LAPLACE TRANSFORMS AND THE AMERICAN STRADDLE

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We address the pricing of American straddle options. We use partial Laplace transform techniques due to Evans et al. (1950) to derive a pair of integral equations giving the locations of the optimal exercise boundaries for an American straddle option with a constant dividend yield.

1. Introduction and analysis

Options are derivative financial instruments which give the holder certain rights. A call option carries the right (but not the obligation) to buy an underlying security at some predetermined price, while a put allows the holder to sell the underlying security. The value $V(S,t)$ of many options can be found using the Black-Scholes partial differential equation (PDE) (see, e.g., [6]),

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r - D_0)S\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

together with appropriate boundary conditions, where S is the price of the underlying security and $t < T$ is the time, with T being the expiry time. The parameters in the above equation are the risk-free rate, r, the dividend yield, D_0, and the volatility, σ; all of them are assumed constant. In addition, we assume that $r > D_0 > 0$.

If an option is European, it can only be exercised at the expiration date. If an option is American, it can be exercised at or before expiry, and a rational investor will exercise the option early if it is to his advantage. There are therefore regions where it is optimal to hold the option

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and others where exercise is optimal, and the need to find the boundary between these regions means that American options are more challenging mathematically than their European counterparts. Indeed, apart from one or two very special cases, closed form solutions have yet to be found for most American options, whereas for European options, solutions can usually be found using error functions or equivalently the cumulative distribution function for the normal distribution. Numerical methods and approximations can however be used to value American options.

In this study, we consider an American straddle, which in this context gives us the right, but not the obligation, to either buy or sell (but not both) an underlying stock at a predetermined price at or before expiry. Thus we have both a put and a call with the same expiry and strike price, but we are allowed to use only one of them. For a European straddle, where exercise is only allowed at expiry, this limitation does not constitute a problem, and a European straddle is worth exactly the same as a European put and call combined. It is important to note that a call and a put with the same exercise price cannot be simultaneously in the money, so for a European straddle when exercise is permitted only at expiry, the option which is currently in the money will be exercised. For an American straddle, by contrast, when early exercise is permitted, it is perfectly possible that the price of the underlying stock moves in such a way that sometimes the call is in the money while at others the put is in the money; and an investor holding a separate call and put would be able to exercise both at different times while an investor holding a straddle can only exercise one of the two, and would therefore have a lower expected return. Because of this limitation, the option value is not simply the sum of the values of a call and a put option. Such an option might be useful if an investor expects a large change in the value of the underlying stock that makes a significant move, but is unsure in terms of the direction of the change, which, as an example, might occur if a company were involved in a major lawsuit or when a major bank or corporation is about to fail. This problem involves two free boundaries: if the option price is sufficiently high, $S \geq S_f^+(t)$, then the holder will exercise the call, while if it is sufficiently low, $S \leq S_f^-(t)$, then the holder will exercise the put, and between these two boundaries, $S_f^-(t) \leq S \leq S_f^+(t)$, the holder would retain the option for the time being. We will tackle this problem using a modified Laplace transform, and the end result of our study is not an exact solution (very few of which exists for American options), but rather a pair of integral equations for the location of the optimal exercise boundaries. Previously, in [2], we looked at the corresponding problems for the call and put options, and derived in each case an integral equation with a general form similar to those found here.

The starting point of our analysis is the Black-Scholes PDE (1.1), together with the pay-off at expiry,

$$V(S,T) = \max(S-E, E-S). \tag{1.2}$$

For the European straddle, the PDE (1.1) can be solved fairly easily. For an American option, we have also the constraint that the price of the option cannot fall below the pay-off from immediate exercise,

$$V(S,t) \geq \max(S-E, E-S), \tag{1.3}$$

with the PDE (1.1) being valid only where $V(S,t) > \max(S-E, E-S)$. There is of course a region in which it is optimal to hold the option to expiry rather than to exercise it, and the boundary of this region is known as the optimal exercise boundary. For this particular problem, there are in fact both an upper boundary $S = S_f^+(t)$ and a lower boundary $S = S_f^-(t)$. In the present analysis, it is convenient to invert these relations, and write instead a single relation, $t = T_f(S)$. We will use a modified Laplace transform to arrive at an integral equation giving the location of this free boundary. Integral equation methods have been used to tackle American options before, including the early works [3, 5] on calls and the recent paper by Kuske and Keller [1] on the put, as well as our own previous work on the put and call [2]. We discuss the differences between those studies and our own in Section 2.

Several properties of the free boundaries are known (e.g., [6]). Firstly, we know that the value of the option and its derivative with respect to S must be continuous across the free boundaries, so that $V = S_f^+(t) - E$ and $(\partial V/\partial S) = 1$ at S_f^+, and $V = E - S_f^-(t)$ and $(\partial V/\partial S) = -1$ at S_f^-. Continuity of these maximizes the value of an American option. The value of the option must be continuous, as if it were greater than the return from immediate exercise the holder would not exercise, and if it were less than that, it would result in an arbitrage opportunity, in that an investor could buy an option and immediately exercise it for a risk-free profit. Similarly, if the delta of the option at the free boundary were greater than the delta of the pay-off, delaying exercise would lead to a higher expected return, while if the delta of the option was less than the delta of the pay-off, exercising earlier would increase the expected return. Secondly, if we evaluate $(\partial V/\partial t)$ right at expiry using (1.1), we can deduce that $S_f^+(T) = S_0 = Er/D_0 > E$ and $S_f^-(T) = E$. In the unusual event that $D_0 > r$, the two locations are reversed. In addition, we know that S_f^+ moves upwards and S_f^- downwards as we move away from expiry. Hence we can deduce that $T_f(S) = T$ for $E \leq S \leq S_0$. Thirdly, we know the position of the boundaries as $T-t \to \infty$. If we consider the perpetual American straddle (which never expires and therefore has no time dependence), the value of this option is $V = AS^{\alpha^+} + BS^{\alpha^-}$ [6], where

$$\alpha^{\pm} = \frac{1}{2\sigma^2}\left[\sigma^2 - 2(r - D_0) \pm \sqrt{4(r - D_0)^2 + 4\sigma^2(r + D_0) + \sigma^4}\right]. \tag{1.4}$$

If we denote the upper and lower boundaries for this perpetual option by S_*^+ and S_*^-, then we require that $V = S_*^+ - E$ and $(\partial V/\partial S) = 1$ at $S = S_*^+$, while $V = E - S_*^-$ and $(\partial V/\partial S) = -1$ at S_*^-. This yields four nonlinear equations, from which we find that the ratio $R = S_*^+/S_*^-$ obeys the equation

$$\alpha^+(\alpha^- - 1)(R^{\alpha^-} + 1)(R^{\alpha^+} + R) = \alpha^-(\alpha^+ - 1)(R^{\alpha^+} + 1)(R^{\alpha^-} + R), \tag{1.5}$$

with $S_*^+ = E\alpha^-(R^{\alpha^+} + 1)/[(\alpha^- - 1)(R^{\alpha^+} + R)]$. In our terms, we require that $T_f(S) \to -\infty$ as $S \to S_*^+$ from below and as $S \to S_*^-$ from above. The upper optimal exercise boundary will lie between the limits, $S_0 \le S_f^+(t) \le S_*^+$, while the lower one will lie between the limits $S_*^- \le S_f^-(t) \le E$.

Having formulated the problem, we now attempt to solve it using a Laplace transform in time. This technique is known to work well with European options, but with American options, one perceived difficulty has been that the Black-Scholes PDE only holds where it is optimal to retain the option. Because of this, we modify the usual definition

$$\mathcal{L}(G)(p) = \int_0^\infty g(t)e^{-pt}\,dt \tag{1.6}$$

somewhat, and define our version as follows for $S_*^- \le S \le S_*^+$:

$$\mathcal{V}(S,p) = \int_{-\infty}^{T_f(S)} V(S,t)e^{pt}\,dt, \tag{1.7}$$

so that the sign of t is reversed from the usual definition, and also the upper limit is $t = T_f(S)$ rather than $t = 0$. This is of course equivalent to setting $V(S,t) = 0$ in the region where it is not optimal to hold. Because of this definition, the price of the option $V(S,t)$ will obey the Black-Scholes equation everywhere where we integrate. We require the real part of p to be positive, that is, $\Re(p) > 0$, for the integral in (1.7) to converge. We know from the definition that $\mathcal{V}(S,p) \to 0$ as $S \to S_*^{\pm}$. We also know that as $p \to \infty$, we have $\mathcal{V}(S,p) \to 0$ and $p\mathcal{V}$ bounded, and in this limit, we can show that

$$\lim_{p\to\infty} p\mathcal{V} = \lim_{p\to\infty} V\left(T_f(S),S\right)e^{pT_f(S)}. \tag{1.8}$$

We can also define an inverse transform

$$V(S,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathcal{V}(S,p)e^{pt}\,dp. \tag{1.9}$$

Given our definition of the forward transform, this inverse is only meaningful where it is optimal to hold the option. In the above, we have adopted the convention that $T_f(S)$ is the location of the free boundary for $S_*^- < S < E$ and $S_0 < S < S_*^+$, while for $E < S < S_0$, we set $T_f = T$ since there it is optimal to hold the option to expiry.

Transform methods in general can be useful when dealing with linear partial differential equations such as (1.1), because they can be used to reduce the dimension of the problem. The appropriate transform to use will obviously depend both on the form of the equation and the geometry of the domain, and for (1.1) it is well known that taking a Laplace transform in time of (1.1) will eliminate the temporal derivative, reducing the problem to an ordinary differential equation; this same technique is regularly used with the heat conduction equation into which the Black-Scholes equation can be transformed. In addition to our earlier work [2] (and that of Knessl (2001)) in applying Laplace transforms to American options, Laplace transforms have been used for path-dependent options before, though we believe that our earlier work was the first to consider an option problem with a free boundary. Geman and Yor (1996) used Laplace transforms to price barrier options, where there are fixed rather than free boundaries, and Geman and Yor (1993) used them to price Asian options, where the pay-off motivation for using Laplace transforms was that they reduced the dimension of the problem.

Applying this modified Laplace transform to the Black-Scholes PDE (1.1), we arrive at the following (nonhomogeneous Euler) ordinary differential equation ODE for the transform of the option price,

$$\left[\frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \left(r - D_0\right)S\frac{\partial}{\partial S} - (p+r)\right]\mathcal{V} + F(S) = 0, \tag{1.10}$$

where the nonhomogeneous term $F(S)$ takes a different value in each of the following regions:

Region (a)

$S_*^- < S < E$, where we have $V(S_f^-(t),t) = E - S_f^-$, $(\partial V/\partial S)(S_f^-(t),t) = -1$, and $T_f < T$, we have

$$F(S) = (E-S)e^{pT_f(S)} + \left[\sigma^2 S^2 - (r-D_0)S(E-S)\right]T_f'(S) \\ - \frac{\sigma^2 S^2}{2}(E-S)T_f''(S). \tag{1.11a}$$

Region (b)

$E < S < S_0$, where $T_f = T$ and

$$F(S) = (S-E)e^{pT}. \tag{1.11b}$$

Region (c)

$S_0 < S < S_*^+$, where $V(S_f^+(t),t) = S - E$, $(\partial V/\partial S)(S_f^+(t),t) = 1$, $T_f < T$, and

$$F(S) = (S-E)e^{pT_f(S)} + \left[\sigma^2 S^2 - (r-D_0)S(S-E)\right]T_f'(S) \\ - \frac{\sigma^2 S^2}{2}(S-E)T_f''(S). \tag{1.11c}$$

The general solution of (1.10) is

$$\mathcal{V} = \frac{2}{\lambda(p)}S^{(1/2\sigma^2)(2D_0-2r+\sigma^2+\lambda(p))}\left[C_+(p) - \int S^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2+\lambda(p))}F(S)\,dS\right] \\ + \frac{2}{\lambda(p)}S^{(1/2\sigma^2)(2D_0-2r+\sigma^2-\lambda(p))}\left[C_-(p) + \int S^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2-\lambda(p))}F(S)\,dS\right], \tag{1.12}$$

where $\lambda(p) = [4(r-D_0)^2 + 4\sigma^2(r+D_0+2p)+\sigma^4]^{1/2}$, and $C_\pm$ are the constants of integration, which may depend on the transform variable p. Applying this solution (1.12) to the three separate regions outlined above, we find that in region (a) in order to get a solution which vanishes as $S \to S_*^-$, we have

$$\mathcal{V}=\frac{2S^{-1}}{\lambda(p)}\int_{S_*^-}^{S}\left(\frac{\tilde{S}}{S}\right)^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2)}\left[\left(\frac{\tilde{S}}{S}\right)^{\lambda(p)/2\sigma^2}-\left(\frac{\tilde{S}}{S}\right)^{-\lambda(p)/2\sigma^2}\right]F(\tilde{S})\,d\tilde{S}, \tag{1.13}$$

and similarly in region (c) in order to get a solution which vanishes as $S\to S_*^+$, we have

$$\mathcal{V}=\frac{2S^{-1}}{\lambda(p)}\int_{S}^{S_*^+}\left(\frac{\tilde{S}}{S}\right)^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2)}\left[\left(\frac{\tilde{S}}{S}\right)^{-\lambda(p)/2\sigma^2}-\left(\frac{\tilde{S}}{S}\right)^{\lambda(p)/2\sigma^2}\right]F(\tilde{S})\,d\tilde{S}, \tag{1.14}$$

while in region (b), we have

$$\begin{aligned}\mathcal{V}=&\frac{2}{\lambda(p)}S^{(1/2\sigma^2)(2D_0-2r+\sigma^2+\lambda(p))}\left[C_+^{(b)}(p)-\int_E^S\tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2+\lambda(p))}F(\tilde{S})\,d\tilde{S}\right]\\&+\frac{2}{\lambda(p)}S^{(1/2\sigma^2)(2D_0-2r+\sigma^2-\lambda(p))}\\&\times\left[C_-^{(b)}(p)+\int_E^S\tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2-\lambda(p))}F(\tilde{S})\,d\tilde{S}\right].\end{aligned} \tag{1.15}$$

We require the transform $\mathcal{V}$ and its derivative with respect to S to be continuous at $S=E$ as we move from region (a) to (b), and also at S_0, as we move from (b) to (c), which tells us that

$$\begin{aligned}C_\pm^{(b)}(p)&=\mp\int_{S_*^-}^{E}\tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2\pm\lambda(p))}F(\tilde{S})\,d\tilde{S}\\&=\pm\int_{S_0}^{S_*^+}\tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2\pm\lambda(p))}F(\tilde{S})\,d\tilde{S}\\&\quad\pm2\sigma^2e^{pT}S_0^{-(1/2\sigma^2)(2D_0-2r+\sigma^2\pm\lambda(p))}\\&\quad\times\left[\frac{E}{2D_0-2r+\sigma^2\pm\lambda(p)}-\frac{S_0}{2D_0-2r-\sigma^2\pm\lambda(p)}\right]\\&\quad\pm2\sigma^2e^{pT}E^{-(1/2\sigma^2)(2D_0-2r-\sigma^2\pm\lambda(p))}\\&\quad\times\left[\frac{1}{2D_0-2r-\sigma^2\pm\lambda(p)}-\frac{1}{2D_0-2r+\sigma^2\pm\lambda(p)}\right].\end{aligned} \tag{1.16}$$

Comparing these two pairs of expressions, we require that

$$\begin{aligned}
&\int_{S_*^-}^{E} \tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2\pm\lambda(p))} F(\tilde{S})\,d\tilde{S} + \int_{S_0}^{S_*^+} \tilde{S}^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2\pm\lambda(p))} F(\tilde{S})\,d\tilde{S} \\
&\qquad = -2\sigma^2 e^{pT} S_0^{-(1/2\sigma^2)(2D_0-2r+\sigma^2\pm\lambda(p))} \\
&\qquad\quad \times \left[\frac{E}{2D_0-2r+\sigma^2\pm\lambda(p)} - \frac{S_0}{2D_0-2r-\sigma^2\pm\lambda(p)}\right] \\
&\qquad\quad \pm 2\sigma^2 e^{pT} E^{-(1/2\sigma^2)(2D_0-2r-\sigma^2\pm\lambda(p))} \\
&\qquad\quad \times \left[\frac{1}{2D_0-2r-\sigma^2\pm\lambda(p)} - \frac{1}{2D_0-2r+\sigma^2\pm\lambda(p)}\right].
\end{aligned} \tag{1.17}$$

The reader's attention is drawn to the fact that there is a "±" in front of $\lambda(p)$ in the exponent of $\tilde{S}$, so that (1.17) is actually a pair of equations, one for either sign.

2. Discussion

This last pair of (1.17) is the main result of this paper. They constitute integral equations for the location of the free boundary, $T_f(S)$, or more specifically, Urysohn equations of the first kind [4]. Since these equations involve the variable p, and must be true for each value of p for which $\Re(p) > 0$, we can think of them as a form of integral transform operating on $T_f(S)$, and inverting this transform would give $T_f(S)$. However, this inversion would appear to be extremely difficult to do analytically because of the term involving $e^{pT_f(S)}$ in $F(S)$ as given in (1.11a), (1.11b), and (1.11c); if this term were absent, we could regard the equations as a form of (finite) Mellin transform. In theory, (1.17) could be solved numerically, but that is outside the range of expertise of the present authors.

As we mentioned briefly in Section 1, other authors have previously used integral equation methods to analyze American options, including the studies [1, 3, 5]. However, those studies tackled the problem in very different ways to that used here, and ended up with equations of a somewhat different form to (1.17). For example, in their recent study, Kuske and Keller [1] used Green's functions to solve the Black-Scholes PDE for the American put, and their result involved an integral equation for $S_f(t)$, whereas we have an integral equation for the inverse of that function, $T_f(S)$. As is the case here, those authors were unable to obtain exact solutions of their integral equations. Studies similar to the present have been performed for both the American put and call [2]; each of these problems involved a single free boundary, and in each case the end result was a single integral equation of the same general form as those found here.

Moving on to the issue of the value of the option, in (1.13), (1.14), and (1.15), we have a series of expressions for $\mathcal{V}(p,S)$, the transform of the option price $V(S,t)$. In theory, given these expressions, we could apply the inverse transform (1.10), and then we would arrive at the option price itself. Unfortunately, these expressions involve $T_f(S)$, the location of the free boundary, which we know only abstractly as the solution of the integral equations (1.17); however, if $T_f(S)$ were known explicitly, taking the inverse Laplace transform would give the value of the option.

References

[1] R. E. Kuske and J. B. Keller, *Optimal exercise boundary for an American put option*, Appl. Math. Fin. **5** (1998), 107–116.

[2] R. Mallier and G. Alobaidi, *Laplace transforms and American options*, Appl. Math. Fin. **7** (2000), no. 4, 241–256.

[3] H. P. McKean Jr., *Appendix: A free boundary problem for the heat equation arising from a problem in mathematical economics*, Industrial Management Review **6** (1965), 32–39.

[4] A. D. Polyanin and V. Manzhirov, *Handbook of Integral Equations*, CRC Press, New York, 1998.

[5] P. Van Moerbeke, *On optimal stopping and free boundary problems*, Arch. Rational Mech. Anal. **60** (1975/76), no. 2, 101–148.

[6] P. Wilmott, *Derivatives: The Theory and Practice of Financial Engineering*, Wiley University Edition, Chichester, 1998.

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