EXISTENCE OF POSITIVE ENTIRE SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS ON $\mathbf{R^N}$

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1. Introduction

In the present paper we are concerned with positive solutions of the following problem:

$$(P) \qquad \begin{cases} -\Delta u + u = g(x,u), \quad x \in R^N, \\ u \in H^1(R^N), \quad N \geq 3, \end{cases}$$

where $g : R^N \times R \to R$ is a continuous mapping. Recently, the existence of positive solutions of the semilinear elliptic problem

$$(P_Q) \qquad \begin{cases} -\Delta u + u = Q(x) \mid u \mid^{p-1} u, \quad x \in R^N, \\ u \in H^1(R^N), \quad N \geq 2, \end{cases}$$

has been studied by several authors, where $1 < p$ for $N = 2$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$ and $Q(x)$ is a positive bounded continuous function. If $Q(x)$ is a radial function, we can find infinity many solutions of problem (P_Q) by restricting our attention to the radial functions (cf. [2, 5]). If $Q(x)$ is nonradial, we encounter a difficulty caused by the lack of a compact embedding of Sobolev type. To overcome this kind of difficulty, P. L. Lions developed the concentrate compactness method [8, 9], and established the following result: Assume that $lim_{|x|\to\infty} Q(x) = \overline{Q}(>0)$ and $Q(x) \geq \overline{Q}$ on R^N. Then the problem (P_Q) has a positive solution. This result is based on the observation that the ground state level c_Q of the functional

$$I_Q(u) = \frac{1}{2}\int_{R^N} (\mid \nabla u \mid^2 + \mid u \mid^2) dx - \frac{1}{p+1}\int_{R^N} Q(x) \mid u \mid^{p+1} dx$$

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is lower than that of

$$I^\infty(u) = \frac{1}{2}\int_{R^N}(\mid \nabla u \mid^2 + \mid u \mid^2)dx - \frac{1}{p+1}\int_{R^N} \mid u \mid^{p+1} dx,$$

then, under additional conditions on g, there exists a positive solution of (P) (cf. Ding and Ni [5], Stuart [14]). In [3], Cao proved the existence of a positive solution of (P_Q) for the case $c_Q \le c_{\overline{Q}}$ under the hypothesis that $\lim_{|x|\to\infty} Q(x) = \overline{Q}$ and $Q(x) \ge 2^{(1-p)/2}\overline{Q}$ on R^N. The difficultly in treating the case $c_Q = c_{\overline{Q}}$ is caused by the fact that we can not apply the concentrate compactness method directly. The argument in [3] is based on Lagrange's method of indeterminate coefficients. That is, if we find a solution u of the minimizing problem

$$\inf\left\{\{I_Q(u) \ : \ u \in V_\lambda\},\right.$$

$$\left. V_\lambda = \left\{\{u \in H^1(R^N), u > 0, \ \int_{R^N} Q(x) \mid u \mid^{p+1} dx = 1\right\}\right\},$$

then cu is a solution of (P_Q) for some $c > 0$. Lagrange's method does not work if g is not the form $Q(x)t^p$. Our purpose in this paper is to consider the existence of a positive solution of (P) for g satisfying $\lim_{|x|\to\infty} g(x,t) = \mid t \mid^{p-1} t$. Our method employed here is based on the singular homology theory.

Throughout this paper, we assume that $g \in C^1(R) \cap C^2(R\backslash\{0\})$ and we impose the following conditions on g:

(g1) There exists a positive number $d < 1$ such that

$$-dt + (1-d) \mid t \mid^{p-1} t \le g(x,t) \le dt + (1+d) \mid t \mid^{p-1} t$$

for all $(x,t) \in R^N \times [0,\infty)$;

(g2) there exists a positive number C such that

$$\mid g_t(x,0) \mid < 1 \ \text{ and } \ 0 < t^3 g_{tt}(x,t) < C(1+ \mid t \mid^{p+1})$$

for all $(x,t) \in R^N \times (0,\infty)$;

(g3)

$$\lim_{|x|\to\infty} g(x,t) = \mid t \mid^{p-1} t$$

uniformly on bounded intervals in $[0,\infty)$,

where $1 < p < (N+2)/(N-2)$ and $g_t(\cdot,\cdot)$ stands for the derivative of g with respect to the second variable.

Remark 1. (1) Throughout the rest of this paper, we assume for the simplicity of the proofs that $g(x,-t) = -g(x,t)$ for $(x,t) \in R^N \times [0,\infty)$. Since we are concerned with positive solutions, this assumption does not effect our result. By this assumption, the functional I is even and if u is a critical point of I, $-u$ is also a critical point of I. (2) Functions of the form $g(x,t) = \Sigma_{i=1}^m q_i(x)t^i + q_p(x)t^p$ satisfy (g1) and (g2) if m is a positive integer with $m < p$, $q_i(x)$ $(1 \le i \le m)$ are sufficiently small and $| q_p(x) - 1 | < 1 + d$. (g3) is satisfied if $\lim_{|x|\to\infty} q_i(x) = 0$ for $1 \le i \le p-1$ and $\lim_{|x|\to\infty} q_p(x) = 1$.

Theorem. *Suppose that* $(g2)$ *and* $(g3)$ *hold. Then there exists* $d_0 > 0$ *such that if* $(g1)$ *holds with* $d < d_0$*, then the problem* (P) *has a positive solution.*

2. Preliminaries

Throughout the rest of this paper, we assume that (g2) and (g3) hold. We put $H = H^1(R^N)$. Then H is a Hilbert space with norm

$$\| u \| = \left(\int_{R^N} (| \nabla u |^2 + | u |^2) dx \right)^{1/2}.$$

The norm of the dual space $H^{-1}(R^N)$ of H is also denoted by $\| \cdot \|$. B_r stands for the open ball centered at 0 with radius r. For subsets A, B of H with $A \subset B$, we denote by $int_B A$ and $\partial_B A$ the relative interior of A in B and the relative boundary of A in B, respectively. For subsets A, B of H, we write $A \cong B$ when A and B have the same homotopy type. The norm and inner product of $L^2(R^N)$ are denoted by $| \cdot |_{L^2}$ and $\langle \cdot, \cdot \rangle$, respectively. For each $x \in R^N$ and $u \in H$, we set $\tau_x u = u(\cdot + x)$. For each functional F on H and $a \in R$, we set

$$F_a = \{u \in H \ : \ F(u) \le a\} \qquad \text{and} \qquad \dot{F}_a = \{u \in H \ : \ F(u) < a\}.$$

We put

$$M = \left\{ u \in H \setminus \{0\} \ : \| u \|^2 = \int_{R^N} u g(x,u) dx \right\},$$
$$M^\infty = \left\{ u \in H \setminus \{0\} \ : \| u \|^2 = \int_{R^N} u^{p+1} dx \right\}.$$

From the assumption (g2), we find that for each $u \in H \setminus \{0\}$,

$$\frac{dI(tu)}{dt}(0) = 0, \qquad \frac{d^2 I(tu)}{dt^2}(0) = | \nabla u |_{L^2}^2 + | u |_{L^2}^2 - \langle g_t(x,0)u, u \rangle > 0,$$

and

$$\frac{d^3 I(tu)}{dt^3}(t) = -\langle g_{tt}(x,tu)u^2, u \rangle < 0 \quad \text{for } t > 0. \tag{2.1}$$

Then, noting that $(dI(tu)/dt)(\lambda) = 0$ if $\lambda u \in M$, we can see that there exists a positive number $\lambda_0(u)$ such that $I_u = \{\lambda u \ : \ \lambda > 0\}$ intersects M at exactly one point $\lambda_0(u)u$. Similarly, we can define a positive number $\lambda_\infty(u)$ by $\lambda_\infty(u)u \in M^\infty$. For simplicity, we write $\lambda_0 u$ and $\lambda_\infty u$ instead of $\lambda_0(u)u$ and $\lambda_\infty(u)u$ respectively, when it is clear in the context what it means. It also follows from the definition of M^∞ that for each $u \in M^\infty$,

$$
\begin{aligned}
I^\infty(u) &= \frac{p-1}{2(p+1)} \int_{R^N} \left(| \nabla u |^2 + | u |^2\right) dx \\
&= \frac{p-1}{2(p+1)} \int_{R^N} | \nabla u |^{p+1} \, dx.
\end{aligned} \tag{2.2}
$$

It is known that there exists a positive radial solution u_∞ of problem

$$
(P_\infty) \qquad \begin{cases} -\Delta u + u = | u |^{p-1} u, \quad x \in R^N \\ u \in H^1(R^N), \end{cases}
$$

such that $c = I^\infty(u_\infty) = \min\{I^\infty(u) \ : \ u \in M^\infty\}$. In [6], Kwong showed that u_∞ is the unique positive solution up to the translation. It then follows as a direct consequence of the concentrate compactness lemma(cf. Lions [8]) that the second critical level of I^∞ is $2c$. That is,

Lemma 2.1. *For each* $0 < \epsilon < c$, $\inf\{\| \nabla I^\infty(u) \| : \ u \in I_{2c-\epsilon} \backslash \dot{I}_{c+\epsilon}\} > 0$.

We put $c_1 = \inf\{I(u) \ : \ u \in M\}$. It then follows from the definition of I and M that if $u \in M$ satisfies $c_1 = I(u)$, then u is a solution of (P). It also follows that u is positive. In fact, if $u^+ = \max\{u, 0\} \not\equiv 0$ and $u^- = -\min\{u, 0\} \not\equiv 0$, then $u^\pm \in M$ and therefore $I(u) = I(u^+) + I(u^-) \geq 2c_1$. This is a contradiction. Then to find a positive solution of problem (P), we will find a critical point of M with critical level c_1. We can see from (g3) that $\lim_{|x|\to\infty} I(u_\infty(\cdot + x)) = c$. Therefore we have that $c_1 \leq c$. Moreover we have

Proposition 2.2. *Suppose that* $(g1)$ *holds with* $d \leq \widetilde{d}_0$, *where* $\widetilde{d}_0$ *is a positive number such that*

$$
\delta = \inf\left\{\frac{1-d}{2} - \frac{(1+d)^2}{(1-d)(p+1)} : 0 \leq d \leq \widetilde{d}_0\right\} > 0.
$$

If $c_1 < c$, *then there exists a positive solution of problem* (P).

Proof. Let $u \in H$. Then by (g1), we have

$$
\begin{aligned}
I(u) &= \frac{1}{2} \int_{R^N} \left(| \nabla u |^2 + | u |^2\right) dx - \int_{R^N} \int_0^{u(x)} g(x, t) dt dx \\
&\geq \int_{R^N} \left(\frac{1}{2}\left(| \nabla u |^2 + (1-d) \, | u |^2\right) - \frac{1+d}{p+1} \, | u |^{p+1}\right) dx.
\end{aligned}
$$

Suppose that $u \in M$. Then again by (g1), we have

$$\| u \|^2 = \int_{R^N} u g(x,u) dx \geq \int_{R^N} \left(-d \mid u \mid^2 + (1-d) \mid u \mid^{p+1}\right) dx.$$

Combining the inequalities above, we have

$$\begin{aligned} I(u) \geq \int_{R^N} \Bigg(& \left(\frac{1}{2} - \frac{1+d}{(1-d)(p+1)}\right) \mid \nabla u \mid^2 \\ & + \left(\frac{1-d}{2} - \frac{(1+d)^2}{(1-d)(p+1)}\right) \mid u \mid^2 \Bigg) dx \\ \geq \delta & \int_{R^N} \left(\mid \nabla u \mid^2 + \mid u \mid^2\right) dx. \end{aligned} \tag{2.3}$$

Let $\{u_n\} \subset M$ be a sequence such that $\lim_{n\to\infty} I(u_n) = c_1$ and $\lim_{n\to\infty} \nabla I(u_n) = 0$. It then follows from (2.3) that $\{u_n\}$ is bounded in H. Then by a parallel argument as in the proof of theorem I.2 of Lions [9], we can see that $\{u_n\}$ converges to $u \in H$ and $\nabla I(u) = 0$ and this completes the proof. ∎

By Proposition 2.2, it is sufficient to consider the case that $c_1 = c$. In the sequel, we assume that $c_1 = c$. We prove Theorem by contradiction, that is, we assume in the following that the functional I does not have nontrivial critical points. Our purpose in the rest of this section is to prove the following Proposition.

Proposition 2.3. *There exists a positive number $d_0 < \widetilde{d}_0$ such that if* (*g*1) *holds with $d \leq d_0$, then for each $0 < \epsilon < c$,*

$$H_*(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = H_*(I_{c+\epsilon}, I_\epsilon)$$

where $H_*(A,B)$ denotes the singular homology group for a pair (A,B) of topological spaces (*cf. Spanier* [11]).

In the following we denote by $M^{0,\infty}$ and M_α $(\alpha > 0)$ the sets defined by

$$M^{0,\infty} = \{t\lambda_0 u + (1-t)\lambda_\infty u \;:\; u \in H\backslash\{0\}, t \in [0,1], \lambda_0 u \in M, \lambda_\infty u \in M^\infty\}$$

and

$$M_\alpha = \{(1+\tau)u \;:\; u \in M^{0,\infty}, \tau \in (-R(u), R(u))\} \tag{2.4}$$

where

$$\begin{aligned} R(u) = \sup \Bigg\{ t > 0 \;:\; & \max\left\{\frac{I(u)}{I((1+\tau)u)}, \frac{I^\infty((1+\tau)u)}{I^\infty(u)}\right\} < 1 + \alpha \\ & \text{for all } \tau \in [-t,t] \Bigg\}. \end{aligned} \tag{2.5}$$

From the definition, $M^\infty, M \subset M^{0,\infty}$ and M_α is an open neighborhood of $M^{0,\infty}$.

Lemma 2.4. *There exist positive numbers d_1 and α_0 such that if (g1) holds with $d \le d_1$, then for each positive number $\alpha < \alpha_0$,*

$$I^\infty_{(7/6)c} \subset I_{(4/3)c} \cup (M_\alpha)^c, \tag{1}$$

$$I_{(4/3)c} \subset I^\infty_{(5/3)c} \cup (M_\alpha)^c, \tag{2}$$

$$I^\infty_{(5/3)c} \subset I_{(11/6)c} \cup (M_\alpha)^c. \tag{3}$$

Proof. The assertions (1), (2) and (3) can be proved by parallel arguments. We give only the proof of (2). Let $d_1 > 0$ such that

$$\frac{4}{5} < \rho = \min\left\{ \left(\frac{(1-d)^2}{2(1+d)} - \frac{1+d}{p+1}\right)\frac{2(p+1)}{p-1},\right.$$
$$\left.\left(\frac{1-d}{1+d}\right)^{2/(p-1)}\left(\frac{2(p+1)}{p-1}\right)\left(\frac{1-d}{2} - \frac{(1+d)}{p+1}\right)\right\}, \text{ for } 0 \le d \le d_0.$$

We assume that (g1) holds with $d \le d_1$. Fix $u \in H\backslash\{0\}$. Then we have from the definitions of M and M^∞ that

$$\| \lambda_0 u \|^2 = \int_{R^N} \lambda_0 u g(x, \lambda_0 u) dx \text{ and } \| \lambda_\infty u \|^2 = \int_{R^N} | \lambda_\infty u |^{p+1} dx. \tag{2.6}$$

By (g1) and (2.6), we have

$$\begin{aligned}
\frac{1-d}{1+d}\int_{R^N} | \lambda_0 u |^{p+1} dx &\le \frac{1}{1+d}\int_{R^N} (\lambda_0 u g(x, \lambda_0 u) + d | \lambda_0 u |^2) dx \\
&\le \frac{1}{1+d}\int_{R^N} (| \nabla\lambda_0 u |^2 dx + (1+d) | \lambda_0 u |^2) dx \\
&\le \int_{R^N} (| \nabla\lambda_0 u |^2 dx + | \lambda_0 u |^2) dx \\
&\le \frac{1}{1-d}\int_{R^N} (| \nabla\lambda_0 u |^2 dx + (1-d) | \lambda_0 u |^2) dx \\
&\le \frac{1}{1-d}\int_{R^N} (\lambda_0 u g(x, \lambda_0 u) - d | \lambda_0 u |^2) dx \\
&\le \frac{1+d}{1-d}\int_{R^N} | \lambda_0 u |^{p+1} dx.
\end{aligned}$$

That is, we have

$$\begin{aligned}
\frac{1-d}{1+d}\int_{R^N} | \lambda_0 u |^{p+1} dx &\le \int_{R^N} (| \nabla\lambda_0 u |^2 dx + | \lambda_0 u |^2) dx \\
&\le \frac{1+d}{1-d}\int_{R^N} | \lambda_0 u |^{p+1} dx.
\end{aligned} \tag{2.7}$$

We find from the second equality of (2.6) and (2.7) that

$$\frac{1-d}{1+d}\lambda_0^{p-1} \leq \lambda_\infty^{p-1} \leq \frac{1+d}{1-d}\lambda_0^{p-1}. \tag{2.8}$$

To prove the assertion, we will show that for $0 < \alpha < \alpha_0$,

$$I_{(4/3)c} \cap M_\alpha \subset I^\infty_{(5/3)c}.$$

Now let $u \in M^{0,\infty}$. From the definition of $M^{0,\infty}$, we have that $\lambda_0 \leq 1 \leq \lambda_\infty$ or $\lambda_\infty \leq 1 \leq \lambda_0$ holds. We first consider the case that $\lambda_\infty \leq 1 \leq \lambda_0$. Since $\lambda_\infty \leq 1$, we have that

$$\| u \|^2 = \int_{R^N} (| \nabla u |^2 + | u |^2) dx \leq \int_{R^N} | u |^{p+1}\, dx.$$

Then we find that

$$I^\infty(u) \leq \frac{p-1}{2(p+1)} \int_{R^N} | u |^{p+1}\, dx. \tag{2.9}$$

On the other hand, recalling that the second equality of (2.6) holds, we obtain from (g1), (2.9) and (2.8) that

$$\begin{aligned} I(u) &\geq \frac{1-d}{2} \int_{R^N} (| \nabla u |^2 + | u |^2) dx - \frac{1+d}{p+1} \int_{R^N} | u |^{p+1}\, dx \\ &= \left(\frac{1-d}{2}\lambda_\infty^{p-1} - \frac{1+d}{p+1} \right) \int_{R^N} | u |^{p+1}\, dx \\ &\geq \left(\frac{(1-d)^2}{2(1+d)} - \frac{1+d}{p+1} \right) \frac{2(p+1)}{p-1} I^\infty(u) \\ &\geq \rho I^\infty(u). \end{aligned} \tag{2.10}$$

We choose a positive number $\alpha_1 < 1$ such that $4/5 < \rho/(1+\alpha_1)^2$. Now suppose that $(1+\tau)u \in M_{\alpha_1}$, $\tau \in R$. Then, by (2.10) , we have

$$\begin{aligned} I((1+\tau)u) \geq (1/(1+\alpha_1))I(u) &\geq (\rho/(1+\alpha_1))I^\infty(u) \\ &\geq (\rho/(1+\alpha_1)^2)I^\infty((1+\tau)u). \end{aligned}$$

Assume that $(1+\tau)u \in I_{(4/3)c}$. Then it follows from the inequalities above that

$$I^\infty((1+\tau)u) \leq (4/3)c(1+\alpha_1)^2/\rho \leq (5/3)c.$$

We next assume that $\lambda_0 \leq 1 \leq \lambda_\infty$. Then by (2.2),

$$I^\infty(u) \leq I^\infty(\lambda_\infty u) = \frac{p-1}{2(p+1)}\lambda_\infty^2 \int_{R^N} (| \nabla u |^2 + | u |^2) dx. \tag{2.11}$$

On the other hand, we have by (2.8) that

$$\lambda_\infty \le \left(\frac{1+d}{1-d}\right)^{1/(p-1)}.$$

Then, noting that $\lambda_\infty^{-(p-1)} \le 1$, we have from (g1) and (2.11) that

$$\begin{aligned} I(u) &\ge \frac{1-d}{2}\int_{R^N}(|\nabla u|^2 + |u|^2)dx - \frac{1+d}{p+1}\int_{R^N}|u|^{p+1}\,dx \\ &\ge \left(\frac{1-d}{2} - \frac{1+d}{p+1}\lambda_\infty^{-(p-1)}\right)\int_{R^N}(|\nabla u|^2 + |u|^2)dx \\ &= \lambda_\infty^{-2}\frac{2(p+1)}{p-1}\left(\frac{1-d}{2} - \lambda_\infty^{-(p-1)}\frac{(1+d)}{p+1}\right)I^\infty(u) \\ &\ge \left(\frac{1-d}{1+d}\right)^{2/(p-1)}\frac{2(p+1)}{p-1}\left(\frac{1-d}{2} - \frac{(1+d)}{p+1}\right)I^\infty(u) \\ &\ge \rho I^\infty(u). \end{aligned} \tag{2.12}$$

Then we have that there exists $\alpha_2 > 0$ such that for all $u \in M_{\alpha_2}$ with $I(u) \le (4/3)c$, $I^\infty(u) \le (5/3)c$. Thus we obtain that the assertion holds with $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. ∎

Throughout the rest of this section we fix the positive number $\alpha < \alpha_0$.

Lemma 2.5. *There exists a continuous mapping* $\gamma_1 : [0,1] \times (I_{(11/6)c} \cup M_\alpha^c) \to I_{(11/6)c} \cup M_\alpha^c$ *such hat*

(i) $\gamma_1(0,x) = x$ *for all* $x \in I_{(11/6)c} \cup M_\alpha^c$,

(ii) $\gamma_1(t,x) = x$ *for all* $(t,x) \in [0,1] \times (I_{(4/3)c} \cup M_\alpha^c)$,

(iii) $I(\gamma_1(t,x)) \le I(\gamma_1(0,x))$ *for all* $(t,x) \in [0,1] \times (I_{(11/6)c} \cup M_\alpha^c)$,

(iv) $\gamma_1(1, I_{(11/6)c} \cup M_\alpha^c) \subset I_{(4/3)c} \cup M_\alpha^c$.

Proof. We set

$$M_o = \{\lambda u \ : \ u \in M, \lambda > 1\} \qquad \text{and} \qquad M_i = \{\lambda u \ : \ u \in M, \lambda < 1\}.$$

Let U be an open set such that

$$(M_\alpha)^c \subset U \qquad \text{and} \qquad U \cap M_{\alpha/2} = \phi.$$

Then since $M \subset M_{\alpha/2}$, we can see that

$$\langle \nabla I(v), v\rangle > 0 \quad \text{on } M_i \cap U \qquad \text{and} \qquad \langle \nabla I(v), v\rangle < 0 \quad \text{on } M_o \cap U.$$

Then by arguing standard way (cf. Lemma 1.6 of Rabinowitz [10]), we can construct a pseudo-gradient vector field $\widetilde{V}$ associated with ∇I such that

(a) $\| \widetilde{V}(u) \| \le 2 \| \nabla I(u) \|$, for $u \in H$;
(b) $\langle \nabla I(u), \widetilde{V}(u)\rangle \ge \| \nabla I(u) \|^2$, for $u \in H$;
(c) $\langle \widetilde{V}(v), v\rangle > 0$ on $M_i \cap U$;
(d) $\langle \widetilde{V}(v), v\rangle < 0$ on $M_o \cap U$.

We put

$$h_1(v) = \| v - M_\alpha^c \| / (\| v - U^c \| + \| v - M_\alpha^c \|) \qquad \text{for } v \in H,$$

$$h_2(v) = \| v - U^c \| / (\| v - U^c \| + \| v - M_\alpha^c \|) \qquad \text{for } v \in H$$

and

$$V(v) = h_1(v)\widetilde{V}(v) + h_2(v)\ \mathrm{sgn}(\langle \widetilde{V}(v), v\rangle)v \qquad \text{for } v \in H. \tag{2.13}$$

Then V is Lipschitz continuous on $I_{(11/6)c} \cup (M_\alpha)^c$. Consider the ordinary differential equation

$$\frac{d\eta}{dt} = -V(\eta), \qquad \eta(0,v) = v \qquad \text{for } v \in I_{(11/6)c} \cup (M_\alpha)^c. \tag{2.14}$$

The solution $\eta : R^+ \times H \to H$ defines a semiflow on H. It follows from the definition of V that $\eta(t,v) \in (M_\alpha)^c$ for $(t,v) \in [0,\infty) \times (M_\alpha)^c$. In fact, if $v \in (M_\alpha)^c$, then for each $t > 0$, $\eta(t,v) = \lambda_t v$, where $\lambda_t \in R$ such that $\lambda_t v \in (M_\alpha)^c$. We also have from (a)-(c) and (2.13) that $\langle V(v), \nabla I(v)\rangle > 0$ on $U \cup I_{(11/6)c}$ and then

$$I(\eta(t,v)) < I(\eta(s,v)) \qquad \text{for } t > s \text{ and } v \in U \cup I_{(11/6)c}.$$

Thus we find that $\eta(t,v) \in I_{(11/6)c} \cup (M_\alpha)^c$ for $(t,v) \in [0,\infty) \times I_{(11/6)c} \cup (M_\alpha)^c$. It follows from Lemma 2.1 that

$$\inf\{\| \nabla I(u) \| :\ u \in I_{(11/6)c} \backslash I_{(4/3)c}\} > 0.$$

Then we have

$$\inf\{\| V(u) \| :\ u \in (U \cup I_{(11/6)c}) \backslash I_{(4/3)c}\} > 0.$$

Therefore, there exists $T > 0$ such that

$$\begin{aligned} \eta(t,v) \in int(I_{(4/3)c} \cup (M_\alpha)^c) &\text{ for all } t > T \\ &\text{and all } v \in I_{(11/6)c} \cup (M_\alpha)^c. \end{aligned} \tag{2.15}$$

Here we put

$$\gamma(t,v) = \eta(t_v \cdot t, v) \qquad \text{for } (t,v) \in [0,1] \times I_{(11/6)c} \cup (M_\alpha)^c,$$

where

$$t_v = \inf\{t \geq 0\ :\ \eta(t,v) \in I_{(4/3)c} \cup (M_\alpha)^c\} \qquad \text{for } v \in I_{(11/6)c} \cup (M_\alpha)^c.$$

Then, by (2.15), we have $\gamma_1 : [0,1] \times I_{(11/6)c} \cup (M_\alpha)^c \to I_{(11/6)c} \cup (M_\alpha)^c$ satisfying the desired properties. ∎

By a parallel argument as in the proof of Lemma 2.5, we have

Lemma 2.6. *There exists a continuous mapping* $\gamma_2 : [0,1] \times I^\infty_{(5/3)c} \cup M^c_\alpha \to I^\infty_{(5/3)c} \cup M^c_\alpha$ *such that*

(v) $\gamma_2(0,x) = x$ *for all* $x \in I^\infty_{(5/3)c} \cup M^c_\alpha$;
(vi) $\gamma_2(t,x) = x$ *for all* $(t,x) \in [0,1] \times (I^\infty_{(7/6)c} \cup M^c_\alpha)$;
(vii) $I^\infty(\gamma_2(t,x)) \le I^\infty(\gamma_2(0,x))$ *for all* $(t,x) \in [0,1] \times (I^\infty_{(5/3)c} \cup M^c_\alpha)$;
(viii) $\gamma_2(1, I^\infty_{(5/3)c} \cup M^c_\alpha) \subset I^\infty_{(7/6)c} \cup M^c_\alpha$.

Lemma 2.7. *For each* $0 < \epsilon < c$, I^∞_ϵ *and* I_ϵ *have the same homotopy type.*

Proof. Let $0 < \epsilon < c$. Then we have by (2.1) that there exist continuous mappings $t_1 : H\backslash\{0\} \to R^+$ and $t_2 : H\backslash\{0\} \to R^+$ such that for each $u \in H\backslash\{0\}$, $t_1(u) < t_2(u)$ and

$$\{I(tu) \ : t \ \geq 0\} \cap I_\epsilon = \{tu \ : \ t \in [0, t_1(u)] \cup [t_2(u), \infty)\}$$

Similarly, there exist continuous mappings $t^\infty_1 : H\backslash\{0\} \to R^+$ and $t^\infty_2 : H\backslash\{0\} \to R^+$ such that for each $u \in H\backslash\{0\}$, $t^\infty_1(u) < t^\infty_2(u)$ and

$$\{I^\infty(tu) \ : \ t \geq 0\} \cap I^\infty_\epsilon = \{tu \ : \ t \in [0, t^\infty_1(u)] \cup [t^\infty_2(u), \infty)\}.$$

Then we find that I^∞_ϵ and I_ϵ have the same homotopy type. ∎

We can now prove Proposition 2.3.

Proof of Proposition 2.3. Let $0 < \epsilon < c$. Then $I^\infty_{c+\epsilon}$ and $I_{c+\epsilon}$ have the same homotopy types as $I^\infty_{(7/6)c}$ and $I_{(7/6)c}$, respectively. We also have that I^∞_ϵ and I_ϵ have the same homotopy types with as $I^\infty_{(1/3)c}$ and $I_{(1/3)c}$, respectively. Then to prove the assertion, it is sufficient to show that

$$H_*(I^\infty_{(7/6)c}, I^\infty_{(1/3)c}) \cong H_*(I_{(7/6)c}, I_{(1/3)c}).$$

We first define a mapping $\widetilde{\gamma} : [0,1] \times (I_{(11/6)c} \cup (M_\alpha)^c) \to I_{(11/6)c} \cup (M_\alpha)^c$ by

$$\widetilde{\gamma}(t,u) = \begin{cases} \gamma_1(2t,u), & \text{for } t \in [0,1/2], \\ \gamma_2(2(t-1/2), \gamma_1(1,u)), & \text{for } t \in (1/2,1]. \end{cases}$$

Then from (iii), we have that

$$\widetilde{\gamma}(t,u) \in I_{(11/6)c} \cup (M_\alpha)^c \tag{2.16}$$

for $(t,u) \in [0,1/2] \times (I_{(11/6)c} \cup (M_\alpha)^c)$. On the other hand, we have, by combining (iv) and (vii) with (3) of Lemma 2.4, that (2.16) holds for $(t,u) \in [1/2,1] \times (I_{(11/6)c} \cup (M_\alpha)^c)$. Thus we have that $\widetilde{\gamma}$ is well defined and a strong deformation retraction from $I_{(11/6)c} \cup (M_\alpha)^c$ onto $I^\infty_{(7/6)c} \cup (M_\alpha)^c$. We next define a mapping $\gamma_3 : [0,1] \times (I^\infty_{(7/6)c} \cup M^c_\alpha) \to I^\infty_{(7/6)c}$. For each $u \in (M_\alpha)^c$ with $I^\infty(u) > (7/6)c$, we set

$$\begin{aligned}
\tau_u^+ &= \min\{\tau > 1 \ : \ I^\infty(\tau u) \le (7/6)c\},\\
\tau_u^- &= \max\{\tau < 1 \ : \ I^\infty(\tau u) \le (7/6)c\},\\
M_o^\infty &= \{\lambda u \ : \ u \in M^\infty, \lambda > 1\}
\end{aligned}$$

and

$$M_i^\infty = \{\lambda u \ : \ u \in M^\infty, \lambda < 1\}.$$

Then we put

$$\gamma_3(t,x) = \begin{cases} t\tau_u^+ u + (1-t)u & \text{if } u \in M_o^\infty \backslash (I^\infty_{(7/6)c} \cup M_\alpha),\\ t\tau_u^- u + (1-t)u & \text{if } u \in M_i^\infty \backslash (I^\infty_{(7/6)c} \cup M_\alpha),\\ u & \text{if } u \in I^\infty_{(7/6)c}. \end{cases}$$

It then easy to see that γ_3 is a strong deformation retraction from $I^\infty_{(7/6)c} \cup (M_\alpha)^c$ to $I^\infty_{(7/6)c}$. Therefore we obtain that $I^\infty_{(7/6)c}$ is a strong deformation retract of $I_{(11/6)c} \cup (M_\alpha)^c$. It then follows that

$$H_*(I^\infty_{(7/6)c}, I^\infty_{(1/3)\epsilon}) = H_*(I_{(11/6)c} \cup (M_\alpha)^c, I^\infty_{(1/3)\epsilon}). \tag{2.17}$$

Then by Lemma 2.7,

$$H_*(I_{(11/6)c} \cup (M_\alpha)^c, I^\infty_{(1/3)\epsilon}) = H_*(I_{(11/6)c} \cup (M_\alpha)^c, I_{(1/3)\epsilon}). \tag{2.18}$$

On the other hand, we can see by a parallel argument as above that $I_{(7/6)c}$ is a strong deformation retract of $I_{(11/6)c} \cup (M_\alpha)^c$. Then from (2.17) and (2.18), we have $H_*(I^\infty_{(7/6)c}, I^\infty_{(1/3)c}) \cong H_*(I_{(7/6)c}, I_{(1/3)c})$, which completes the proof. ∎

3. Proof of the Theorem

We start with the following proposition.

Proposition 3.1. *For each positive number $\epsilon < c$,*

$$H_q(I^\infty_{c+\epsilon}, I^\infty_\epsilon) = \begin{cases} 2 & \text{if } q = 0,\\ 0 & \text{if } q \neq 0. \end{cases}$$

Proposition 3.1 was proved in [6]. For completeness, we give the proof of it in the appendix. We next consider a triple $(U, K, \epsilon) \subset H \times H \times R^+$ satisfying the following conditions:

(1) $U \cap (-U) = \phi$;
(2) $\{\tau_x u_\infty \ : | \ x \ |\ge r\} \subset intK$ for some $r > 0$;
(3) $cl(I_{c+\epsilon} \cap K) \subset int_{I_{c+\epsilon}}(I_{c+\epsilon} \cap U)$;
(4) I_ϵ is a strong deformation retract of $I_{c+\epsilon} \backslash (K \cup (-K))$;
(5) $H_{N-1}(I_{c+\epsilon} \cap U) = 1, \quad H_1(I_{c+\epsilon} \cap U) = 0$;
(6) $H_{N-1}((I_{c+\epsilon} \cap U)\backslash K) = 2$ or $H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$ holds.

Proposition 3.2. *There exists a triple* $(U, K, \epsilon) \subset H \times H \times R^+$ *which satisfies* $(1) - (6)$.

The proof of Proposition 3.2 is given in Section 4.

Lemma 3.3. *Suppose that there exist a triple* $(U, K, \epsilon) \subset H \times H \times R^+$ *satisfying* $(1) - (6)$. *Suppose, in addition, that* $H_{N-1}((I_{c+\epsilon} \cap U)\backslash K) \geq 2$. *Then* $H_N(I_{c+\epsilon}, I_\epsilon) \geq 2$.

Proof. We put $\widetilde{K} = K \cup (-K)$. Since I_ϵ is a strong deformation retract of $I_{c+\epsilon}\backslash\widetilde{K}$, we find that

$$H_q(I_{c+\epsilon}\backslash\widetilde{K}, I_\epsilon) \cong H_q(I_\epsilon, I_\epsilon) \cong 0.$$

Then we have from the exactness of the singular homology groups of the triple $(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}, I_\epsilon)$ that

$$0 \to H_q(I_{c+\epsilon}, I_\epsilon) \to H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \to 0.$$

That is,

$$H_q(I_{c+\epsilon}, I_\epsilon) \cong H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}).$$

From (1) and (3), we find

$$H_q(I_{c+\epsilon}, I_{c+\epsilon}\backslash\widetilde{K}) \cong H_q(W, W\backslash K) \oplus H_q(-W, (-W)\backslash(-K)),$$

where $W = I_{c+\epsilon} \cap U$. Then since $H_{N-1}(W\backslash K) \geq 2$, we have from (5) and the exactness of the sequence

$$\begin{aligned} &\to H_q(W, W\backslash K) \to H_{q-1}(W\backslash K) \\ &\qquad\qquad \to H_{q-1}(W) \to H_{q-1}(W, W\backslash K) \to, \end{aligned} \tag{3.1}$$

with $q = N$, that $H_N(I_{c+\epsilon}, I_\epsilon) \cong H_N(W, W\backslash K) \oplus H_N(W, W\backslash K) \geq 2$. ■

Lemma 3.4. *Suppose that* $(U, K, \epsilon) \subset H \times H \times R^+$ *satisfies* $(1) - (6)$. *Suppose in addition that* $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$. *Then* $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ *or* $H_0(I_{c+\epsilon}, I_\epsilon) = 2$ *holds.*

Proof. From the argument in the proof of Proposition 3.2, we have

$$H_1(I_{c+\epsilon}, I_\epsilon) \cong H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U)\backslash K) \oplus H_1(I_{c+\epsilon} \cap U, (I_{c+\epsilon} \cap U)\backslash K).$$

Then since $H_1(I_{c+\epsilon} \cap U) = 0$ and $H_0(I_{c+\epsilon} \cap U) = H_0((I_{c+\epsilon} \cap U)\backslash K) = 1$, the assertion follows from the exactness of the sequence (3.1) with $q = 1$. ■

We can now prove the Theorem.

Proof of the Theorem. Let (U, K, ϵ) satisfy $(1) - (6)$. We have by Proposition 2.3 and Proposition 3.1 that $H_1(I_{c+\epsilon}, I_\epsilon) = 2$ and $H_q(I_{c+\epsilon}, I_\epsilon) = 0$ for $q \neq 1$. Now suppose that $(I_{c+\epsilon} \cap U)\backslash K$ is disconnected. Then since $H_0((I_{c+\epsilon} \cap U)\backslash K) \geq 2$, we find by (6) that $H_{N-1}(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. On the other hand, if $U\backslash K$ is connected, then $H_0((I_{c+\epsilon} \cup U)\backslash K) = 1$. Then by Lemma 3.4, we have $H_1(I_{c+\epsilon}, I_\epsilon) = 0$ or $H_0(I_{c+\epsilon}, I_\epsilon) = 2$. This is a contradiction. Thus we obtain that there exists a positive solution of (P). ■

4. Proof of Proposition 3.2

We shall construct a triple (U, K, ϵ) satisfying (1) - (6). First we state the following lemma.

Lemma 4.1. *If* $0 < \epsilon < c < d < 2c$ *and* $\{u_n\} \subset I_d \backslash I_\epsilon$ *is a sequence such that* $\nabla I(u_n) \to 0$*, then* $u_n \to \tau_{x_n} u_\infty$ *where* $\{x_n\} \subset R^N$ *with* $\lim_{n\to\infty} \mid x_n \mid = \infty$.

Since we are assuming that I has no critical point in $\dot{I}_{2c} \backslash I_c$, the assertion of Lemma 4.1 is a direct consequence of the arguments in [8, 9]. Thus, we omit the proof (cf. also [3]).

We fix a positive number $\rho < 1$. Recalling that the mappings $t \to I^\infty((\pm t + 1)u_\infty)$ are decreasing as t varies from 0 to ± 1, we have $I_c^\infty \cap \{tu_\infty \ : \ t \in [-\rho + 1, \rho + 1]\} = \{u_\infty\}$. Then we can choose positive numbers r_0 and δ such that

$$\{tv \ : \ t \in [-\rho + 1, -\rho/2 + 1] \cup [\rho/2 + 1, \rho + 1], v \in S_0\} \subset I^\infty_{c-\delta} \tag{4.1}$$

where $S_0 = (u_\infty + B_{r_0}) \cap M^\infty$. We note that S_0 is a contractible neighborhood of u_∞ in M^∞. We may choose r_0 so small that

$$S_0 \subset I^\infty_{(4/3)c}. \tag{4.2}$$

Next, we fix a contractible neighborhood $\widetilde{S}_0$ of u_∞ in M^∞ such that $\widetilde{S}_0 \subset int_{M^\infty} S_0$. We put

$$\begin{aligned} D_0 &= \{\tau_x v \ : \ v \in S_0, x \in R^N \text{ with } \mid x \mid \geq R_0\}, \\ \widetilde{D}_0 &= \{\tau_x v \ : \ v \in \widetilde{S}_0, x \in R^N \text{ with } \mid x \mid \geq 2R_0\}, \end{aligned}$$

where R_0 is a positive number. Then $\widetilde{D}_0 \subset D_0 \subset M^\infty$. Now we define subsets U, K of H by

$$\begin{aligned} U &= \{tv \ : \ t \in [-\rho + 1, \rho + 1], v \in D_0\}, \\ K &= \{tv \ : \ t \in [-\rho/2 + 1, \rho/2 + 1], v \in \widetilde{D}_0\}. \end{aligned} \tag{4.3}$$

Since $\{\tau_x u_\infty \ : \ x \in R^N\} \cap \{\tau_x(-u_\infty) \ : \ x \in R^N\} = \phi$, by choosing r_0 and ρ sufficiently small, we have that $U \cap (-U) = \phi$. That is, (1) holds. Since (4.1) holds and $\lim_{|x|\to\infty} I(\tau_x u_\infty) = c$, we can choose R_0 so large that

$$\{tv \ : \ t \in [-\rho + 1, -\rho/2 + 1] \cup [\rho/2 + 1, \rho + 1], v \in D_0\} \subset I_c. \tag{4.4}$$

We also have by (4.2) that R_0 can be chosen so large that $U \subset I_{(6/5)c}$. It follows from the defintion of U and K that

$$\{\tau_x u_\infty \ : \mid x \mid \geq 3R_0\} \subset intK \subset K \subset intU. \tag{4.5}$$

That is, (2) holds with $r = 3R_0$. From the definition, it is obvious that (3) holds. As a direct consequence of (3) of Lemma 4.1 and (4.5) , we have

$$\inf\{\| \nabla I(v) \| : \ v \in I_d \backslash (I_\epsilon \cup K \cup (-K))\} > 0 \tag{4.6}$$

for all $0 < \epsilon < c < d < 2c$. Then by deformation lemma(cf. [3]), there exists $\epsilon_0 > 0$ such that for each $0 < \epsilon < \epsilon_0$, I_ϵ is a strong deformation retract of $I_{c+\epsilon} \backslash (K \cup (-K))$. That is, (4) holds for all $0 < \epsilon < \epsilon_0$.

We will see that there exists $0 < \epsilon < \epsilon_0$ such that (U, K, ϵ) satisifes (5) and (6). Here we note that

$$c_2 = \inf\{I(\lambda_0(v)v) \ : \ v \in D_0 \backslash \widetilde{D}_0\} > c_1 \tag{4.7}$$

In fact, if $c_2 = c_1$, there exists a sequence $\{u_n\} \subset M$ such that $u_n = \lambda_0(v_n)v_n$, $v_n \in D_0 \backslash \widetilde{D}_0$ for each $n \geq 1$ and that $\lim_{n\to\infty} I(u_n) = c$. This implies that $\nabla I(u_n) \to 0$ and then by Lemma 4.1, $u_n \to \tau_{x_n} u_\infty$, where $\{x_n\} \subset R^N$ with $\lim \mid x_n \mid = \infty$. This implies that $v_n \to \tau_{x_n} u_\infty$ and this contradicts to the definition of $\{v_n\}$. Here we choose a positive number ϵ such that $\epsilon < c_2 - c$. Here we define subsets of M and H. Noting that

$$\lim_{|x|\to\infty} I(\tau_x u_\infty) = c$$

We can choose contractible neighborhoods S_1, S_2 of u_∞ in M^∞ and positive numbers R_1, R_2 such that $S_2 \subset int_{M^\infty} S_1 \subset S_0$, $R_1 < R_2$ and

$$U_i = \{t\tau_x v \ : \ t \in [-\rho + 1, \rho + 1], \mid x \mid \geq R_i, v \in S_i\} \subset I_{c+\epsilon}.$$

We also set

$$U_{1,+} = \{tv \ : \ t \in [-\rho + 1, -\rho/2 + 1], v \in D_0\},$$
$$U_{1,-} = \{tv \ : \ t \in [\rho/2 + 1, \rho + 1], v \in D_0\}$$

and

$$U_{2,+} = \{tv \ : \ t \in [-\rho + 1, -\rho/4 + 1], v \in D_0\},$$
$$U_{2,-} = \{tv \ : \ t \in [\rho/4 + 1, \rho + 1], v \in D_0\}.$$

Then from the definitions above and (4.2) , we have that

$$\widetilde{U}_2 = U_2 \cup U_{2,+} \cup U_{2,-} \subset \widetilde{U}_1 = U_1 \cup U_{1,+} \cup U_{1,-} \subset I_{c+\epsilon},$$

and

$$\widetilde{U}_1 \cong \widetilde{U}_2 \cong \{\tau_x u_\infty \ : \mid x \mid \geq R_1\} \cong S^{N-1}. \tag{4.8}$$

Then we have that (5) holds, as a direct consequence of the following lemma 4.5.

Lemma 4.2. *$\widetilde{U}_1$ is a deformation reatract of $I_{c+\epsilon} \cap U$.*

Proof. To prove the assertion it is sufficient to show the existence of a semiflow $\eta : [0, \infty) \times (I_{c+\epsilon} \cap U) \to I_{c+\epsilon} \cap U$ such that for each $v \in I_{c+\epsilon} \cap U$, there exists $t_v \geq 0$ satsifying $\eta(t, v) \in int_{I_{c+\epsilon} \cap U} \widetilde{U}_1$ for all $t \geq t_v$. In fact, if there exists such a semiflow, we can construct a strong deformation retraction as in the proof of Lemma 2.5. By (4.7) and the definition of I,

$$I(v) > c + \epsilon \qquad \text{for } v \in \partial_{M^\infty} D_0,$$

and we have

$$D_2 = \{v \in D_0 \ : \ I(v) \leq c + \epsilon\} \subset int_{M^\infty} D_0,$$

Here we fix an open neighborhood D_1 of D_2 in M^∞ such that

$$D_2 \subset int_{M^\infty} D_1 \subset cl(D_1) \subset int_{M^\infty} D_0$$

and set

$$W_i = \{tv \ : \ t \in [-\rho + 1, \rho + 1], v \in D_i\}, \text{ i = 1,2.}$$

Then

$$U_1 \subset W_2 \subset W_1 \subset I_{c+\epsilon} \cap U.$$

We note that

$$I(\lambda_0(v)v) > c + \epsilon \qquad \text{for } v \in D_0 \backslash D_2. \tag{4.9}$$

Let V_1 be a Lipschitz continuous vector field associate with ∇I and V_2 be a vector field defined on $(I_{c+\epsilon} \cap U) \backslash W_2$ by

$$V_2(u) = \begin{cases} u & \text{if } \lambda_0(u) > 1 \\ -u & \text{if } \lambda_0(u) < 1. \end{cases}$$

Since $\lambda_0(u) \neq 1$ on $(I_{c+\epsilon} \cap U) \backslash W_2$ by (4.9), we can see that V_2 is well defined and continuous on $(U \cap I_{c_1+\bar{\epsilon}}) \backslash W_2$. We now set

$$V(u) = \| U_{2,-} \cup U_{2,+} - u \| \, (\| W_1^c - u \| \, V_1(u) + \| W_2 - u \| \, V_2(u))$$

Then V is a Lipschitz continuous vector field on $I_{c+\epsilon} \cap U$ and the solution η of (2.14) defines a semiflow. We shall see that

$$\eta(t, v) \in I_{c+\epsilon} \cap U \qquad \text{for all } (t, v) \in [0, \infty) \times (I_{c+\epsilon} \cap U). \tag{4.10}$$

We first note that from the definition of V, $\langle \nabla I(v), V(v) \rangle > 0$ on $I_{c+\epsilon} \cap U$. Then it follows that $\eta(t, v) \leq \eta(s, v)$ for all $t > s \geq 0$ and $v \in I_{c+\epsilon} \cap U$. Since $W_1 \backslash (U_{1,-} \cup U_{1,+}) \subset int(I_{c+\epsilon} \cap U)$, to show (4.10), it is sufficient to show that (4.10) holds for all $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$. If $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$, then from the definition of V, we can see that $\eta(t,v) \in W_1^c \cap (I_{c+\epsilon} \cap U)$ for $t \geq 0$ and then (4.10) holds. Moreover we have that for each $v \in W_1^c \cap (I_{c+\epsilon} \cap U)$, $\eta(t,v) \in U_{1,-} \cup U_{1,+}$ for t sufficiently large. On the other hand, it follows from the definition of V that

$$\inf\{\| V(u) \| \ : \ u \in (I_{c+\epsilon} \cap U)\backslash \widetilde{U}_2\} > 0. \tag{4.11}$$

Then we can see that for any $v \in I_{c+\epsilon} \cap U$, there exists $t_v \geq 0$ such that $\eta(t,v) \in \widetilde{U}_1$ for all $t \geq t_v$. This completes the proof. ∎

We lastly show that (6) holds. (6) is a consequence of the following Lemma.

Lemma 4.3. *If $(I_{c+\epsilon} \cap U)\backslash K$ is disconnected, then $H_{N-1}((I_{c+\epsilon} \cap U)\backslash K) = 2$.*

Proof. Let $V_\pm$ be the components of $(I_{c+\epsilon} \cap U)\backslash K$ containing $U_{1,\pm}$, respectively. We will see that $(I_{c+\epsilon} \cap U)\backslash K$ consists of exactly two components $V_\pm$ and that $V_\pm \cong S^{N-1}$. Let $v \in D_0$. Then from the definition of M and U, we have that

$$\begin{aligned}\{tv \ : \ t \in [-\rho+1, \rho+1]\} &\cap (I_{c+\epsilon}\backslash K) \\ &= \{tv \ : \ t \in [-\rho+1, t_1(v)] \cup [t_2(v), \rho+1]\},\end{aligned} \tag{4.12}$$

where $-\rho/2+1 \leq t_1(v) \leq t_2(v) \leq \rho/2+1$. This implies that if $t_1(v) = t_2(v)$ for some $v \in D_0$, then $(I_{c+\epsilon} \cap U)\backslash K$ is connected. Therefore $t_1(v) < t_2(v)$ for all $v \in D_0$. Then, again by (4.12), $(I_{c+\epsilon} \cap U)\backslash K \cong U_{1,+} \cup U_{1,-}$. Then since $U_{1,\pm} \cong S^{N-1}$, the assertion follows. ∎

5. Appendix

We put $\mathcal{C} = \cup\{\tau_x u_\infty \ : \ x \in R^N\}$ and

$$T_{u_\infty}(\mathcal{C}) = \{\lim_{t\to 0}(u_\infty(\cdot + tx) - u_\infty(\cdot))/t \ : \ x \in R^N\}.$$

It is obvious that $\dim T_{u_\infty}(\mathcal{C}) = N$. We denote by $\widetilde{H}$ the subspace such that $\widetilde{H} \oplus T_{u_\infty}(\mathcal{C})$. Then $H = \tau_x\widetilde{H} \oplus \tau_x T_{u_\infty}(\mathcal{C})$ for each $x \in R^N$. For each $r > 0$, we set $B_r^0 = B_r \cap \widetilde{H}$. Since $\mathcal{C}$ is a smooth N-manifold, we have that there exists a positive number $r_0 <\| u_\infty \| /4$ such that for $x, y \in R^N$ with $x \neq y$,

$$\tau_x(u_\infty + B_{r_0}^0) \cap \tau_y(u_\infty + B_{r_0}^0) = \phi \tag{5.1}$$

We choose a closed contractible neighborhood S_0 of u_∞ in $M^\infty \cap (u_\infty + B_{r_0}^0)$ and $0 < \rho < 1$ such that

$$\sup\{I^\infty((\pm\rho/2+1)v) \ : \ v \in S_0\} < c. \tag{5.2}$$

Since $I(v) > c$ for all $v \in S_0 \backslash \{u_\infty\}$, we have that

$$\inf\{I^\infty(v) \ : \ v \in \partial_{M^\infty \cap (u_\infty + B^0_{r_0})} S_0\} > c. \tag{5.3}$$

Here we recall that mappings $t \to I^\infty((\pm t + 1)v)$ are decreasing as t varies from 0 to $\pm\rho$. Then from (5.2), we have

$$\begin{aligned} I_c^\infty \cap \{tv \ : \ t \in [-\rho+1, \rho+1]\} \\ =\{tv \ : \ t \in [-\rho+1, \lambda_-(v)]\} \cup \{tv \ : \ t \in [\lambda_+(v), \rho+1]\} \end{aligned} \tag{5.4}$$

where

$$\begin{cases} \lambda_-(v) < 1 < \lambda_+(v) & \text{for } v \in S_0 \backslash \{u_\infty\} \\ \lambda_-(v) = \lambda_+(v) = 1 & \text{for } v = u_\infty. \end{cases}$$

That is, for each $v \in S_0 \backslash \{u_\infty\}$, the set $I_c^\infty \cap \{tv \ : \ t \in [-\rho+1, \rho+1]\}$ consists of two intervals, and each interval has one end point in one of the sets

$$V_\pm = \{(\pm\rho + 1)v \ : \ v \in S_0\}.$$

Then noting that $\lambda_-(\cdot)$ and $\lambda_+(\cdot)$ are continuous and $V_\pm$ are contractible, we have from observations above that

$$I_c^\infty \cap (V \backslash \{u_\infty\}) \cong V_- \cup V_+ \cong \{0, 1\} \ \text{ and } \ I_c^\infty \cap V \cong [0, 1] \tag{5.5}$$

Now let $0 < \epsilon < c$. First we note that

$$I^\infty(u) = \tau_x \cdot I^\infty(u) = I^\infty(\tau_x u) \qquad \text{for all } x \in R^N \text{ and } u \in H.$$

Then we have that $I_c^\infty \cap (\cup\{\tau_x V \ : \ x \in R^N\})$ and $I_\epsilon^\infty \cap (\cup\{\tau_x V \ : \ x \in R^N\})$ have the same homotopy type with that of $I_c^\infty \cap V$ and $I_\epsilon^\infty \cap V$, respectively. On the other hand, by the same argument for the second deformation lemma in Chang [4], we have that I_c^∞ is a strong deformation rectraction of $I_{c+\epsilon}^\infty$. Then we find

$$H_q(I_{c+\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_{c-\epsilon}^\infty).$$

We also have by the deformation property that

$$H_q(I_c^\infty \backslash \mathcal{C}, I_{c-\epsilon}^\infty) \cong H_q(I_{c-\epsilon}^\infty, I_{c-\epsilon}^\infty) \cong 0.$$

From the exactness of the singular homology groups, we have

$$\begin{aligned} H_q(I_c^\infty \backslash \mathcal{C}, I_{c-\epsilon}) \to H_q(I_c^\infty, I_{c-\epsilon}^\infty) \\ \to H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \to H_{q-1}(I_c^\infty \backslash \mathcal{C}, I_{c-\epsilon}^\infty) \to \cdots \end{aligned}$$

and we find

$$0 \to H_q(I_c^\infty, I_{c-\epsilon}^\infty) \to H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \to 0.$$

That is,

$$H_q(I_c^\infty, I_{c-\epsilon}^\infty) \cong H_q(I_c^\infty, I_c^\infty \backslash \mathcal{C}).$$

Then from the excision property of homology groups and (5.5), we have

$$\begin{aligned} H_*(I_{c+\epsilon}^\infty, I_\epsilon^\infty) &\cong H_*(I_c^\infty, I_c^\infty \backslash \mathcal{C}) \\ &\cong H_*(I_c^\infty \cap (\cup_x \tau_x V), I_c^\infty \cap ((\cup_x \tau_x V) \backslash \mathcal{C})) \\ &\cong H_*(I_c^\infty \cap V, I_c^\infty \cap (V \backslash \{u_\infty\})) \\ &\cong H_*([0,1], \{0,1\}). \end{aligned}$$

This completes the proof. ∎

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