ON THE SOLUTIONS OF NONLINEAR INITIAL-BOUNDARY VALUE PROBLEMS

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We deal with the general initial-boundary value problem for a second-order nonlinear nonstationary evolution equation. The associated operator equation is studied by the Fredholm and Nemitskii operator theory. Under local Hölder conditions for the nonlinear member, we observe quantitative and qualitative properties of the set of solutions of the given problem. These results can be applied to different mechanical and natural science models.

1. Introduction

The generic properties of solutions of the second-order ordinary differential equations were studied by Brüll and Mawhin in [2], Mawhin in [7], and by Šeda in [8]. Such questions were solved for nonlinear diffusional-type problems with the Dirichlet-, Neumann-, and Newton-type conditions in [5, 6].

In this paper, we study the set structure of classic solutions, bifurcation points and the surjectivity of an associated operator to a general second-order nonlinear evolution problem by the Fredholm operator theory. The present results allow us to search the generic properties of nonparabolic models which describe mechanical, physical, reaction-diffusion, and ecology processes.

2. The formulation of the problem and basic notions

Throughout this paper, we assume that the set $\Omega \subset \mathbb{R}^n$ for $n \in \mathbb{N}$ is a bounded domain with the sufficiently smooth boundary $\partial\Omega$. The real number T is positive and $Q := (0,T] \times \Omega$, $\Gamma := (0,T] \times \partial\Omega$.

We use the notation D_t for $\partial/\partial t$, D_i for $\partial/\partial x_i$, D_{ij} for $\partial^2/\partial x_i \partial x_j$, where $i, j = 1,\dots,n$, and $D_0 u$ for u. The symbol $\operatorname{cl} M$ means the closure of a set M in $\mathbb{R}^n$.

We consider the nonlinear differential equation (possibly of a nonparabolic type)

$$D_t u - A(t,x,D_x)u + f(t,x,u,D_1 u,\dots,D_n u) = g(t,x) \tag{2.1}$$

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for $(t,x) \in Q$, where the coefficients a_{ij}, a_i, a_0, for $i,j = 1,\dots,n$, of the second-order linear operator

$$A(t,x,D_x)u = \sum_{i,j=1}^{n} a_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} a_i(t,x)D_i u + a_0(t,x)u \tag{2.2}$$

are continuous functions from the space $C(\operatorname{cl} Q,\mathbb{R})$. The function f is from the space $C(\operatorname{cl} Q \times \mathbb{R}^{n+1},\mathbb{R})$ and $g \in C(\operatorname{cl} Q,\mathbb{R})$.

Together with (2.1), we consider the following general homogeneous boundary condition:

$$B_3(t,x,D_x)u|_\Gamma := \sum_{i=1}^{n} b_i(t,x)D_i u + b_0(t,x)u|_\Gamma = 0, \tag{2.3}$$

where the coefficients b_i, for $i = 1,\dots,n$, and b_0 are continuous functions from $C(\operatorname{cl}\Gamma,\mathbb{R})$.

Furthermore, we require for the solution of (2.1) to satisfy the homogeneous initial condition

$$u|_{t=0} = 0 \quad \text{on } \operatorname{cl}\Omega. \tag{2.4}$$

Remark 2.1. In the case where $b_i = 0$, for $i = 1,\dots,n$, and $b_0 = 1$ in (2.3), we get the Dirichlet problem studied in [5].

If we consider the vector function $\nu := (0,\nu_1,\dots,\nu_n) : \operatorname{cl}\Gamma \to \mathbb{R}^{n+1}$ and the value $\nu(t,x)$ which means the unit inner normal vector to $\operatorname{cl}\Gamma$ at the point $(t,x) \in \operatorname{cl}\Gamma$ and we let $b_i = \nu_i$ for $i = 1,\dots,n$ on $\operatorname{cl}\Gamma$, then problem (2.1), (2.3), (2.4) represents the Newton or Neuman problem investigated in [6].

Our considerations are concerned with a broad class of nonparabolic operators.

In the following definitions, we will use the notations

$$\begin{aligned}
\langle u\rangle^{s}_{t,\mu,Q} &:= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\mu}},\\
\langle u\rangle^{y}_{x,\nu,Q} &:= \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{|u(t,x)-u(t,y)|}{|x-y|^{\nu}},\\
\langle f\rangle^{s,y,v}_{t,x,u} &:= |f(t,x,u_0,u_1,\dots,u_n) - f(s,y,v_0,v_1,\dots,v_n)|,\\
\langle f\rangle^{s,y,v(s,y)}_{t,x,u(t,x)} &:= |f[t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)],\\
&\qquad - f[s,y,v(s,y),D_1v(s,y),\dots,D_nv(s,y)]|,
\end{aligned} \tag{2.5}$$

where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ are from $\mathbb{R}^n$, $|x-y| = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$, and $\mu,\nu \in \mathbb{R}$.

We will need the following Hölder spaces (see [4, page 147]).

Definition 2.2. Let $\alpha \in (0,1)$.

(1) By the symbol $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ we denote the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ which have continuous derivatives $D_i u$ for $i = 1,\dots,n$ on $\mathrm{cl}\,Q$, and the norm

$$\begin{aligned} \|u\|_{(1+\alpha)/2,1+\alpha,Q} := {} & \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \langle u \rangle^{s}_{t,(1+\alpha)/2,Q} \\ & + \sum_{i=1}^{n} \langle D_i u \rangle^{s}_{t,\alpha/2,Q} + \sum_{i=1}^{n} \langle D_i u \rangle^{y}_{x,\alpha/2,Q} \end{aligned} \tag{2.6}$$

is finite.

(2) The symbol $C_{(t,x)}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ means the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ for which there exist continuous derivatives $D_t u$, $D_i u$, $D_{ij} u$ on $\mathrm{cl}\,Q$, $i, j = 1,\dots,n$, and the norm

$$\begin{aligned} \|u\|_{(2+\alpha)/2,2+\alpha,Q} = {} & \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_t u(t,x) \right| \\ & + \sum_{i,j=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ij} u(t,x) \right| + \sum_{i=1}^{n} \langle D_i u \rangle^{s}_{t,(1+\alpha)/2,Q} + \langle D_t u \rangle^{s}_{t,\alpha/2,Q} \\ & + \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{s}_{t,\alpha/2,Q} + \langle D_t u \rangle^{y}_{x,\alpha,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{y}_{x,\alpha,Q} \end{aligned} \tag{2.7}$$

is finite.

(3) The symbol $C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ means the vector space of continuous functions $u : \mathrm{cl}\,Q \to \mathbb{R}$ for which the derivatives D_t, $D_i u$, $D_t D_i u$, $D_{ij} u$, $D_{ijk} u$, $i, j, k = 1,\dots,n$, are continuous on $\mathrm{cl}\,Q$, and the norm

$$\begin{aligned} \|u\|_{(3+\alpha)/2,3+\alpha,Q} := {} & \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i u(t,x) \right| + \sum_{i,j=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ij} u(t,x) \right| \\ & + \sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_t D_i u(t,x) \right| + \sum_{i,j,k=1}^{n} \sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_{ijk} u(t,x) \right| \\ & + \langle D_t u \rangle^{s}_{t,(1+\alpha)/2,Q} + \sum_{i,j=1}^{n} \langle D_{ij} u \rangle^{s}_{t,(1+\alpha)/2,Q} \\ & + \sum_{i=1}^{n} \langle D_t D_i u \rangle^{s}_{t,\alpha/2,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u \rangle^{s}_{t,\alpha/2,Q} \\ & + \sum_{i=1}^{n} \langle D_t D_i u \rangle^{y}_{x,\alpha,Q} + \sum_{i,j,k=1}^{n} \langle D_{ijk} u \rangle^{y}_{x,\alpha,Q} \end{aligned} \tag{2.8}$$

is finite.

The above-defined norm spaces are Banach ones.

Definition 2.3 (the smoothness condition $(S_3^{1+\alpha})$). Let $\alpha \in (0,1)$. The differential operators $A(t,x,D_x)$ from (2.1) and $B_3(t,x,D_x)$ from (2.3) satisfy *the smoothness condition* $(S_3^{1+\alpha})$ if, respectively,

(i) the coefficients a_{ij}, a_i, a_0 from (2.1), for $i,j = 1,\dots,n$, belong to the space $C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ and $\partial\Omega \in C^{3+\alpha}$,

(ii) the coefficients b_i from (2.3), for $i = 1,\dots,n$, belong to the space $C_{t,x}^{(2+\alpha)/2,2+\alpha}(\mathrm{cl}\,\Gamma,\mathbb{R})$.

Definition 2.4 (the complementary condition (C)). If at least one of the coefficients b_i, for $i = 1,\dots,n$, of the differential operator $B_3(t,x,D_x)$ in (2.3) is not zero, then $B_3(t,x,D_x)$ satisfies the *complementary condition (C)*.

Now, we are prepared to formulate hypotheses for deriving fundamental lemmas.

Definition 2.5. (1) Fredholm conditions.

(A1) Consider the operator $A_3 : X_3 \to Y_3$, where

$$A_3 u = D_t u - A(t,x,D_x)u, \quad u \in X_3, \tag{2.9}$$

and the operators $A(t,x,D_x)$ and $B_3(t,x,D_x)$ satisfy the smoothness condition $(S_3^{1+\alpha})$ for $\alpha \in (0,1)$ and the complementary condition (C). Here, we consider the vector spaces

$$\begin{aligned}
D(A_3) &:= \{u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\mathrm{cl}\,Q,\mathbb{R});\ B_3(t,x,D_x)u|_\Gamma = 0,\ u|_{t=0}(x) = 0 \text{ for } x \in \mathrm{cl}\,Q\},\\
H(A_3) &:= \{v \in C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R});\ B_3(t,x,D_x)v(t,x)|_{t=0,x\in\partial\Omega} = 0\}
\end{aligned} \tag{2.10}$$

and Banach subspaces (of the given Hölder spaces)

$$\begin{aligned}
X_3 &= \left(D(A_3), \|\cdot\|_{(3+\alpha)/2,3+\alpha,Q}\right),\\
Y_3 &= \left(H(A_3), \|\cdot\|_{(1+\alpha)/2,1+\alpha,Q}\right).
\end{aligned} \tag{2.11}$$

(A2) There is a second-order linear homeomorphism $C_3 : X_3 \to Y_3$ with

$$C_3 u = D_t u - C(t,x,D_x)u, \quad u \in X_3, \tag{2.12}$$

where

$$C(t,x,D_x)u = \sum_{i,j=1}^{n} c_{ij}(t,x)D_{ij}u + \sum_{i=1}^{n} c_i(t,x)D_i u + c_0(t,x)u, \tag{2.13}$$

satisfying the smoothness condition $(S_3^{1+\alpha})$. The operator C_3 is not necessarily a parabolic one.

(2) Local Hölder and compatibility conditions.

Let $f := f(t,x,u_0,u_1,\dots,u_n) : \operatorname{cl} Q \times \mathbb{R}^{n+1} \to \mathbb{R}$, $\alpha \in (0,1)$, and let p, q, p_r, for $r = 0,1,\dots,n$ be nonnegative constants. Here, D represents any compact subset of $(\operatorname{cl} Q) \times \mathbb{R}^{n+1}$. For f, we need the following assumptions:

(B1) let $f \in C^1(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$ and let the first derivatives $\partial f/\partial x_i$, $\partial f/\partial u_j$ be locally Hölder continuous on $\operatorname{cl} Q \times \mathbb{R}^{n+1}$ such that

$$\begin{aligned} \left\langle \frac{\partial f}{\partial x_i} \right\rangle_{t,x,u}^{s,y,v} &\leq p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r \left| u_r - v_r \right|, \\ \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{t,x,u}^{s,y,v} &\leq p|t-s|^{\alpha/2} + q|x-y|^{\alpha} + \sum_{r=0}^{n} p_r \left| u_r - v_r \right|, \end{aligned} \tag{2.14}$$

for $i = 1,\dots,n$, $j = 0,1,\dots,n$, and any D;

(B2) let $f \in C^3(\operatorname{cl} Q \times \mathbb{R}^{n+1}, \mathbb{R})$ and let the local growth conditions for the third derivatives of f hold on any D:

$$\begin{aligned} \left\langle \frac{\partial^3 f}{\partial \tau \partial x_i \partial u_j} \right\rangle_{t,x,u}^{t,x,v} &\leq \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial \tau \partial u_j \partial u_k} \right\rangle_{t,x,u}^{t,x,v} &\leq \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial x_i \partial x_l \partial u_j} \right\rangle_{t,x,u}^{t,x,v} &\leq \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial x_i \partial u_j \partial u_k} \right\rangle_{t,x,u}^{t,x,v} &\leq \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \\ \left\langle \frac{\partial^3 f}{\partial u_j \partial u_k \partial u_r} \right\rangle_{t,x,u}^{t,x,v}, &\leq \sum_{s=0}^{n} p_s \left| u_s - v_s \right|^{\beta_s}, \end{aligned} \tag{2.15}$$

where $\beta_s > 0$ for $s = 0,1,\dots,n$ and $i,l = 1,\dots,n$; $j,k,r = 0,1,\dots,n$;

(B3) the equality of compatibility

$$\sum_{i=1}^{n} b_i(t,x) D_i f(t,x,0,\dots,0) + b_0(t,x) f(t,x,0,\dots,0)|_{t=0,x\in\partial\Omega} = 0 \tag{2.16}$$

holds.

(3) Almost coercive condition.

Let, for any bounded set $M_3 \subset Y_3$, there exist a number $K > 0$ such that for all solutions $u \in X_3$ of problem (2.1), (2.3), (2.4) with the right-hand sides $g \in M_3$, the following alternative holds:

(C1) either

(α_1) $\|u\|_{(1+\alpha)/2,1+\alpha,Q} \leq K$, $f := f(t,x,u_0) : \mathrm{cl}\,Q \times \mathbb{R} \to \mathbb{R}$, and the coefficients of the operators A_3 and C_3 (see (2.1) and (A2)) satisfy the equations

$$a_{ij} = c_{ij}, \quad a_i = c_i, \quad \text{for } i,j = 1,\dots,n, \qquad a_0 \neq c_0 \quad \text{on } \mathrm{cl}\,Q, \tag{2.17}$$

or

(α_2) $\|u\|_{(2+\alpha)/2,2+\alpha,Q} \leq K$, $f := f(t,x,u_0,u_1,\dots,u_n) : \mathrm{cl}\,Q \times \mathbb{R}^{n+1} \to \mathbb{R}$, and the coefficients of the operators A_3 and C_3 satisfy the relations

$$a_{ij} = c_{ij} \quad \text{for } i,j = 1,\dots,n, \qquad a_i \neq c_i \quad \text{for at least one } i = 1,\dots,n \tag{2.18}$$

on $\mathrm{cl}\,Q$.

Remark 2.6. (1) Especially, condition (A2) is satisfied for the diffusion operator

$$C_3 u = D_t u - \triangle u, \quad u \in X_3, \tag{2.19}$$

or for any uniformly parabolic operator C_3 with sufficiently smooth coefficients. However, the operator C_3 is not necessarily uniform parabolic.

(2) The local Hölder conditions in (B1) and (B2) admit sufficiently strong growths of f in the last variables $u_0, u_1, \dots, u_n$. For example, they include exponential and power-type growths.

Definition 2.7. (1) A couple $(u,g) \in X_3 \times Y_3$ will be called *the bifurcation point of the mixed problem* (2.1), (2.3), (2.4) if u is a solution of that mixed problem and there exists a sequence $\{g_k\} \subset Y_3$ such that $g_k \to g$ in Y_3 as $k \to \infty$, and problem (2.1), (2.3), (2.4) for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in \mathbb{N}$ and $u_k \to u$, $v_k \to u$ in X_3 as $k \to \infty$.

(2) The set of all solutions $u \in X_3$ of (2.1), (2.3), (2.4) (or the set of all functions $g \in Y_3$) such that (u,g) is a bifurcation point of problem (2.1), (2.3), (2.4) will be called *the domain of bifurcation (the bifurcation range)* of that problem.

3. Fundamental lemmas

Lemma 3.1. *Let conditions (A1) and (A2) hold (see Definition 2.5). Then,*

(1) $\dim X_3 = +\infty$;

(2) *the operator* $A_3 : X_3 \to Y_3$ *is a linear bounded Fredholm operator of the zero index.*

Proof. (1) To prove the first part of this lemma, we use the decomposition theorem from [9, page 139].

Let X be a linear space and let $x^* : X \to \mathbb{R}$ be a linear functional on X such that $x^* \neq 0$. Furthermore, let $M = \{x \in X;\ x^*(x) = 0\}$ and let $x_0 \in X - M$. Then, every element $x \in X$ can be expressed by the formula

$$x = \left[\frac{x^*(x)}{x^*(x_0)}\right] x_0 + m, \quad m \in M, \tag{3.1}$$

that is, there is a one-dimensional subspace L_1 of X such that $X = L_1 \oplus M$.

If we now let

$$M_1 := \left\{u \in C_{t,x}^{(3+\alpha)/2,3+\alpha}(\operatorname{cl} Q, \mathbb{R}) =: H^{3+\alpha};\ B_3(t,x,D_x)u|_\Gamma = 0\right\}, \tag{3.2}$$

which is the linear subspace of $H^{3+\alpha}$, then there exists a linear subspace L_1 of $H^{3+\alpha}$ with $\dim L_1 = 1$ such that $H^{3+\alpha} = L_1 \oplus M_1$. Similarly, if we take $M_2 := \{u \in M_1;\ u|_{t=0} = 0 \text{ on } \operatorname{cl}\Omega\}$, then there is a subspace L_2 of M_1 with $\dim L_2 = 1$ such that $M_1 = L_2 \oplus M_2$. Hence, we have $H^{3+\alpha} = L_1 \oplus L_2 \oplus D(A_3)$. Since $\dim H^{3+\alpha} = +\infty$, we get that $\dim X_3 = +\infty$.

(2) (a) In the first step, we prove the boundedness of the linear operator A_3. To this end, we observe the norm $\|A_3u\|_{(1+\alpha)/2,1+\alpha,Q}$ for $u \in D(A_3)$. From the assumption $(\mathrm{S}_3^{1+\alpha})$ we get for $k = 0,1,\dots,n$,

$$\sup_{(t,x)\in \operatorname{cl} Q} \left| D_k A_3 u(t,x) \right| \le K_1 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_1 > 0. \tag{3.3}$$

Applying again the smoothness assumption $(\mathrm{S}_3^{1+\alpha})$, the mean value theorem for the functions u and D_iu, and the boundedness of Q, we obtain for the second member of the above-mentioned norm the following estimation:

$$\begin{aligned} \langle A_3u\rangle^s_{t,(1+\alpha)/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{\left|A_3u(t,x) - A_3u(s,x)\right|}{|t-s|^{(1+\alpha)/2}} \\ &\le K_2 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_2 > 0. \end{aligned} \tag{3.4}$$

For the third member of the norm (2.6), we estimate for $k = 1,\dots,n$ as follows:

$$\begin{aligned} \langle D_kA_3u\rangle^s_{t,\alpha/2,Q} &= \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{\left|D_kA_3u(t,x) - D_kA_3u(s,x)\right|}{|t-s|^{\alpha/2}} \\ &\le K_3 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_3 > 0. \end{aligned} \tag{3.5}$$

An estimation of the last member in (2.6) for A_3u is given by the following inequality for $k = 1,\dots,n$:

$$\begin{aligned} \langle D_kA_3u\rangle^y_{x,\alpha/2,Q} &= \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{\left|D_kA_3u(t,x) - D_kA_3u(t,y)\right|}{|x-y|^{\alpha/2}} \\ &\le K_4 \|u\|_{(3+\alpha)/2,3+\alpha,Q}, \quad K_4 > 0. \end{aligned} \tag{3.6}$$

From the estimations (3.3), (3.4), (3.5), and (3.6), we can conclude that

$$\|A_3u\|_{Y_3} = \|A_3u\|_{(1+\alpha)/2,1+\alpha,Q} \le K(n,T,\alpha,\Omega,a_{ij},a_i,a_0) \|u\|_{X_3}. \tag{3.7}$$

(b) To prove that A_3 is a Fredholm operator with the zero index, we express it in the form

$$A_3u = C_3u + \left[C(t,x,D_x) - A(t,x,D_x)\right]u =: C_3u + T_3u, \tag{3.8}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism and C is the linear operator from (A2). By the decomposition Nikol̆skii theorem [10, page 233], it is sufficient to show that $T_3 : X_3 \to Y_3$ is a linear completely continuous operator.

The complete continuity of T_3 can be proved by the Ascoli-Arzelá theorem (see [11, page 141]).

From $(\mathrm{S}_3^{1+\alpha})$, the uniform boundedness of the operator

$$T_3u = \sum_{i,j=1}^{n} [c_{ij}(t,x) - a_{ij}(t,x)]D_{ij}u + \sum_{i=1}^{n} [c_i(t,x) - a_i(t,x)]D_iu \\ + [c_0(t,x) - a_0(t,x)]u \tag{3.9}$$

follows by the same way as the boundedness of the operator A_3 in the previous part (1). Thus, for all $u \in M \subset X_3$, where M is a set bounded by the constant $K_1 > 0$, we obtain the estimate

$$||T_3u||_{Y_3} \le K(n,\alpha T,\Omega,a_{ij},c_{ij},a_i,c_i,a_0,c_0)\|u\|_{X_3} \le KK_1. \tag{3.10}$$

Using the smoothness condition of the operators A and C, we get the inequalities

$$\begin{aligned}
|T_3u(t,x) - T_3u(s,y)| \le{}& \sum_{i,j=1}^{n} |[c_{ij} - a_{ij}](t,x) - [c_{ij} - a_{ij}](s,y)|\,|D_{ij}u(t,x)| \\
&+ \sum_{i,j=1}^{n} |c_{ij}(s,y) - a_{ij}(s,y)|\,|D_{ij}u(t,x) - D_{ij}u(s,y)| \\
&+ \sum_{i=1}^{n} |[c_i - a_i](t,x) - [c_i - a_i](s,y)|\,|D_iu(t,x)| \\
&+ \sum_{i=1}^{n} |c_i(s,y) - a_i(s,y)|\,|D_iu(t,x) - D_iu(s,y)| \\
&+ |[c_0 - a_0](t,x) - [c_0 - a_0](s,y)|\,|u(t,x)| \\
&+ |c_0(s,y) - a_0(s,y)|\,|u(t,x) - u(s,y)| \\
\le{}& 4K_1Kn^2[|t-s|^{\alpha/2} + |x-y|^{\alpha}] \\
&+ 2K_1Kn[(|t-s|^{\alpha/2} + |x-y|^{\alpha}) + (|t-s|^{(1+\alpha)/2} + |x-y|)] \\
&+ 2K_1K[(|t-s|^{\alpha/2} + |x-y|^{\alpha}) + (|t-s| + |x-y|)],
\end{aligned} \tag{3.11}$$

where K_1, K are positive constants. Hence, the equicontinuity of $T_3M \subset Y_3$ follows. This finishes the proof of Lemma 3.1. □

Lemma 3.1 implies the following alternative.

Corollary 3.2. *Let L mean the set of all second-order linear differential operators*

$$A_3 = D_t - A(t,x,D_x) : X_3 \longrightarrow C_{t,x}^{(1+\alpha)/2,1+\alpha}(\mathrm{cl}\,Q,\mathbb{R}) \tag{3.12}$$

satisfying conditions (C) and $(\mathrm{S}_2^{1+\alpha})$. *Then, for each* $A_3 \in L$, *the mixed homogeneous problem* $A_3 u = 0$ *on* Q, *(2.3), and (2.4) has a nontrivial solution or any* $A_3 \in L$ *is a linear bounded Fredholm operator of the zero-index mapping* X_3 *onto* Y_3.

The following lemma establishes the complete continuity of the Nemitskii operator from the nonlinear part of (2.1).

Lemma 3.3. *Let assumptions (B1) and (B3) be satisfied. Then the Nemitskii operator* $N_3 : X_3 \to Y_3$ *defined by*

$$(N_3 u)(t,x) = f\left[t,x,u(t,x),D_1 u(t,x),\dots,D_n u(t,x)\right] \tag{3.13}$$

for $u \in X_3$ *and* $(t,x) \in \mathrm{cl}\,Q$ *is completely continuous.*

Proof. Let $M_3 \subset X_3$ be a bounded set. By the Ascoli-Arzelá theorem, it is sufficient to show that the set $N_3(M_3)$ is uniformly bounded and equicontinuous. We will use assumption (B3) to prove the inclusion $N_3(M_3) \subset Y_3$.

Take $u \in M_3$. According to assumption (B1), we obtain the local boundedness of the function f and of its derivatives $\partial f/\partial x_i$ on $(\mathrm{cl}\,Q) \times \mathbb{R}^{n+1}$ for $i = 1,\dots,n$. From this and from the equation

$$D_i(N_3 u)(t,x) = \left\{ D_i f[\cdot] + \sum_{l=0}^{n} \frac{\partial f}{\partial u_l}[\cdot] D_i D_l u \right\}[\cdot,\cdot,u,D_1 u,\dots,D_n u](t,x), \tag{3.14}$$

we have the estimation

$$\sup_{(t,x)\in \mathrm{cl}\,Q} \left| D_i(N_3 u)(t,x) \right| \le K_1 \tag{3.15}$$

for $i = 0,1,\dots,n$ with a positive sufficiently large constant K_1 not depending on $u \in M_3$.

Using the differentiability of f and the mean value theorem in the variable t for the difference of the derivatives of u, we can write

$$\langle N_3 u \rangle_{t,(1+\alpha)/2,Q}^{s} \le K_1. \tag{3.16}$$

Similarly, by (2.14), we have

$$\langle D_i N_3 u \rangle_{t,\alpha/2,Q}^{s} \le K_1, \qquad \langle D_i N_3 u \rangle_{x,\alpha,Q}^{y} \le K_1, \tag{3.17}$$

for $i = 1,\dots,n$ and $u \in M_3$. The previous estimations yield the inequality

$$\|N_3 u\|_{Y_3} \le K_1 \tag{3.18}$$

for all $u \in M_3$.

With respect to (B1), for any $u \in M_3$ and $(t,x),(s,y) \in \mathrm{cl}\,Q$ such that $|t-s|^2 + |x-y|^2 < \delta^2$ with a sufficiently small $\delta > 0$, we have

$$\left| N_3u(t,x) - N_3u(s,y) \right| < \epsilon, \quad \epsilon > 0, \tag{3.19}$$

which is the equicontinuity of $N_3(M_3)$. This finishes the proof of Lemma 3.3. □

Lemma 3.4. *Let assumptions (A1), (A2), (B1), (B3), and (C1) hold. Then the operator $F_3 = A_3 + N_3 : X_3 \to Y_3$ is coercive.*

Proof. We need to prove that if the set $M_3 \subset Y_3$ is bounded in Y_3, then the set of arguments $F_3^{-1}(M_3) \subset X_3$ is bounded in X_3.

In both cases (α_1) and (α_2), we get for all $u \in F_3^{-1}(M_3)$,

$$\left|\left|N_3u\right|\right|_{(1+\alpha)/2,1+\alpha,Q} \le K_1, \tag{3.20}$$

where $K_1 > 0$ is a sufficiently large constant. Hence,

$$\left|\left|A_3u\right|\right|_{Y_3} \le K_1 \tag{3.21}$$

for any $u \in F_3^{-1}(M_3)$.

Hypothesis (A2) ensures the existence and uniqueness of the solution $u \in X_3$ of the linear equation

$$C_3u = y, \tag{3.22}$$

and for any $y \in Y_3$,

$$\|u\|_{X_3} \le K_1 \|y\|_{Y_3}. \tag{3.23}$$

If we write

$$\begin{aligned} C_3u = A_3u &+ \sum_{i,j=1}^{n} \left[a_{ij}(t,x) - c_{ij}(t,x)\right] D_{ij}u \\ &+ \sum_{i=1}^{n} \left[a_i(t,x) - c_i(t,x)\right] D_iu + \left[a_0(t,x) - c_0(t,x)\right]u, \end{aligned} \tag{3.24}$$

then in both cases and for each $u \in F_3^{-1}(M_3)$, we obtain

$$\|y\|_{Y_3} \le \left|\left|C_3u\right|\right|_{Y_3} \le K_1, \tag{3.25}$$

whence, by inequality (3.23), we can conclude that the operator F_3 is coercive. □

Lemma 3.5. *Let the Nemitskii operator $N_3 : X_3 \to Y_3$ from (3.13) satisfy conditions (B2) and (B3). Then the operator N_3 is continuously Fréchet-differentiable, that is, $N_3 \in C^1(X_3, Y_3)$ and it is completely continuous.*

Proof. From (B2), we obtain (B1) which implies by Lemma 3.3 the complete continuity of N_3. To obtain the first part of the assertion of this lemma, we need to prove that the Fréchet derivative $N_3' : X_3 \to L(X_3, Y_3)$ defined by the equation

$$N_3'(u)h(t,x) = \sum_{j=0}^{n} \frac{\partial f}{\partial u_j}(t,x,u(t,x),D_1u(t,x),\dots,D_nu(t,x)]D_jh(t,x) \tag{3.26}$$

for $u,h \in X_3$ is continuous on X_3. Thus, we must prove, for every $v \in X_3$, that

$$\forall \epsilon > 0 \;\; \exists \delta(\epsilon,v) > 0, \quad \forall u \in X_3, \; \|u-v\|_{X_3} < \delta : \sup_{h\in X_3,\, \|h\|_{X_3}\le 1} \big|\big|[N_3'(u)-N_3'(v)]h\big|\big|_{Y_3} < \epsilon. \tag{3.27}$$

Using the norms (2.6), (2.8) and the estimation $\|u-v\|_{X_3} < \delta$, we have for the first term of (3.27) by the mean value theorem,

$$\begin{aligned}
&\sum_{i=0}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} \big|D_i[N_3'(u)-N_3'(v)]h(t,x)\big| \\
&\quad\le \sum_{i,j=0}^{n} \sup_{(t,x)\in \mathrm{cl}\, Q} \Bigg[\left\langle \frac{\partial^2 f}{\partial x_i \partial u_j} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \big|D_jh(t,x)\big| \\
&\qquad\qquad + \sum_{k=0}^{n} \left\langle \frac{\partial^2 f}{\partial u_j \partial u_k} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \big|D_{ik}u\big| \cdot \big|D_jh\big|(t,x) \\
&\qquad\qquad + \sum_{k=0}^{n} \left| \frac{\partial^2 f}{\partial u_j \partial u_k}(t,x,v(t,x),\dots) \right| \big|D_{ik}u - D_{ik}v\big| \big|D_jh\big|(t,x) \\
&\qquad\qquad + \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{t,x,u(t,x)}^{t,x,v(t,x)} \big|D_{ij}h(t,x)\big| \Bigg] < K\delta, \quad K > 0.
\end{aligned} \tag{3.28}$$

For the second term of (3.27), we estimate as follows:

$$\begin{aligned}
&\big\langle [N_3'(u)-N_3'(v)]h \big\rangle_{t,(1+\alpha)/2,Q}^{s} \\
&\quad \le \sum_{j=0}^{n} \sup_{\mathrm{cl}\, Q,\, t\ne s} |t-s|^{-(1+\alpha)/2} \Bigg[\left| \int_s^t D_\tau \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)} d\tau \right| \big|D_jh(t,x)\big| \\
&\qquad\qquad + \left\langle \frac{\partial f}{\partial u_j} \right\rangle_{s,x,u(s,x)}^{s,x,v(s,x)} \left| \int_s^t D_\tau D_j h(\tau,x)\, d\tau \right| \Bigg] \\
&\quad \le K\delta, \quad K > 0.
\end{aligned} \tag{3.29}$$

Here, we have used the mean value theorem for $\partial^2 f/\partial\tau\partial u_j$, $\partial^2 f/\partial u_j \partial u_k$, and $\partial f/\partial u_j$ for $j,k = 0,1,\dots,n$.

The third term of (3.27) gives by (2.15),

$$
\begin{aligned}
&\sum_{i=1}^{n}\langle D_i\{[N_3'(u)-N_3'(v)]h\}\rangle_{t,\alpha/2,Q}^{s}\\
&\quad\le \sum_{i=1}^{n}\sum_{j=0}^{n}\sup_{\mathrm{cl}\,Q,t\neq s}|t-s|^{-\alpha/2}\\
&\qquad\qquad \times\Bigg\{\bigg|\int_s^t D_\tau\Big\langle\frac{\partial^2 f}{\partial x_i\partial u_j}\Big\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\bigg|\,|D_jh(t,x)|\\
&\qquad\qquad\quad+\Big\langle\frac{\partial^2 f}{\partial x_i\partial u_j}\Big\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}\bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\bigg|\\
&\qquad\qquad\quad+\sum_{k=0}^{n}\Bigg[\bigg|\int_s^t D_\tau\Big\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\Big\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\bigg|\,|D_{ik}u|\,|D_jh|(t,x)\\
&\qquad\qquad\qquad+\bigg|\int_s^t D_\tau\Big[\frac{\partial^2 f}{\partial u_j\partial u_k}(\tau,x,v,\ldots)d\tau\Big]\bigg|\\
&\qquad\qquad\qquad\times|D_{ik}u(t,x)-D_{ik}v(t,x)|\,|D_jh(t,x)|\\
&\qquad\qquad\qquad+\Big\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\Big\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ik}u(t,x)-D_{ik}u(s,x)|\,|D_jh(t,x)|\\
&\qquad\qquad\qquad+\bigg|\frac{\partial^2 f}{\partial u_j\partial u_k}(s,x,v,\ldots)\bigg|\\
&\qquad\qquad\qquad\times|D_{ik}u(t,x)-D_{ik}v(t,x)-[D_{ik}u(s,x)-D_{ik}v(s,x)]|\,|D_jh(t,x)|\\
&\qquad\qquad\qquad+\Big\langle\frac{\partial^2 f}{\partial u_j\partial u_k}\Big\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ik}u(s,x)|\,\bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\bigg|\\
&\qquad\qquad\qquad+\bigg|\frac{\partial^2 f}{\partial u_j\partial u_k}(s,x,v,\ldots)\bigg|\,|D_{ik}u(s,x)-D_{ik}v(s,x)|\\
&\qquad\qquad\qquad\times\bigg|\int_s^t D_\tau D_jh(\tau,x)d\tau\bigg|\\
&\qquad\qquad\qquad+\bigg|\int_s^t D_\tau\Big\langle\frac{\partial f}{\partial u_j}\Big\rangle_{\tau,x,u(\tau,x)}^{\tau,x,v(\tau,x)}d\tau\bigg|\,|D_{ij}h(t,x)|\\
&\qquad\qquad\qquad+\Big\langle\frac{\partial f}{\partial u_j}\Big\rangle_{s,x,u(s,x)}^{s,x,v(s,x)}|D_{ij}h(t,x)-D_{ij}h(s,x)|\Bigg]\Bigg\}\\
&\quad\le K\bigg(\sum_{s=0}^{n}\delta^{\beta_s}+\delta\bigg),\quad K>0.
\end{aligned}
\tag{3.30}
$$

Making the corresponding changes, the last term of (3.27), by condition (B2), gives the required estimation:

$$\sum_{i=1}^{n} \langle D_i\{[N_3'(u) - N_3'(v)]h\}\rangle_{x,\alpha,Q}^{y}. \tag{3.31}$$

This finishes the proof of Lemma 3.5. □

4. Generic properties for continuous operators

On a mutual equivalence between the solution of the given initial-boundary value problem and an operator equation, we have the following lemma.

Lemma 4.1. *Let $A_3 : X_3 \to Y_3$ be the linear operator from Lemma 3.1, let $N_3 : X_3 \to Y_3$ be the Nemitskii operator from Lemma 3.3, and let $F_3 = A_3 + N_3 : X_3 \to Y_3$. Then,*

(1) *the function $u \in X_3$ is a solution of the initial-boundary value problem (2.1), (2.3), (2.4) for $g \in Y_3$ if and only if $F_3 u = g$;*
(2) *the couple $(u,g) \in X_3 \times Y_3$ is the bifurcation point of the initial-boundary value problem (2.1), (2.3), (2.4) if and only if $F_3(u) = g$ and $u \in \Sigma$, where Σ means the set of all points of X_3 at which F_3 is not locally invertible.*

Proof. (1) The first equivalence directly follows from the definition of the operator F_3 and of the mixed problem (2.1), (2.3), (2.4).

(2) If (u,g) is a bifurcation point of the mixed problem (2.1), (2.3), (2.4) and u_k, v_k, and g_k for $k = 1,2,\ldots$ have the same meaning as in Definition 2.7, then with respect to (1) we have $F_3(u) = g$, $F_3(u_k) = g_k = F_3(v_k)$. Thus, F_3 is not locally injective at u. Hence, F_3 is not locally invertible at u, that is, $u \in \Sigma$. Conversely, if F_3 is not locally invertible at u and $F_3(u) = g$, then F_3 is not locally injective at u. Indirectly, from Definition 2.7, we see that the couple (u,g) is a bifurcation point of (2.1), (2.3), (2.4). □

Lemma 4.2. *Let*

(i) *the operator $A(t,x,D_x) \neq 0$ from (2.1) and the operator $B_3(t,x,D_x)$ from (2.3) satisfy the smoothness condition $(\mathrm{S}_3^{1+\alpha})$;*
(ii) *the nonlinear part f of (2.1) belong to $C(\mathrm{cl}\, Q \times \mathbb{R}^{n+1}, \mathbb{R})$;*
(iii) *the operator $A_3 + N_3 : X_3 \to Y_3$ be nonconstant.*

Then, for any compact set of the right-hand sides $g \in Y_3$ from (2.1), the set of all solutions of problem (2.1), (2.3), (2.4) is compact (possibly empty).

Proof. Following the proof of Lemma 3.1, we see that $\dim X_3 = +\infty$ and the linear operator $A_3 : X_3 \to Y_3$ is continuous and accordingly closed. From hypothesis (ii) the Nemitskii operator $N_3 : X_3 \to Y_3$ given in (4.9) is closed too. By [8, Proposition 2.1], the operator $F_3 = A_3 + N_3 : X_3 \to Y_3$ is proper, and with respect to Lemma 4.1 we get our assertion. □

Theorem 4.3. *Under assumptions (A1), (A2) and (B1), (B3), the following statements hold for problem (2.1), (2.3), (2.4):*

(a) *the operator* $F_3 = A_3 + N_3 : X_3 \to Y_3$ *is continuous;*
(b) *for any compact set of the right-hand sides* $g \in Y_3$ *from (2.1), the corresponding set of all solutions is a countable union of compact sets;*
(c) *for* $u_0 \in X_3$, *there exist neighborhoods* $U(u_0)$ *of* u_0 *and* $U(F_3(u_0))$ *of* $F_3(u_0) \in Y_3$ *such that for each* $g \in U(F_3(u_0))$, *there is a unique solution of (2.1), (2.3), (2.4) if and only if the operator* F_3 *is locally injective at* u_0.

Moreover, if (C1) is assumed, then

(d) *for each compact set of* Y_3, *the corresponding set of all solutions is compact (possibly empty).*

Proof. Assertion (a) is evident by Lemmas 3.1 and 3.3.

Using the Nikoľskii theorem for A_3, we can write

$$F_3 = C_3 + (T_3 + N_3), \tag{4.1}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism and is proper (see [8, Proposition 2.1]) and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous mapping.

Now take the compact sets $K \subset Y_3$ and $F_3^{-1}(K)$. Then there exists a sequence of the closed and bounded sets $M_n \subset F_3^{-1}(K) \subset X_3$ for $n = 1,2,\ldots$ such that $\bigcup_{n=1}^{\infty} M_n = F_3^{-1}(K)$.

According to [8, Proposition 2.2], the restrictions $F_3|_{M_n}$ for $n = 1,2,\ldots$ are proper mappings and $\left(F_3|_{M_n}\right)^{-1}(K) = M_n$ is a compact set. Hence, the operator F_3 is σ-proper, which gives the result (b).

Assertion (d) is a direct consequence of [8, Proposition 2.2].

Suppose now that F_3 is injective in a neighborhood $U(u_0)$ of $u_0 \in X_3$. From the decomposition (4.1) the mapping

$$C_3^{-1} F_3 = I + C_3^{-1}(T_3 + N_3), \tag{4.2}$$

where $I : X \to Y$ is the identity, is completely continuous and injective in $U(u_0)$. On the basis of the Schauder domain invariance theorem (see [3, page 66]), the set $C_3^{-1}F_3(U(u_0))$ is open in X_3 and the restriction $C_3^{-1}F_3|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $C_3^{-1}F_3(U(u_0))$. Therefore, F_3 is locally invertible. From Lemma 4.1 we obtain (c).

The most important properties of the mapping F_3, whereby A_3 is a linear bounded Fredholm operator of zero index, N_3 is completely continuous, and F_3 is coercive, give the following theorem. □

Theorem 4.4. *If hypotheses (A1), (A2), (B1), (B3), and (C1) are satisfied, then for the initial-boundary value problem (2.1), (2.3), (2.4), the following statements hold.*

(e) *For each* $g \in Y_3$, *the set* S_{3g} *of all solutions is compact (possibly empty).*
(f) *The set* $R(F_3) = \{g \in Y_3 :$ *there exists at least one solution of the given problem*$\}$ *is closed and connected in* Y_3.
(g) *The domain of bifurcation* D_{3b} *is closed in* X_3 *and the bifurcation range* R_{3b} *is closed in* Y_3. $F_3(X_3 - D_{3b})$ *is open in* Y_3.
(h) *If* $Y_3 - R_{3b} \neq \emptyset$, *then each component of* $Y_3 - R_{3b}$ *is a nonempty open set (i.e., a domain).*

The number n_{3g} *of solutions is finite, constant (it may be zero) on each component of the set* $Y_3 - R_{3b}$, *that is, for every* g *belonging to the same component of* $Y_3 - R_{3b}$.

(i) *If* $R_{3b} = 0$, *then the given problem has a unique solution* $u \in X_3$ *for each* $g \in Y_3$ *and this solution continuously depends on* g *as a mapping from* Y_3 *onto* X_3.

(j) *If* $R_{3b} \neq \emptyset$, *then the boundary of the* F_3*-image of the set of all points from* X_3 *in which the operator* F_3 *is locally invertible is a subset of the* F_3*-image of the set of all points from* X_3 *in which* F_3 *is not locally invertible, that is,*

$$\partial F_3(X_3 - D_{3b}) \subset F_3(D_{3b}) = R_{3b}. \tag{4.3}$$

Proof. Statement (e) follows immediately from Theorem 4.3(d).

(f) Let the sequence $\{g_n\}_{n\in\mathbb{N}} \subset R(F_3) \subset Y_3$ converge to $g \in Y_3$ as $n \to \infty$. By Theorem 4.3(d), there is a compact set of all solutions $\{u_\gamma\}_{\gamma\in I} \subset X_3$ (I is an index set) of the equations $F_3(u) = g_n$ for all $n = 1, 2, \dots$. Then there exists a sequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset \{u_\gamma\}_{\gamma\in I}$ converging to $u \in X_3$ for which $F_3(u_{n_k}) = g_{n_k} \to g$. Since the operator F_3 is proper, whence it is closed, we have $F_3(u) = g$. Hence, $g \in R(F_3)$ and $R(F_3)$ is a closed set.

The connectedness of $R(F_3) = F_3(X_3)$ follows from the fact that $R(F_3)$ is a continuous image of the connected set X_3.

(g) According to Lemma 4.1(2), $D_{3b} = \Sigma_3$ and $R_{3b} = F_3(D_{3b})$. Since $X_3 - \Sigma_3$ is an open set, D_{3b} and its continuous image R_{3b} are closed sets in X_3 and Y_3, respectively.

Since $X_3 - D_{3b}$ is a set of all points in which the mapping F_3 is locally invertible, then it ensures that to each $u_0 \in X_3 - D_{3b}$ there is a neighborhood $U_1(F_3(u_0)) \subset F_3(X_3 - D_{3b})$, which means that the set $F_3(X_3 - D_{3b})$ is open.

(h) The set $Y_3 - R_{3b} = Y_3 - F_3(D_{3b}) \neq 0$ is open in Y_3, then each of its components is nonempty and open.

The second part of (h) follows from Ambrosetti theorem [1, page 216].

(i) Since $R_{3b} = \emptyset$, the mapping F_3 is locally invertible in X_3. From [8, Proposition 2.2], we get that F_3 is a proper mapping. Then the global inverse mapping theorem [12, page 174] proves this statement.

(j) By (f) and (g), we have ($\Sigma_3 = D_{3b}$)

$$F_3(X_3) = F_3(\Sigma_3) \cup F_3(X_3 - \Sigma_3) = F_3(\Sigma_3) \cup \overline{F_3(X_3 - \Sigma_3)} = \overline{F(X_3)}. \tag{4.4}$$

Furthermore, $\partial F_3(X_3 - \Sigma_3) = \overline{F(X_3 - \Sigma_3)} - F(X_3 - \Sigma_3)$, and thus the previous equality implies assertion (j). □

Theorem 4.5. *Under assumption (A1), (A2), (B1), (B3), and (C1), each of the following conditions is sufficient for the solvability of problem (2.1), (2.3), (2.4) for each* $g \in Y_3$:

(k) *for each* $g \in R_{3b}$, *there is a solution* u *of (2.1), (2.3), (2.4) such that* $u \in X_3 - D_{3b}$;

(l) *the set* $Y_3 - R_{3b}$ *is connected and there is a* $g \in R(F_3) - R_{3b}$.

Proof. First of all, we see that conditions (k) and (l) are mutually equivalent to the following conditions:

(k′) $F_3(D_{3b}) \subset F_3(X_3 - D_{3b})$,

(l′) $Y_3 - R_{3b}$ is a connected set and

$$F_3(X_3 - D_{3b}) - R_{3b} \neq \emptyset, \tag{4.5}$$

respectively ($D_{3b} = \Sigma_3$).

Then it is sufficient to show that conditions (k′) and (l′), respectively, are sufficient for the surjectivity of the operator $F_3 : X_3 \to Y_3$.

(k′) From the first equality of (4.4), we obtain $F_3(X_3) = F_3(X_3 - D_{3b})$. Hence, $R(F_3)$ is an open as well as a closed subset of the connected space Y_3. Thus, $R(F_3) = Y_3$.

(l′) By Theorem 4.4(h), $\operatorname{card} F_3^{-1}(\{q\}) = \text{const} =: k \geq 0$ for every $q \in Y_3 - R_{3b}$.

If $k = 0$, then $F_3(X_3) = R_{3b}$ and $F_3(X_3 - D_{3b}) \subset R_{3b}$. This is a contradiction to (4.5). Then $k > 0$ and $R(F_3) = Y_3$. □

The other surjectivity theorem is true.

THEOREM 4.6. *Let hypotheses (A1), (A2), (B1), (B3), and (C1) hold and*

(i) *there exists a constant $K > 0$ such that all solutions $u \in X_3$ of the initial-boundary value problem for the equation*

$$C_3u + \mu[A_3u - C_3u + N_3u] = 0, \quad \mu \in (0,1), \tag{4.6}$$

with data (2.3), (2.4), fulfil one of conditions (α_1) and (α_2) of the almost coercive condition (C1), then

(m) *problem (2.1), (2.3), (2.4) has at least one solution for each $g \in Y_3$;*

(n) *the number n_{3g} of solutions of (2.1), (2.3), (2.4) is finite, constant, and different from zero on each component of the set $Y_3 - R_{3b}$ (for all g belonging to the same component of $Y_3 - R_{3b}$).*

Proof. (m) It is sufficient to prove the surjectivity of the mapping $F_3 : X_3 \to Y_3$. By Lemma 3.1, we can write

$$F_3 = A_3 + N_3 = C_3 + (T_3 + N_3), \tag{4.7}$$

where $C_3 : X_3 \to Y_3$ is a linear homeomorphism from X_3 onto Y_3 and $T_3 + N_3 : X_3 \to Y_3$ is a completely continuous operator. Then the operator

$$C_3^{-1}F_3 = I + C_3^{-1}(T_3 + N_3) : X_3 \longrightarrow X_3 \tag{4.8}$$

is completely continuous and condensing (see [12, page 496]). The set $\Sigma_3 = D_{3b}$ is the set of all points $u \in X_3$ where $C_3^{-1}F_3$, as well as F_3, is not locally invertible.

Denote $S_1 \subset X_3$ a bounded set. Then $C_3(S_1) =: S$ is bounded in Y_3, and by Lemma 3.4, $F_3^{-1}(S) = F_3^{-1}(C_3(S_1)) = (C_3^{-1} \circ F_3)^{-1}(S_1)$ is a bounded set in X_3. Thus, the operator $C_3^{-1} \circ F_3$ is coercive.

Now we show that condition (i) implies the conditions from [8, Theorem 3.2, Corollary 3.3, and Remark 3.1] for $F(u) = C_3^{-1} \circ F_3(u)$ and $C(u) = G(u) = u$, $u \in X_3$.

In fact, as $C_3^{-1} \circ F_3(u) = ku$ if and only if $F_3(u) = kC_3(u)$, we get for $k < 0$,

$$C_3u + (1-k)^{-1}[A_3u - C_3u + N_3u] = 0, \tag{4.9}$$

where $(1-k)^{-1} \in (0,1)$.

In case (α_1), there is a constant $K > 0$ such that for all solutions $u \in X_3$ of (4.9),

$$\|u\|_{(1+\alpha)/2,1+\alpha,Q} \le K, \tag{4.10}$$

and in case (α_2),

$$\|u\|_{(2+\alpha)/2,2+\alpha,Q} \le K. \tag{4.11}$$

Furthermore, by the same method as in Lemma 3.4, we get the estimation

$$\|u\|_{X_3} < K_1, \quad K_1 > 0, \tag{4.12}$$

for all solutions $u \in X_3$ of $C_3^{-1} \circ F_3u = ku$. Hence, we get the surjectivity of F_3 and thus (m).

(n) From Theorem 4.4(h) and the surjectivity of F_3, it follows that there is $n_{3g} \neq 0$. This finishes the proof of Theorem 4.6. □

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