NONUNIQUENESS THEOREM FOR A SINGULAR CAUCHY-NICOLETTI PROBLEM

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The problem of nonuniqueness for a singular Cauchy-Nicoletti boundary value problem is studied. The general nonuniqueness theorem ensuring the existence of two different solutions is given such that the estimating expressions are nonlinear, in general, and depend on suitable Lyapunov functions. The applicability of results is illustrated by several examples.

1. Introduction

The nonuniqueness of a regular or singular Cauchy problem for ordinary differential equations is studied in several papers such as [3, 4, 5, 13, 14, 15, 16, 17]. Most of these results can also be found in the monograph [1]. The uniqueness of solutions of Cauchy initial value problem for ordinary differential equations with singularity is investigated in [7, 8, 9, 12]. The topological structure of solution sets to a large class of boundary value problems for ordinary differential equations is studied in [2]. First results on the nonuniqueness for a singular Cauchy-Nicoletti boundary value problem are given in [10, 11, 12] by Kiguradze, where sufficient conditions for the nonuniqueness are written in the form of one-sided inequalities for the components in the right-hand side $f(t,x_1,\dots,x_n)$ of the corresponding equation. An expression for the estimation of the jth component $f_j(t,x_1,\dots,x_n)$ of f depends on t and x_j and is linear in $|x_j|$.

In [6], we studied the nonuniqueness for a singular Cauchy problem. Our criteria involve vector Lyapunov functions and the estimations need not be linear. The present paper deals with the nonuniqueness of the singular Cauchy-Nicoletti problem and extends the results of [6] to this more general problem.

Supposing $-\infty \le a < A \le \infty$, $b > 0$, we will use the following notations throughout the paper: $\mathbb{R}^k$ and $\mathbb{R}^+$ denote k-dimensional real Euclidean space and the interval $[0,\infty)$, respectively. $|\cdot|$ is used for the notation of Hölder's 1-norm (the sum of the absolute values of components). $x = (x_1,\dots,x_n)$ denotes a variable vector from $\mathbb{R}^n$ with components $x_1,\dots,x_n$, while $x_0 = (x_{01},\dots,x_{0n})$ stands for a fixed vector from $\mathbb{R}^n$ with components $x_{01},\dots,x_{0n}$. N is equal to the set $\{1,\dots,n\}$. l denotes a fixed number from the set $\{1,\dots,n\}$.

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$i_1, i_2, \dots, i_l$ are fixed integers such that $1 \le i_1 < i_2 < \cdots < i_l \le n$. I is set to be equal to $\{i_1, \dots, i_l\}$. $\Pr x$ denotes a projection of x such that $\Pr x = (x_{i_1}, \dots, x_{i_l})$, while $\Pr^\star x$ denotes a complementary projection to $\Pr x$. Clearly, $\Pr^\star x = (x_{j_1}, \dots, x_{j_{n-l}})$, where $1 \le j_1 < \cdots < j_{n-l} \le n$, $\{i_1, \dots, i_l\} \cap \{j_1, \dots, j_{n-l}\} = \emptyset$. $R^k_{\alpha,\beta;b}(x_0)$ and $\tilde{R}^k_{a,A}$ are used for the notation of the set $\{(t,x) \in \mathbb{R}^{k+1} : \alpha < t < \beta, |x - x_0| \le b\}$ and the set $\{(t,x) \in \mathbb{R}^{k+1} : a < t < A, x \in \mathbb{R}^k\}$, respectively. The symbol $\hat{R}^n_{a,A}$ will be used for the set $\{(t,x) \in \mathbb{R}^{n+1} : a \le t \le A, x \in \mathbb{R}^n\}$. $\Delta(\alpha,\beta)$ denotes the interval $(\min(\alpha,\beta), \max(\alpha,\beta))$.

The notation $C[\Gamma, \Omega]$ is used for the notation of the class of all continuous mappings $\Gamma \to \Omega$. $AC[[a,A], \mathbb{R}^k]$ and $\widetilde{AC}[[a,A], \mathbb{R}^k]$ denote the class of all absolutely continuous mappings $[a,A] \to \mathbb{R}^k$ and the class of all mappings from $C[[a,A], \mathbb{R}^k]$ which are absolutely continuous on any interval $[\alpha,\beta]$, where $a < \alpha < \beta < A$, respectively. The class of all Lebesgue-integrable mappings $[a,A] \to \mathbb{R}^+$ is denoted by $L[[a,A], \mathbb{R}^+]$. $\mathscr{L}_\tau[\hat{R}^n_{a,A}, \mathbb{R}^{+k}]$ stands for the class of all functions $V(t,x) : \hat{R}^n_{a,A} \to \mathbb{R}^{+k}$ with the following property: $V(t,\cdot)$ is uniformly continuous, and if $a < \alpha < \beta < A$, $\tau \notin [\alpha,\beta]$, then $V(t,x(t))$ is absolutely continuous on $[\alpha,\beta]$ for any absolutely continuous function $x : [\alpha,\beta] \to \mathbb{R}^n$. $K_{\sigma_1,\dots,\sigma_p}[\hat{R}^k_{a,A}, \mathbb{R}^m]$ denotes the class of all mappings $\hat{R}^k_{a,A} \to \mathbb{R}^m$ which satisfy Carathéodory conditions on $R^k_{\alpha,\beta;\varrho}(0)$ for any $\alpha, \beta, a \le \alpha < \beta \le A$, $\sigma_j \notin [\alpha,\beta]$ $(j = 1, \dots, p)$, $\varrho \in (0,\infty)$, $\sigma_1, \dots, \sigma_p$ being numbers from $[a,A]$. $N_0(a,A;\tau_1,\dots,\tau_n)$ is used for the notation of the class $\{\Lambda = (\lambda_{ij}(t))^n_{i,j=1} : \lambda_{ij} \in L[[a,A], \mathbb{R}^+]\}$ such that the system of differential inequalities $|x_i'(t)| \le \sum_{j=1}^n \lambda_{ij}(t)|x_j(t)|$, $t \in [a,A]$, $i \in N$, possesses no nontrivial solution $x(t) = (x_1(t), \dots, x_n(t)) \in AC[[a,A], \mathbb{R}^n]$ satisfying $x_i(\tau_i) = 0$ $(i = 1, \dots, n)$.

The fundamental role in the proof of our main theorem will be played by the following theorem by Kiguradze, which is adapted from [12] (see also [10]) in a simplified form.

Kiguradze Theorem. *Let $a \le \tau_i \le A$, $\hat{x}_{0i} \in \mathbb{R}$ for $i = 1, \dots, n$. Suppose that $f \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A}, \mathbb{R}^n]$. Assume that the components f_i of f satisfy*

$$f_i(t,x) \operatorname{sgn}\left[(t-\tau_i)(x_i - \hat{x}_{0i})\right] \le \sum_{j=1}^n \lambda_{ij}(t)|x_j| + \mu_i(t) \quad (i = 1, \dots, n) \tag{1.1}$$

for $(t,x) = (t, x_1, \dots, x_n) \in \tilde{R}^n_{a,A}$, where $\hat{x}_{0i} = 0$ if $\tau_i \in \{\sigma_1, \dots, \sigma_p\}$. Suppose that $\Lambda(t) = (\lambda_{ij}(t))^n_{i,j=1} \in N_0(a,A;\tau_1,\dots,\tau_n)$, $\mu_i \in L[[a,A], \mathbb{R}^+]$. Then the Cauchy-Nicoletti problem

$$x' = f(t,x), \qquad x_i(\tau_i) = 0 \quad (i = 1, \dots, n) \tag{1.2}$$

has at least one solution $x(t) = (x_1(t), \dots, x_n(t)) \in AC[[a,A], \mathbb{R}^n]$.

2. Results

Consider a Cauchy-Nicoletti boundary value problem

$$x' = f(t,x), \qquad x_i(t_i) = x_{0i} \quad (i = 1, \dots, n), \tag{2.1}$$

where $f(t,x) = (f_1(t,x_1,\dots,x_n), \dots, f_n(t,x_1,\dots,x_n))$, $f \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A}, \mathbb{R}^n]$, $x_{0i} \in \mathbb{R}$, and $t_i \in [a,A]$ $(i \in N)$.

Theorem 2.1. *Suppose that there are numbers* $c_i \in \mathbb{R}$ $(i \in N)$, $B_i \in [a,A] \setminus \{t_i,\sigma_1,\dots,\sigma_p\}$ $(i \in I)$, *a matrix function* $\Lambda = (\lambda_{ij})_{i,j=1}^n \in N_0(a,A;\tau_1,\dots,\tau_n)$ *and functions* $\mu_i \in L[[a,A],\mathbb{R}^+]$ $(i \in N)$ *such that* $c_i = x_{0i}$ *for* $i \in N \setminus I$ *and*

$$f_i(t,x)[\operatorname{sgn}(t-\tau_i)(x_i-c_i)] \le \sum_{j=1}^n \lambda_{ij}(t)|x_j| + \mu_i(t) \quad (i \in N) \tag{2.2}$$

holds for $(t,x) = (t,x_1,\dots,x_n) \in \tilde{\mathbb{R}}^n_{a,A}$, *where* $\tau_i = t_i$ *or* $\tau_i = B_i$ *whenever* $i \in N \setminus I$ *or* $i \in I$, *respectively.*

Assume that

(i) *there exist vector functions* $g_i = (g_{i1},\dots,g_{ik_i}) \in K_{a,A,t_i,B_i}[\hat{\mathbb{R}}^{k_i}_{a,A},\mathbb{R}^{k_i}]$ $(i \in I)$ *such that* $\operatorname{sgn}(t-t_i)g_{ij}(t,u_1,\dots,u_{j-1},\cdot,u_j,\dots,u_{k_i})$ *is nondecreasing for* $j = 1,\dots,k_i$ *and there is a solution* $\varphi_i(t) = (\varphi_{i1}(t),\dots,\varphi_{ik_i}(t))$ *of*

$$u_i' = g_i(t,u_1,\dots,u_{k_i}) \tag{2.3}$$

satisfying

$$\varphi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t\to t_i}\varphi_i(t) = 0, \qquad \liminf_{t\to B_i}\varphi_i(t) > 0 \tag{2.4}$$

for $i \in I$;

(ii) $V_i(t,x) = (V_{i1}(t,x),\dots,V_{ik_i}(t,x)) \in \mathcal{L}_{t_i}[\hat{R}^n_{a,A},\mathbb{R}^{+k_i}]$ $(i \in I)$ *are such that there exists* $y_0 \in \mathbb{R}^l$ *with the property*

$$\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \operatorname{Pr} y = y_0\} < \liminf_{t\to B_i}\varphi_{ij}(t) \quad (j = 1,\dots,k_i) \tag{2.5}$$

$$|V_i(t,x)| \ge \Psi_i(|x_i - z_i(t)|) \quad \text{for } t \in \Delta(t_i,B_i), \tag{2.6}$$

where $\Psi_i \in C[\mathbb{R}^+,\mathbb{R}^+]$, $z_i \in C[(a,A),\mathbb{R}]$ *are such that*

$$\Psi_i(0) = 0, \qquad \Psi_i(u) > 0 \quad \text{for } u > 0, \qquad \lim_{t\to t_i} z_i(t) = x_{0i} \tag{2.7}$$

for $i \in I$;

(iii) *there exist positive functions* $\varepsilon_{ik} \in C[(a,A),\mathbb{R}^+]$ $(i \in I; k = 1,\dots,k_i)$ *such that*

$$\begin{aligned} &\operatorname{sgn}(B_i - t_i)V_{ij}'(t,x(t)) \\ &\quad \ge \operatorname{sgn}(B_i - t_i)g_{ij}(t,\varphi_{i1}(t),\dots,\varphi_{i,j-1}(t),V_{ij}(t,x(t)),\varphi_{i,j+1}(t),\dots,\varphi_{ik_i}(t)) \end{aligned} \tag{2.8}$$

holds for $i \in I$, $j = 1,\dots,k_i$, *and for any solution* $x(t)$ *of (2.1) a.e. on any interval* $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ *for which*

$$\begin{aligned} V_{ik}(t,x(t)) < \varphi_{ik}(t) + \varepsilon_{ik}(t) \quad &\text{on } (\alpha_{i1},\alpha_{i2}) \ (k = 1,\dots,k_i), \\ V_{ij}(t,x(t)) > \varphi_{ij}(t) \quad &\text{on } (\alpha_{i1},\alpha_{i2}). \end{aligned} \tag{2.9}$$

Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on $[a,A]$, *either of which satisfies* $V_i(t,x(t)) \le \varphi_i(t)$ *for* $t \in \Delta(t_i,B_i)$ *and* $i \in I$.

Proof. Without loss of generality, it can be assumed that $I = \{1,\dots,l\}$,

$$\Pr x = (x_1,\dots,x_l), \qquad \Pr^* x = (x_{l+1},\dots,x_n). \tag{2.10}$$

For any $i \in I$ and $j \in \{1,\dots,k_i\}$, denote

$$L_{ij} = \liminf_{t\to B_i} \varphi_{ij}(t), \qquad S_{ij} = \sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \Pr y = y_0\}. \tag{2.11}$$

According to (2.5) and to the uniform continuity of $V_{ij}(B_i,\cdot)$, we have a relation

$$\begin{aligned} V_{ij}(B_i,y^*) &\le V_{ij}(B_i,y) + V_{ij}(B_i,y^*) - V_{ij}(B_i,y) \\ &\le \frac{1}{2}(L_{ij}+S_{ij}) + \frac{1}{4}(L_{ij}-S_{ij}) = \frac{3}{4}L_{ij} + \frac{1}{4}S_{ij} < L_{ij} \end{aligned} \tag{2.12}$$

for $y \in \mathbb{R}^n$, $\Pr y = y_0$, and for $y^* \in \mathbb{R}^n$ sufficiently close to y. Hence it can be supposed without loss of generality that $y_0 \neq \Pr x_0$.

Further, the uniform continuity of $V_{ij}(B_i,\cdot)$ implies that the inequality

$$\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \Pr y = y_0 - \lambda(y_0 - \Pr x_0)\} < \liminf_{t\to B_i} \varphi_{ij}(t) \quad (i \in I; j = 1,\dots,k_i) \tag{2.13}$$

holds provided that $\lambda > 0$ is sufficiently small. Therefore, we can choose $\tilde{x}_1,\tilde{x}_2 \in \mathbb{R}^l$, $\tilde{x}_1 \neq \tilde{x}_2$, such that

$$\max_{k=1,2}\left[\sup\{V_{ij}(B_i,y) : y \in \mathbb{R}^n, \Pr y = \tilde{x}_k\}\right] < \liminf_{t\to B_i} \varphi_{ij}(t) \quad (i = 1,\dots,l; j = 1,\dots,k_i). \tag{2.14}$$

Choose $\tilde{\xi} \in \{\tilde{x}_1,\tilde{x}_2\}$ arbitrary. Put $\xi = x_0 - (\tilde{\xi}, \Pr^* x_0)$, $X = x - x_0 + \xi$, and $f^*(t,X) = f(t,x_0 + X - \xi)$ for $(t,X) = (t,X_1,\dots,X_n) \in \hat{R}^n_{a,A}$.

Clearly $f^* \in K_{\sigma_1,\dots,\sigma_p}[\hat{R}^n_{a,A},\mathbb{R}^n]$. By using (2.2), we obtain

$$\begin{aligned} f_i^*(t,X)\operatorname{sgn}\left[(t-\tau_i)(X_i + \tilde{\xi}_i - c_i)\right] &\le \sum_{j=1}^{l} \lambda_{ij}(t)\left|X_j + \tilde{\xi}_j\right| + \sum_{j=l+1}^{n} \lambda_{ij}(t)\left|X_j + x_{0j}\right| + \mu_i(t) \\ &\le \sum_{j=1}^{n} \lambda_{ij}(t)\left|X_j\right| + \tilde{\mu}_i(t) \end{aligned} \tag{2.15}$$

for $(t,X) \in \tilde{R}^n_{a,A}$, $i = 1,\dots,l$, and

$$\begin{aligned} f_i^*(t,X)\operatorname{sgn}\left[(t-\tau_i)X_i\right] &\le \sum_{j=1}^{l} \lambda_{ij}(t)\left|X_j + \tilde{\xi}_j\right| + \sum_{j=l+1}^{n} \lambda_{ij}(t)\left|X_j + x_{0j}\right| + \mu_i(t) \\ &\le \sum_{j=1}^{n} \lambda_{ij}(t)\left|X_j\right| + \tilde{\mu}_i(t) \end{aligned} \tag{2.16}$$

for $(t,X) \in \tilde{R}^n_{a,A}$, $i = l+1,\dots,n$, where

$$\tilde{\mu}_i(t) = \sum_{j=1}^{l} \lambda_{ij}(t)\,|\,\tilde{\xi}_j\,| + \sum_{j=l+1}^{n} \lambda_{ij}(t)\,|\,x_{0j}\,| + \mu_i(t) \tag{2.17}$$

for $i = 1,2,\dots,n$. As $\tilde{\mu}_i \in L[[a,A],\mathbb{R}^+]$ holds, Kiguradze theorem implies that the boundary value problem

$$X' = f^*(t,X), \qquad X_i(\tau_i) = 0 \quad (i = 1,\dots,n) \tag{2.18}$$

has at least one solution $X(t) \in AC[[a,A],\mathbb{R}^n]$. Hence $x(t) = X(t) + x_0 - \xi$ is a solution of

$$\begin{gathered} x' = f(t,x), \qquad x_i(\tau_i) = \tilde{\xi}_i \quad (i = 1,\dots,l), \\ x_i(\tau_i) = x_{0i} \quad (i = l+1,\dots,n). \end{gathered} \tag{2.19}$$

Now we will prove that $\lim_{t\to t_i} x_i(t) = x_{0i}$ for $i = 1,\dots,l$. Put $m_i(t) = V_i(t,x(t))$, $m_{ij}(t) = V_{ij}(t,x(t))$ for $i = 1,\dots,l$ and $j = 1,\dots,k_i$. In view of (2.14), the inequality

$$m_i(t) < \varphi_i(t) \tag{2.20}$$

holds for $t \in (a,A)$ sufficiently close to B_i. Suppose for definiteness that $t_i < B_i$, that is, $\Delta(t_i,B_i) = (t_i,B_i)$ for some $i \in \{1,\dots,l\}$. We will show that $m_i(t) \le \varphi_i(t)$ for $t \in (t_i,B_i)$. Assume on the contrary that there is a $\tau \in (t_i,B_i)$ such that $m_i(\tau) \le \varphi_i(\tau)$ is not true. Since $x(t)$ is continuous and (2.20) holds for $t \in (a,A)$ sufficiently close to B_i, there exist $j \in \{1,\dots,k_i\}$ and an interval $J_i = (\tau_{i1},\tau_{i2})$ such that $\tau < \tau_{i1} < \tau_{i2} < B_i$,

$$\begin{gathered} m_{ij}(\tau_{i2}) = \varphi_{ij}(\tau_{i2}), \\ \varphi_{ij}(s) < m_{ij}(s) < \varphi_{ij}(s) + \varepsilon_{ij}(s) \quad \text{for } s \in J_i, \\ m_{ik}(s) < \varphi_{ik}(s) + \varepsilon_{ik}(s) \quad \text{for } s \in J_i,\ k = 1,\dots,k_i. \end{gathered} \tag{2.21}$$

Using (2.8), we get

$$m'_{ij}(s) \ge g_{ij}(s,\varphi_{i1}(s),\dots,\varphi_{i,j-1}(s),m_{ij}(s),\varphi_{i,j+1}(s),\dots,\varphi_{ik_i}(s)) \tag{2.22}$$

a.e. on J_i. As $g_{ij}(t,u_1,\dots,u_{j-1},\cdot,u_{j+1},\dots,u_n(s))$ is nondecreasing, we have

$$m'_{ij}(s) \ge g_{ij}(s,\varphi_{i1}(s),\dots,\varphi_{ik_i}(s)) = \varphi'_{ij}(s) \tag{2.23}$$

a.e. on J_i. Therefore, the function $m_{ij}(t) - \varphi_{ij}(t)$ is nondecreasing on J_i, which is a contradiction to $m_{ij}(\tau_{i2}) = \varphi_{ij}(\tau_{i2})$. Thus

$$0 \le m_i(t) \le \varphi_i(t) \quad \text{for } t \in (t_i,B_i). \tag{2.24}$$

Now the condition $\lim_{t\to t_i+} \varphi_i(t) = 0$ implies $\lim_{t\to t_i+} m_i(t) = 0$. With respect to the continuity of $x_i(t)$ on $[a,A]$, we have $x_i(t_i) = \lim_{t\to t_i} x_i(t) = x_{0i}$. The inequality (2.24) implies $V_i(t,x(t)) \le \varphi_i(t)$ for $t \in \Delta(t_i,B_i)$. □

Corollary 2.2. *Let $c_i \in \mathbb{R}$ $(i \in N)$, $B_i \in [a,A] \setminus \{t_i, \sigma_1, \dots, \sigma_p\}$ $(i \in I)$, a matrix function $\Lambda = (\lambda_{ij})_{i,j=1}^n \in N_0(a,A;\tau_1,\dots,\tau_n)$, and functions $\mu_i \in L[[a,A],\mathbb{R}^+]$ $(i \in N)$ be such that $c_i = x_{0i}$ for $i \in N \setminus I$ and condition (2.2) is fulfilled, where $\tau_i = t_i$ or $\tau_i = B_i$ whenever $i \in N \setminus I$ or $i \in I$, respectively.*

Assume that

(i) *there exist functions $g_i \in K_{a,A,t_i,B_i}[\hat{R}^1_{a,A},\mathbb{R}]$ $(i \in I)$ such that $\operatorname{sgn}(t-t_i)g_i(t,\cdot)$ are nondecreasing and there are solutions $\varphi_i(t)$ of*

$$u_i' = g_i(t,u_i) \tag{2.25}$$

satisfying (2.4);

(ii) *there are $z_i \in \widetilde{AC}[[a,A],\mathbb{R}]$ and $\varepsilon = (\varepsilon_{i_1},\dots,\varepsilon_{i_l}) \in C[(a,A),\mathbb{R}^{+^l}]$ such that $z_i(t_i) = x_{0i}$ $(i \in I)$ and the estimation*

$$\operatorname{sgn}(B_i - t_i)\operatorname{sgn}(x_i - z_i(t))\left(f_i(t,x) - z_i'(t)\right) \geq \operatorname{sgn}(B_i - t_i) g_i(t, |x_i - z_i(t)|)(i \in I) \tag{2.26}$$

is fulfilled on $\hat{\Omega} = \{(t,x) : \varphi_i(t) < |x_i - z_i(t)| < \varphi_i(t) + \varepsilon_i(t), t \in \Delta(t_i,B_i)\}$ for almost all $t \in \Delta(t_i,B_i)$. Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on $[a,A]$, either of which satisfies $|x_i(t) - z_i(t)| \leq \varphi_i(t)$ for $t \in \Delta(t_i,B_i)$ and $i \in I$.

Proof. Without loss of generality, it can be supposed that $I = \{1,\dots,l\}$ and $\operatorname{Pr} x = (x_1,\dots,x_l)$. Put $k_i = 1$ and $V_i(t,x(t)) = V_{i1}(t,x) = |x_i - z_i(t)|$ for $i = 1,\dots,l$. Then

$$\begin{aligned}
\operatorname{sgn}(B_i - t_i) V_{i1}'(t,x(t)) &\geq \operatorname{sgn}(B_i - t_i)\left(f_i(t,x(t)) - z_i'(t)\right)\operatorname{sgn}(x_i(t) - z_i(t)) \\
&\geq \operatorname{sgn}(B_i - t_i) g_i(t, |x_i(t) - z_i(t)|) \\
&= \operatorname{sgn}(B_i - t_i) g_i(t, V_{i1}(t,x(t)))
\end{aligned} \tag{2.27}$$

holds for any solution $x(t)$ of (2.1) a. e. on any interval $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ for which $\varphi_i(t) < V_i(t,x(t)) < \varphi_i(t) + \varepsilon_i(t)$ on $(\alpha_{i1},\alpha_{i2})$. The assumptions of Theorem 2.1 are satisfied. □

Example 2.3. Let $f_1,\dots,f_n \in K_0[\hat{R}^n_{0,1},\mathbb{R}]$ be such that

$$\begin{aligned}
f_1(t,x_1,\dots,x_n)\operatorname{sgn} x_1 &\geq \delta(t)|x_1|^\gamma, \\
-f_j(t,x_1,\dots,x_n)\operatorname{sgn} x_j &\leq \sum_{k=1}^{j} \lambda_{jk}(t)|x_k| + \mu_j(t) \quad (j = 2,\dots,n)
\end{aligned} \tag{2.28}$$

for $(t,x_1,\dots,x_n) \in \tilde{R}^n_{0,1}$, where $\gamma \in (0,1)$ and $\delta, \lambda_{jk}, \mu_j \in L[[0,1],\mathbb{R}^+]$, δ being a positive function. Consider the boundary value problem

$$\begin{aligned}
x_1' &= f_1(t,x_1,\dots,x_n), \quad x_1(0) = 0, \\
x_2' &= f_2(t,x_1,\dots,x_n), \quad x_2(1) = 0, \\
&\vdots \\
x_n' &= f_n(t,x_1,\dots,x_n), \quad x_n(1) = 0.
\end{aligned} \tag{2.29}$$

Put $t_1 = 0, t_2 = t_3 = \cdots = t_n = 1$,

$$g_1(t,u) = \begin{cases} \delta(t)u^\gamma & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \tag{2.30}$$

$\lambda_{1k}(t) \equiv 0$ $(k = 1,\dots,n)$, $\lambda_{jk}(t) \equiv 0$ $(j = 2,\dots,n; k = j+1,\dots,n)$, and $\mu_1(t) \equiv 0$. Let $B_1 = 1$. Then $\tau_1 = \tau_2 = \cdots = \tau_n = 1$,

$$\begin{gathered} f_1(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B_1)x_1] \leq 0, \\ f_j(t,x_1,\dots,x_n)\operatorname{sgn}[(t-1)x_j] \leq \sum_{k=1}^{n} \lambda_{jk}(t)|x_k| + \mu_j(t) \quad (j = 2,\dots,n), \end{gathered} \tag{2.31}$$

and the equation $u_1' = g_1(t,u)$ has a positive solution

$$\varphi_1(t) = \left[(1-\gamma)\int_0^t \delta(s)\,ds\right]^{1/(1-\gamma)} \tag{2.32}$$

in $(0,1]$ such that $\lim_{t\to 0}\varphi_1(t) = 0$. The assumptions of Corollary 2.2 are fulfilled with $I = \{1\}$, $c_1 = 0$, and $z(t) = z_1(t) \equiv 0$. Therefore, the considered boundary value problem has at least two different solutions on $[a,A]$. Moreover, the first component $x_1(t)$ of these solutions satisfies $|x_1(t)| \leq \varphi_1(t)$ for $t \in (0,1]$.

Corollary 2.4. *Suppose that $-\infty < a < A < \infty$, $c \in \mathbb{R}$, $\lambda \in L[[a,A],\mathbb{R}^+]$, and $\mu \in L[[a,A],\mathbb{R}^+]$. Let $B \in [a,A] \setminus \{t_n,\sigma_1,\dots,\sigma_p\}$ be such that*

$$\tilde{f}(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B)(x_n-c)] \leq \lambda(t)|x_n| + \mu(t) \tag{2.33}$$

for $(t,x) \in \tilde{R}^n_{a,A}$. Assume that

(i) *there exists a function $q \in K_{a,A,t_n,B}[\hat{R}^1_{a,A},\mathbb{R}]$ such that $\operatorname{sgn}(t-t_n)q(t,\cdot)$ is nondecreasing and there is a solution $\varphi(t)$ of*

$$u' = q(t,u) \tag{2.34}$$

satisfying

$$\varphi(t) > 0 \quad \text{for } t \in \Delta(t_n,B), \qquad \lim_{t\to t_n}\varphi(t) = 0, \qquad \liminf_{t\to B}\varphi(t) > 0; \tag{2.35}$$

(ii) *there are $z \in \widetilde{AC}[[a,A],\mathbb{R}]$ and $\varepsilon \in C[(a,A),\mathbb{R}^+]$ such that $z(t_n) = x_{0n}$ and*

$$\operatorname{sgn}(B-t_n)\operatorname{sgn}(x_n - z(t))\left(\tilde{f}(t,x_1,\dots,x_n) - z'(t)\right) \geq \operatorname{sgn}(B-t_n)q\left(t,|x_n - z(t)|\right) \tag{2.36}$$

holds on $\hat{\Omega} = \{(t,x_1,\dots,x_n) : \varphi(t) < |x_n - z(t)| < \varphi(t) + \varepsilon(t), t \in \Delta(t_n,B)\}$ *for almost all* $t \in \Delta(t_n,B)$. *Then the boundary value problem*

$$v^{(n)} = \tilde{f}(t,v,v',\dots,v^{(n-1)}),\\ v(t_1) = x_{01}, \quad v'(t_2) = x_{02},\dots, \quad v^{(n-1)}(t_n) = x_{0n} \tag{2.37}$$

has at least two different solutions on $[a,A]$.

Proof. Put $I = \{n\}$, $k_1 = 1$, $\Pr x = x_n$, $c_n = c$, $g_n(t,u) = q(t,u)$, $\varphi_n(t) = \varphi(t)$, $c_i = x_{0i}$ for $i = 1,\dots,n-1$, $\mu_i(t) = 0$ for $i = 1,\dots,n-1$, $\mu_n(t) = \mu(t)$, $B_n = B$, and

$$\lambda_{ij}(t) = \begin{cases} 1 & \text{for } 1 \le i = j-1 \le n-1, \\ \lambda(t) & \text{for } i = j = n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.38}$$

Considering the system

$$\begin{aligned} x_1' &= x_2, & x_1(t_1) &= x_{01}, \\ x_2' &= x_3, & x_2(t_2) &= x_{02}, \\ &\vdots & &\vdots \\ x_{n-1}' &= x_n, & x_{n-1}(t_{n-1}) &= x_{0n-1}, \\ x_n' &= \tilde{f}(t,x_1,x_2,\dots,x_n), & x_n(t_n) &= x_{0n}, \end{aligned} \tag{2.39}$$

and applying Corollary 2.2, we get

$$\begin{aligned} f_n(t,x_1,\dots,x_n)\operatorname{sgn}[(t-B_n)(x_n-c_n)] &\le \sum_{j=1}^{n} \lambda_{nj}(t)|x_j| + \mu_n(t), \\ f_i(t,x_1,\dots,x_n)\operatorname{sgn}[(t-t_i)(x_i-c_i)] \le |x_{i+1}| &\le \lambda_{i,i+1}|x_{i+1}| \\ &= \sum_{j=1}^{n} \lambda_{ij}(t)|x_j| + \mu_i(t) \end{aligned} \tag{2.40}$$

for $i = 1,\dots,n-1$. The result follows from Corollary 2.2. □

Example 2.5. Let $\gamma \in (0,1)$. Consider the boundary value problem

$$v'' = p_1(t,v)|v'|^{\gamma}\operatorname{sgn} v' + p_2(t,v,v'), \qquad v(0) = 0, \qquad v'(1) = 0, \tag{2.41}$$

where $p_1 \in K_1[\hat{R}^1_{0,1},\mathbb{R}]$ and $p_2 \in K_1[\hat{R}^2_{0,1},\mathbb{R}]$ are such that

$$x_2 p_2(t,x_1,x_2) \le 0 \quad \text{for } (t,x_1,x_2) \in (0,1)\times\mathbb{R}^2,\\ p_1(t,x_1) \le -\delta(t) \quad \text{for } (t,x_1) \in (0,1)\times\mathbb{R}, \tag{2.42}$$

$\delta \in L[[0,1],\mathbb{R}]$ being a positive function. Since

$$-p_1(t,x_1)|x_2|^{\gamma} - p_2(t,x_1,x_2)\operatorname{sgn} x_2 \ge \delta(t)|x_2|^{\gamma}, \tag{2.43}$$

the assumptions of Corollary 2.4 are fulfilled with $n = 2$, $a = 0$, $A = 1$, $t_1 = 0$, $t_2 = 1$, $c = 0$, $B = 0$, $z(t) \equiv 0$, $\lambda(t) \equiv 0$, $\mu(t) \equiv 0$, and

$$q(t,u) = \begin{cases} -\delta(t)u^{\gamma} & \text{for } u \geq 0, \\ 0 & \text{for } u < 0, \end{cases} \qquad \varphi(t) = \left[(1-\gamma)\int_t^1 \delta(s)\,ds\right]^{1/1-\gamma}. \tag{2.44}$$

Therefore, problem (2.41) has at least two different solutions on $[0,1]$.

Corollary 2.6. *Let the assumptions of Corollary 2.2 be fulfilled with the exception that the conditions (i), (ii) are replaced by (i′), (ii′):*

(i′) *there exist functions* $h_i, q_i \in K_{a,A,t_i,B_i}[\hat{R}^1_{a,A}, \mathbb{R}]$ $(i \in I)$ *such that functions* $\operatorname{sgn}(t - t_i)h_i(t,\cdot)$ *and* $\operatorname{sgn}(t-t_i)q_i(t,\cdot)$ *are nondecreasing for* $i \in I$ *and there are solutions* $\varphi_i(t)$, $\psi_i(t)$ *of* $u_i' = h_i(t,u_i)$ *and* $v_i' = q_i(t,v_i)$, *respectively, satisfying*

$$\begin{aligned} &\varphi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t\to t_i}\varphi(t) = 0, \qquad \liminf_{t\to B_i}\varphi(t) > 0, \\ &\psi_i(t) > 0 \quad \text{for } t \in \Delta(t_i,B_i), \qquad \lim_{t\to t_i}\psi(t) = 0, \qquad \liminf_{t\to B_i}\psi(t) > 0 \end{aligned} \tag{2.45}$$

for $t \in I$;

(ii′) *there are* $z_i \in \widetilde{AC}[[a,A],\mathbb{R}]$ *and* $\varepsilon = (\varepsilon_{i_1},\ldots,\varepsilon_{i_l}) \in C[(a,A),\mathbb{R}^{+l}]$ *such that* $z_i(t_i) = x_{0i}$ *and the inequalities*

$$\begin{aligned} \operatorname{sgn}(B_j - t_j)\left[\left(f_j(t,x) - z_j'(t)\right) - h_j\left(t,(x_j - z_j(t))_+\right)\right] \geq 0 \quad (j \in I) \\ \operatorname{sgn}(B_j - t_j)\left[-\left(f_j(t,x) - z_j'(t)\right) - q_j\left(t,(x_j - z_j(t))_-\right)\right] \geq 0 \quad (j \in I) \end{aligned} \tag{2.46}$$

are fulfilled on $\hat{\Omega} = \{(t,x) : \varphi_j(t) < x_j - z_j(t) < \varphi_j(t) + \varepsilon_j(t), t \in \Delta(t_j,B_j)\}$ *and* $\hat{\hat{\Omega}} = \{(t,x) : \psi_j(t) < z_j(t) - x_j < \psi_j(t) + \varepsilon_j(t), t \in \Delta(t_j,B_j)\}$, *respectively, for almost all* $t \in \Delta(t_j,B_j)$. *Then the Cauchy-Nicoletti boundary value problem (2.1) has at least two different solutions on* $[a,A]$.

Proof. Without loss of generality, it can again be assumed that $I = \{1,\ldots,l\}$ and $\operatorname{Pr} x = (x_1,\ldots,x_l)$. Put $k_i = 2$, $g_{i1}(t,u) = h_i(t,u)$, $g_{i2}(t,v) = q_i(t,v)$, $\varphi_{i1}(t) = \varphi_i(t)$, $\varphi_{i2}(t) = \psi_i(t)$, $V_{i1}(t,x) = (x_i - z_i(t))_+$, $V_{i2}(t,x) = (x_i - z_i(t))_-$, and $V_i(t,x) = (V_{i1}(t,x), V_{i2}(t,x))$ for $i \in I$. Then we have

$$\begin{aligned} \operatorname{sgn}(B_i - t_i)V_{i1}'(t,x(t)) &\geq \operatorname{sgn}(B_i - t_i)\left(f_i(t,x(t)) - z_i'(t)\right) \\ &\geq \operatorname{sgn}(B_i - t_i)g_{i1}\left(t, V_{i1}(t,x(t))\right), \\ \operatorname{sgn}(B_i - t_i)V_{i2}'(t,x(t)) &\geq -\operatorname{sgn}(B_i - t_i)\left(f_i(t,x(t)) - z_i'(t)\right) \\ &\geq \operatorname{sgn}(B_i - t_i)g_{i2}\left(t, V_{i2}(t,x(t))\right) \end{aligned} \tag{2.47}$$

for any solution $x = x(t)$ of (2.1) a.e. on any interval $(\alpha_{i1},\alpha_{i2}) \subseteq \Delta(t_i,B_i)$ for which

$$V_{i1}(t,x(t)) < \varphi_i(t) + \varepsilon_i(t), \qquad V_{i2}(t,x(t)) < \psi_i(t) + \varepsilon_i(t) \tag{2.48}$$

on $(\alpha_{i1}, \alpha_{i2})$, $i = 1, \ldots, l$, and

$$V_{i1}(t, x(t)) > \varphi_i(t) \quad \text{or} \quad V_{i2}(t, x(t)) > \psi_i(t) \tag{2.49}$$

on $(\alpha_{i1}, \alpha_{i2})$, respectively. The statement follows from Theorem 2.1. □

Corollary 2.7. *Let the assumptions of Corollary 2.4 be fulfilled with the exception that conditions (i), (ii) are replaced by the following:*

(i′) *there exist functions* $h \in K_{a,A,t_n,B}[\hat{R}^1_{a,A}, \mathbb{R}]$ *and* $q \in K_{a,A,t_n,B}[\hat{R}^1_{a,A}, \mathbb{R}]$ *such that* $\operatorname{sgn}(t - t_n)h(t, \cdot)$ *and* $\operatorname{sgn}(t - t_n)q(t, \cdot)$ *are nondecreasing and there are solutions* $\varphi(t)$, $\psi(t)$ *of* $u' = h(t, u)$ *and* $v' = q(t, v)$, *respectively, satisfying*

$$\begin{gathered} \varphi(t) > 0, \quad \psi(t) > 0 \quad \text{for } t \in \Delta(t_n, B), \quad \lim_{t \to t_n} \varphi(t) = \lim_{t \to t_n} \psi(t) = 0, \\ \liminf_{t \to B} \varphi(t) > 0, \qquad \liminf_{t \to B} \psi(t) > 0; \end{gathered} \tag{2.50}$$

(ii′) *there are* $z \in \widetilde{AC}[[a, A], \mathbb{R}]$ *and* $\varepsilon \in C[(a, A), \mathbb{R}^+]$ *such that* $z(t_n) = x_{0n}$ *and*

$$\begin{gathered} \operatorname{sgn}(B - t_n)\big[\tilde{f}(t, x_1, \ldots, x_n) - z'(t) - h(t, (x_n - z(t))_+)\big] \ge 0, \\ \operatorname{sgn}(B - t_n)\big[-\tilde{f}(t, x_1, \ldots, x_n) + z'(t) - q(t, (x_n - z(t))_-)\big] \ge 0 \end{gathered} \tag{2.51}$$

hold on $\hat{\Omega} = \{(t, x_1, \ldots, x_n) : \varphi(t) < x_n - z(t) < \varphi(t) + \varepsilon(t),\ t \in \Delta(t_n, B)\}$ *and* $\hat{\hat{\Omega}} = \{(t, x_1, \ldots, x_n) : \psi(t) < z(t) - x_n < \psi(t) + \varepsilon(t),\ t \in \Delta(t_n, B)\}$, *respectively, for almost all* $t \in \Delta(t_n, B)$. *Then the Cauchy-Nicoletti boundary value problem (2.37) has at least two different solutions on* $[a, A]$.

Proof. Corollary 2.7 follows from Corollary 2.6 in the same way as Corollary 2.4 follows from Corollary 2.2. □

Example 2.8. Let $p_1 \in K_1[\hat{R}^2_{0,1}, \mathbb{R}]$ and $p_2 \in K_1[\hat{R}^2_{0,1}, \mathbb{R}]$ be such that

$$\begin{gathered} p_1(t, x_1, x_2) \le -\delta_1(t)\vartheta_1(x_2) \quad \text{for } (t, x_1, x_2) \in (0, 1) \times \mathbb{R} \times (0, \infty), \\ p_1(t, x_1, x_2) \ge \delta_2(t)\vartheta_2(|x_2|) \quad \text{for } (t, x_1, x_2) \in (0, 1) \times \mathbb{R} \times (-\infty, 0), \\ x_2 p_2(t, x_1, x_2) \le 0 \quad \text{for } (t, x_1, x_2) \in (0, 1) \times \mathbb{R}^2, \end{gathered} \tag{2.52}$$

where δ_1, δ_2 are positive functions such that $\delta_j \in L[[0, 1], \mathbb{R}]$ and $\vartheta_j \in C[[0, \infty), \mathbb{R}^+]$ $(j = 1, 2)$ are nondecreasing and positive on $(0, \infty)$ and satisfying $\vartheta_1(0) = \vartheta_2(0) = 0$, $\int_0^1 \delta_1(s)\, ds < \int_0^\infty 1/\vartheta_1(s)\, ds < \infty$, and $\int_0^1 \delta_2(s)\, ds < \int_0^\infty 1/\vartheta_2(s)\, ds < \infty$.

Consider the boundary value problem

$$w'' = p_1(t, w, w') + p_2(t, w, w'), \qquad w(0) = 0, \qquad w'(1) = 0. \tag{2.53}$$

It holds that

$$
\begin{aligned}
&-\big[p_1(t,x_1,x_2)+p_2(t,x_1,x_2)+\delta_1(t)\vartheta_1(x_2)\big]\geq 0\\
&\qquad\text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(0,\infty),\\
&-\big[-p_1(t,x_1,x_2)-p_2(t,x_1,x_2)+\delta_2(t)\vartheta_2(-x_2)\big]\geq 0\\
&\qquad\text{for } (t,x_1,x_2)\in(0,1)\times\mathbb{R}\times(-\infty,0).
\end{aligned}
\tag{2.54}
$$

The problems

$$
\begin{aligned}
u' &= -\delta_1(t)\vartheta_1(u), \qquad u(1)=0,\\
v' &= -\delta_2(t)\vartheta_2(v), \qquad v(1)=0
\end{aligned}
\tag{2.55}
$$

have positive solutions on $[0,1)$ and condition (2.54) implies

$$
\big[p_1(t,x_1,x_2)+p_2(t,x_1,x_2)\big]\operatorname{sgn}x_2\leq 0. \tag{2.56}
$$

Therefore, the assumptions of Corollary 2.7 are fulfilled with $a=0$, $A=1$, $c=0$, $z(t)\equiv 0$, $B=0$, $t_1=0$, $t_2=1$, $\lambda(t)\equiv 0$, $\mu(t)\equiv 0$, and

$$
\begin{aligned}
h(t,u) &= \begin{cases} -\delta_1(t)\vartheta_1(u) & \text{for } (t,u)\in(0,1)\times(0,\infty),\\ 0 & \text{for } (t,u)\in(0,1)\times(-\infty,0], \end{cases}\\
q(t,v) &= \begin{cases} -\delta_2(t)\vartheta_2(v) & \text{for } (t,v)\in(0,1)\times(0,\infty),\\ 0 & \text{for } (t,v)\in(0,1)\times(-\infty,0]. \end{cases}
\end{aligned}
\tag{2.57}
$$

Hence problem (2.53) has at least two solutions on $[0,1]$.

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