 163213, 47 pp.

[69] J. Vybíral, *Sobolev and Jawerth embeddings for spaces with variable smoothness and integrability*, Ann. Acad. Sci. Fenn. Math. **34** (2009), no. 2, 529–544.

[70] F. Weisz, *Summability of Multi-dimensional Fourier Series and Hardy Spaces*, Mathematics and its Applications **541**, Kluwer Academic Publishers, Dordrecht, 2002.

[71] ———, *Summability of multi-dimensional trigonometric Fourier series*, Surv. Approx. Theory **7** (2012), 1–179.

[72] ———, *Convergence and Summability of Fourier Transforms and Hardy Spaces*, Birkhäuser, Basel, 2017.

[73] J. Xu, *Variable Besov and Triebel-Lizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 2, 511–522.

[74] X. Yan, D. Yang, W. Yuan and C. Zhuo, *Variable weak Hardy spaces and their applications*, J. Funct. Anal. **271** (2016), no. 10, 2822–2887.

[75] D. Yang, Y. Liang and L. D. Ky, *Real-variable Theory of Musielak-Orlicz Hardy Spaces*, Lecture Notes in Mathematics **2182**, Springer, Cham, 2017.

[76] D. Yang, D. Yang and G. Hu, *The Hardy Space H^1 with Non-doubling Measures and Their Applications*, Lecture Notes in Mathematics **2084**, Springer, Cham, 2013.

[77] D. Yang, W. Yuan and C. Zhuo, *Musielak-Orlicz Besov-type and Triebel-Lizorkin-type spaces*, Rev. Mat. Complut. **27** (2014), no. 1, 93–157.

[78] D. Yang and Y. Zhou, *Boundedness of sublinear operators in Hardy spaces on RD-spaces via atoms*, J. Math. Anal. Appl. **339** (2008), no. 1, 622–635.

[79] ———, *A boundedness criterion via atoms for linear operators in Hardy spaces*, Constr. Approx. **29** (2009), no. 2, 207–218.

[80] ———, *New properties of Besov and Triebel-Lizorkin spaces on RD-spaces*, Manuscripta Math. **134** (2011), no. 1-2, 59–90.

[81] D. Yang, C. Zhuo and E. Nakai, *Characterizations of variable exponent Hardy spaces via Riesz transforms*, Rev. Mat. Complut. **29** (2016), no. 2, 245–270.

[82] D. Yang, C. Zhuo and W. Yuan, *Besov-type spaces with variable smoothness and integrability*, J. Funct. Anal. **269** (2015), no. 6, 1840–1898.

[83] C. Zhuo, Y. Sawano and D. Yang, *Hardy spaces with variable exponents on RD-spaces and applications*, Dissertationes Math. (Rozprawy Mat.) **520** (2016), 1–74.

[84] C. Zhuo, D. Yang and Y. Liang, *Intrinsic square function characterizations of Hardy spaces with variable exponents*, Bull. Malays. Math. Sci. Soc. **39** (2016), no. 4, 1541–1577.

[85] C. Zhuo, D. Yang and W. Yuan, *Interpolation between $H^{p(\cdot)}(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$: real method*, J. Geom. Anal. (2017) or arXiv:1703.05527.

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