THE APOLLONIAN METRIC: LIMITS OF THE COMPARISON AND BILIPSCHITZ PROPERTIES

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The Apollonian metric is a generalization of the hyperbolic metric. It is defined in arbitrary domains in $\mathbb{R}^n$. In this paper, we derive optimal comparison results between this metric and the j_G metric in a large class of domains. These results allow us to prove that Euclidean bilipschitz mappings have small Apollonian bilipschitz constants in a domain G if and only if G is a ball or half-space.

1. Introduction

The Apollonian metric is a generalization of the hyperbolic metric introduced by Beardon [2]. It is defined in arbitrary domains in $\mathbb{R}^n$ and is Möbius invariant. Another advantage over the well-known quasihyperbolic metric [8] is that it is simpler to evaluate. On the downside, points cannot generally be connected by geodesics of the Apollonian metric. This paper is the last in a series of four papers on the Apollonian metric, the first three being [9, 10, 11]. Other authors who have approached this metric from the same perspective, providing the incentive for this investigation, are Rhodes [13], Seittenranta [14], Gehring and Hag [5, 6], and Ibragimov [12]. As becoming of a concluding paper, we will return here to the beginning and take a new look at the comparison and bilipschitz properties considered in [10]. Using results from [9], we are able to answer a question posed to the author by M. Vuorinen, which led to the start of this investigation, namely: under what circumstances are Euclidean bilipschitz with small distortion also Apollonian bilipschitz mappings with small distortion? This question can be seen as a step towards answering the question asked in [2] by Beardon about the isometries of the Apollonian metric, since the comparison condition has previously been shown to imply quite some regularity of the Apollonian metric (cf., e.g., the proof of [10, Theorem 1.6]).

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We start by stating the main results and, at the same time, we sketch the structure of the rest of the paper. The notation used conforms largely to that of [1, 17], the reader can consult Section 2.

We will be considering domains (open connected nonempty sets) G in the Möbius space $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$. The Apollonian metric for $x, y \in G \subsetneq \mathbb{R}^n$ is defined by

$$\alpha_G(x,y) := \sup_{a,b\in\partial G} \log \frac{|a-x|}{|a-y|}\frac{|b-y|}{|b-x|} \tag{1.1}$$

(with the understanding that if $a = \infty$, then we set $|a-x|/|a-y| = 1$ and similarly for b; see also Section 2.2). It is in fact a metric if and only if the complement of G is not contained in a hyperplane, as was noted in [2, Theorem 1.1].

To define the comparison property, we need the j_G metric, which is a modification from [16] of a metric introduced in [7]. This metric is defined for $x, y \in G \subsetneq \mathbb{R}^n$ by

$$j_G(x,y) := \log\left(1 + \frac{|x-y|}{\min\{d(x,\partial G), d(y,\partial G)\}}\right). \tag{1.2}$$

Definition 1.1. A domain $G \subsetneq \mathbb{R}^n$ has the *comparison property* if there exists a constant K such that $j_G/K \le \alpha_G \le 2j_G$.

The upper bound from the previous definition always holds and the constant 2 is the best possible, as was proved in [2, Theorem 3.2]. Next, we define the exterior ball condition, which played an important role in [10] that dealt with the comparison property. Several related conditions from the literature were reviewed in [10, Section 3].

Definition 1.2. Let $G \subsetneq \mathbb{R}^n$ and $L \ge 1$. A domain G is said to satisfy the *L-exterior ball condition* (*L-EB condition*) if, for every $x \in \partial G \setminus \{\infty\}$ and $r > 0$, there exists a point $z \in \overline{B^n}(x,r)$ such that $B^n(z, r/L) \subset G^c$.

In [10], it was shown that every EB domain has the comparison property. Unfortunately, the constant in that paper was $9L$, whereas we would like to have a constant that tends to 1 as $L \to 1$, since it is known [14, Theorem 4.2] that this constant equals 1 for 1-EB domains. In fact, we can calculate the optimal constant for every $L \ge 1$.

THEOREM 1.3. *If $G \subsetneq \mathbb{R}^n$ has the L-EB property, then G has the comparison property with constant $L + \sqrt{L^2 - 1}$. This constant is the best possible one depending only on L.*

In Section 5, we consider the Apollonian bilipschitz modulus which was introduced in [10]. For $L \geq 1$ and $G \subsetneq \mathbb{R}^n$, we define

$$\alpha_L(G) := \sup_f \sup_{x,y\in G} \left\{ \frac{\alpha_{f(G)}(f(x), f(y))}{\alpha_G(x,y)}, \frac{\alpha_G(x,y)}{\alpha_{f(G)}(f(x), f(y))} \right\}, \tag{1.3}$$

where the first supremum is taken over all L-bilipschitz mappings $f : G \to \mathbb{R}^n$ (with the understanding that terms with zero denominators are ignored). Notice that the second supremum is the Apollonian bilipschitz modulus of f, that is, the least constant for which f is Apollonian bilipschitz. Hence, $\alpha_L(G) < \infty$ if and only if every L-bilipschitz mapping is Apollonian bilipschitz as well, with uniformly bounded constant.

The next result answers the question stated in the first paragraph of this paper regarding getting small Apollonian bilipschitz constants.

Theorem 1.4. *If $G \subsetneq \mathbb{R}^n$ is a domain, then*

$$\lim_{L\to 1} \alpha_L(G) = 1 \tag{1.4}$$

if and only if G is a ball or half-space.

2. Notation and terminology

Sections 2.1, 2.2, and 2.3 contain fairly standard material and can be perused by the seasoned reader. Sections 2.4 and 2.5, on the other hand, contain material specific to the Apollonian metric.

2.1. The Möbius space. We denote by $\{e_1, e_2, \dots, e_n\}$ the standard basis of $\mathbb{R}^n$ and by n the dimension of the Euclidean space under consideration and we assume that $n \geq 2$. For $x \in \mathbb{R}^n$, we denote by x_i its ith coordinate. The following notation is used for balls, spheres, and the upper half-space ($x \in \mathbb{R}^n$ and $0 < r < \infty$):

$$\begin{aligned} B^n(x,r) &:= \{y \in \mathbb{R}^n : |x-y| < r\}, \qquad S^{n-1}(x,r) := \partial B^n(x,r), \\ B^n &:= B^n(1), \qquad S^{n-1} := S^{n-1}(1), \qquad H^n := \{y \in \mathbb{R}^n : y_n > 0\}. \end{aligned} \tag{2.1}$$

We use the notation $\overline{\mathbb{R}^n} := \mathbb{R}^n \cup \{\infty\}$ for the one-point compactification of $\mathbb{R}^n$. We define the spherical (chordal) metric q in $\overline{\mathbb{R}^n}$ by means of the canonical projection onto the Riemann sphere. We consider $\overline{\mathbb{R}^n}$ as the metric space $(\overline{\mathbb{R}^n}, q)$, hence, its balls are the (open) balls of $\mathbb{R}^n$, half-spaces, and complements of closed balls. If $G \subset \overline{\mathbb{R}^n}$, we denote by ∂G, G^c, and $\overline{G}$ its boundary, complement, and closure, respectively, all with respect to $\overline{\mathbb{R}^n}$. In contrast to topological operations, we consider metric operations with respect to the ordinary Euclidean metric.

2.2. Möbius mappings. The cross ratio $|a,b,c,d|$ is defined by

$$|a,b,c,d| := \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)} \left(= \frac{|a-c||b-d|}{|a-b||c-d|} \right) \tag{2.2}$$

for $a \neq b$, $c \neq d$, and $a,b,c,d \in \overline{\mathbb{R}^n}$, where the second equality holds if $a,b,c,d \in \mathbb{R}^n$. A homeomorphism $f : \overline{\mathbb{R}^n} \to \overline{\mathbb{R}^n}$ is a Möbius mapping if

$$|f(a), f(b), f(c), f(d)| = |a,b,c,d| \tag{2.3}$$

for every quadruple $a,b,c,d \in \overline{\mathbb{R}^n}$ with $a \neq b$ and $c \neq d$. For more information on Möbius mappings, see, for example, [1, Section 3]. Using the cross ratio, we can express the Apollonian metric as

$$\alpha_G(x,y) = \sup_{a,b \in \partial G} |a,y,x,b| \tag{2.4}$$

for $x,y \in G \subset \overline{\mathbb{R}^n}$. This means, in particular, that α_G is Möbius invariant, as was noted in [2, Introduction (2)].

2.3. Some miscellaneous notation and terminology. (i) For $x \in G \subsetneq \mathbb{R}^n$, we denote $\delta(x) := d(x, \partial G) := \min\{|x-z| : z \in \partial G\}$.

(ii) For $x,y,z \in \mathbb{R}^n$, we denote by $\widehat{xyz}$ the smallest angle between the vectors $x-y$ and $z-y$.

(iii) For $x,y \in \mathbb{R}^n$, we denote by xy the line through x and y and by $[x,y]$ the closed segment between x and y.

2.4. The Apollonian balls approach. The Apollonian balls approach has previously been used in [2, 3] and [14, Theorem 4.1] although this presentation is from [10, Section 5.1]. The notation of this section will be used practically in every proof in this paper.

For $x,y \in G \subsetneq \mathbb{R}^n$, we define

$$q_x := \sup_{b \in \partial G} \frac{|b-y|}{|b-x|}, \qquad q_y := \sup_{a \in \partial G} \frac{|a-x|}{|a-y|}. \tag{2.5}$$

The numbers q_x and q_y are called the *Apollonian parameters* of x and y (with respect to G). By definition, $\alpha_G(x,y) = \log(q_x q_y)$. The *Apollonian balls* are defined by

$$\begin{aligned} B_x &:= \left\{ z \in \mathbb{R}^n : \frac{|z-x|}{|z-y|} < \frac{1}{q_x} \right\}, \\ B_y &:= \left\{ z \in \mathbb{R}^n : \frac{|z-y|}{|z-x|} < \frac{1}{q_y} \right\}. \end{aligned} \tag{2.6}$$

We collect the following immediate results regarding these balls:

(1) $B_x \subset G$ and $\overline{B_x} \cap \partial G \neq \emptyset$, similarly for B_y;
(2) if i_x and i_y denote the inversions in the spheres ∂B_x and ∂B_y, then $y = i_x(x) = i_y(x)$;
(3) since $\infty \notin G$, we have $q_x, q_y \geq 1$; if $\infty \notin \overline{G}$, then $q_x, q_y > 1$;
(4) if x_0 denotes the center of B_x and r_x its radius, then

$$\left|x - x_0\right| = \frac{|x-y|}{q_x^2 - 1} = \frac{r_x}{q_x}; \tag{2.7}$$

(5) we have $q_x - 1 \leq |x-y|/\delta(x) \leq q_x + 1$.

2.5. Quasi-isotropy. We define the concept of quasi-isotropy which is a kind of local comparison property. It was introduced in [11] and was the focus of [9]; however, it was originally conceived of by the author in order to prove Theorem 1.4 on the Apollonian bilipschitz modulus.

This property is the weakest regularity property of the Apollonian metric which we consider. Thus, we will show in Lemma 5.5 that the Apollonian bilipschitz constant $\alpha_L(G)$ is always greater than or equal to the quasi-isotropy constant. Similarly, it was shown in [11, Section 4] that if G has the comparison property with constant K, then G is $2K$-quasi-isotropic. This means that the quasi-isotropy constant gives us a lower bound for the comparison constant, a fact that we will use in the proof of Theorem 1.3.

Definition 2.1. A metric space (G,d) with $G \subset \mathbb{R}^n$ is K-*quasi-isotropic* if

$$\limsup_{r \to 0} \frac{\sup\{d(x,z) : |x-z| = r\}}{\inf\{d(x,y) : |x-y| = r\}} \leq K \tag{2.8}$$

for every $x \in G$. A metric which is 1-quasi-isotropic is said to be *isotropic*, whereas a metric that is not K-quasi-isotropic, for any K, is said to be *anisotropic*. The function qi is defined on the set of domains in $\mathbb{R}^n$ so that $qi(G)$ is the least constant for which α_G is quasi-isotropic or $qi(G) = \infty$ if α_G is anisotropic.

We will only use quasi-isotropy in a very tangential manner in this paper, hence, we will not expose here any methods for calculating the quasi-isotropy constant. For a presentation of such techniques, the reader is referred to [9].

3. The comparison constant of an exterior ball domain

In this section, we calculate the exact value of the comparison constant for EB domains. We start with a geometrical lemma which is similar to [9, Lemma 3.6], except that we now consider the Apollonian balls about two points instead of the Apollonian spheres through one point.

Lemma 3.1. *Let $x, y \in G \subsetneq \mathbb{R}^n$ and let B_x and B_y be the corresponding Apollonian balls. If $B := B^n(b,r)$ is a ball with $r > d(B_x, B_y)/2$ which does not intersect $B_x \cup B_y$,* *then*

$$|x-b| \geq \sqrt{r^2 + \frac{2rq_y|x-y|}{(q_xq_y-1)}}. \tag{3.1}$$

Proof. We may assume that B is tangent to both B_x and B_y, since otherwise $|x-b|$ is smaller for some other ball with the same radius, or we can choose another ball with the same distance to x but with a larger r.

Denote the centers of B_x and B_y by x_0 and y_0 and set $\theta := \widehat{y_0x_0b}$, $s := |x-x_0|$, and $w := |x_0-y_0|$. Using the cosine rule in the triangles y_0x_0b and xx_0b, we get that

$$\begin{aligned}(r+r_y)^2 &= (r+r_x)^2 + w^2 - 2(r+r_x)w\cos\theta,\\ |x-b|^2 &= (r+r_x)^2 + s^2 - 2(r+r_x)s\cos\theta.\end{aligned} \tag{3.2}$$

Combining these equations to eliminate $\cos\theta$, we get that

$$|x-b|^2 = \frac{w-s}{w}(r+r_x)^2 + \frac{s}{w}(r+r_y)^2 + s(s-w). \tag{3.3}$$

It follows from the definition of q_x and q_y (cf. Result (4) in Section 2.4) that $s = |x-y|/(q_x^2-1)$, $r_x = |x-y|q_x/(q_x^2-1)$, $r_y = |x-y|q_y/(q_y^2-1)$, and

$$\begin{aligned} w &= |x_0-x| + |x-y| + |y-y_0| = \left(\frac{1}{q_x^2-1} + 1 + \frac{1}{q_y^2-1}\right)|x-y| \\ &= \frac{q_x^2q_y^2-1}{(q_x^2-1)(q_y^2-1)}|x-y|. \end{aligned} \tag{3.4}$$

Using these in (3.3), we get that

$$\begin{aligned} |x-b|^2 &= \frac{q_y^2(q_x^2-1)(r+r_x)^2}{q_x^2q_y^2-1} + \frac{(q_y^2-1)(r+r_y)^2}{q_x^2q_y^2-1} - \frac{|x-y|^2}{q_x^2-1}\left(1+\frac{1}{q_y^2-1}\right) \\ &= \Bigg[(q_y^2q_x^2-1)r^2 + 2(q_xq_y+1)q_yr|x-y| \\ &\qquad + \frac{q_x^2q_y^4-q_y^2}{(q_x^2-1)(q_y^2-1)}|x-y|^2\Bigg]\frac{1}{q_x^2q_y^2-1} - \frac{q_y^2|x-y|^2}{(q_x^2-1)(q_y^2-1)} \\ &= r^2 + 2\frac{q_y|x-y|r}{q_xq_y-1}. \end{aligned} \tag{3.5}$$

□

We also need the following lemma which is a variant of [14, Theorem 4.2].

Lemma 3.2. *Let $x, y \in G \subsetneq \mathbb{R}^n$ and let B_x and B_y be the Apollonian balls. If the convex hull of $B_x \cup B_y$ does not intersect ∂G, then $j_G(x,y) \le \alpha_G(x,y)$.*

The proof of Theorem 1.3 is quite similar to the proof of [9, Theorem 1.4(1)] in which we estimated the quasi-isotropy constant of an EB domain. However, since there is a gap between the Apollonian balls, it follows that for a large enough $\alpha_G(x,y)$, the EB property becomes worthless. Thus, we proceed in two steps this time; first, considering points with small Apollonian distance (the next lemma) and then using an ad hoc measure to take care of the rest of the points in the proof of Theorem 1.3.

Lemma 3.3. *Let $G \subsetneq \mathbb{R}^n$ be an L-EB domain for $L > 1$ and let $x, y \in G$ be points such that*

$$\alpha_G(x,y) < \log\left(2\sqrt{\frac{L+1}{L-1}} - 1\right). \tag{3.6}$$

Then

$$\alpha_G(x,y) \ge \left(L - \sqrt{L^2-1}\right) j_G(x,y). \tag{3.7}$$

Proof. Fix $x, y \in G$, satisfying inequality (3.6), let B_x and B_y be the Apollonian balls, and let q_x and q_y be the Apollonian ball parameters, as described in Section 2.4.

If there are no points of ∂G in the convex hull C of $B_x \cup B_y$, then $\alpha_G(x,y) \ge j_G(x,y)$ by Lemma 3.2 and there is nothing to prove. We may thus assume that $C \cap \partial G \ne \emptyset$. Fix $r > d(B_x, B_y)/2$. For $\zeta \in C$, let B be the ball with radius r, tangent to B_x and B_y for which the distance $h := d(\zeta, B)$ is minimal. Then every ball with radius r and center in $B^n(\zeta, h+r)$ intersects G. This means that if ζ was a boundary point of G, then G would not be EB with constant smaller than $(h+r)/r = |\zeta - b|/r$, where b denotes the center of the ball. Since we know that G is L-EB, it follows that $|\zeta - b| \le Lr$ for $\zeta \in C \cap \partial G$.

Combining this with Lemma 3.1, we find that

$$\begin{aligned}\delta(x) &= \inf_{\zeta \in C \cap \partial G} |x-b| - |\zeta - b| \ge |x-b| - Lr \\ &= \sqrt{r^2 + \frac{2rq_y|x-y|}{(q_xq_y-1)}} - Lr =: f(r)\end{aligned} \tag{3.8}$$

for all $r > d(B_x, B_y)/2$. We choose r so as to maximize the lower bound.

We find that $df/dr = (r+c)/\sqrt{r^2+2rc} - L$, where we denote that $c := q_y|x-y|/(q_xq_y-1)$. Hence, f has a maximum at $r_0 = c(L/\sqrt{L^2-1} - 1)$. We need to check that $r_0 > d(B_x, B_y)/2$ so that Lemma 3.1 is applicable for this value of r.

The inequality $r_0 > d(B_x, B_y)/2$ is equivalent to

$$\begin{aligned} 2\left(\frac{L}{\sqrt{L^2-1}}-1\right)\frac{q_y|x-y|}{q_xq_y-1} &> |x-y| - d(x,\partial B_x) - d(y,\partial B_y) \\ &= |x-y|\left(1-\frac{1}{q_x+1}-\frac{1}{q_y+1}\right) \\ &= |x-y|\frac{q_xq_y-1}{(q_x+1)(q_y+1)}. \end{aligned} \tag{3.9}$$

We thus have to show that

$$2\left(\frac{L}{\sqrt{L^2-1}}-1\right) > \frac{(q_xq_y-1)^2}{q_y(q_x+1)(q_y+1)}. \tag{3.10}$$

The denominator of the right-hand side of this estimate equals $(q_xq_y+q_y)(q_y+1)$. Since $q_y \ge 1$, we get the lower bound $2(q_xq_y+1)$ for this denominator. Since $\exp\alpha_G(x,y) = q_xq_y$, we see that it suffices to show that

$$4\left(\frac{L}{\sqrt{L^2-1}}-1\right)(\exp\alpha_G(x,y)+1) > (\exp\alpha_G(x,y)-1)^2. \tag{3.11}$$

Solving this second-degree equation in $\exp\alpha_G(x,y)$ gives

$$\exp\alpha_G(x,y) < 2\sqrt{\frac{(L+1)}{(L-1)}}-1, \tag{3.12}$$

which is the assumption of the lemma.

We then set $r = r_0$ in the estimate (3.8), which gives

$$\frac{|x-y|}{\delta(x)} \le \left(L+\sqrt{L^2-1}\right)\frac{q_xq_y-1}{q_x}. \tag{3.13}$$

We can derive a similar estimate for $\delta(y)$ and so we find that

$$\frac{|x-y|}{\min\{\delta(x),\delta(y)\}} \le \left(L+\sqrt{L^2-1}\right)\frac{q_xq_y-1}{\min\{q_x,q_y\}}. \tag{3.14}$$

Using the Bernoulli inequality, we find that

$$\begin{aligned} \frac{j_G(x,y)}{L+\sqrt{L^2-1}} &\le \log\left(1+\frac{|x-y|}{\left(L+\sqrt{L^2-1}\right)\min\{\delta(x),\delta(y)\}}\right) \\ &\le \log\left(1+\frac{q_xq_y-1}{\min\{q_x,q_y\}}\right) \le \log(q_xq_y) = \alpha_G(x,y), \end{aligned} \tag{3.15}$$

which was to be shown. □

Proof of Theorem 1.3. We first note that if $L = 1$, then G is convex and the claim follows from [14, Theorem 4.2]. We assume then that $L > 1$ and denote $d := 2\sqrt{(L+1)/(L-1)} - 1$. The bound $L + \sqrt{L^2 - 1}$ on the comparison constant holds by Lemma 3.3 if $\alpha_G(x, y) \le \log d$.

Suppose then that $x, y \in G$ are such that $\alpha_G(x, y) \ge \log d$. By result (5) in Section 2.4, we always have

$$\frac{|x-y|}{\min\{\delta(x), \delta(y)\}} \le \max\{1+q_x, 1+q_y\}. \tag{3.16}$$

Hence, we find that

$$\frac{j_G(x,y)}{\alpha_G(x,y)} \le \frac{\log(2+\max\{q_x, q_y\})}{\log(q_x q_y)} \le \frac{\log(2+q_x q_y)}{\log(q_x q_y)} \le \frac{\log(2+d)}{\log d}. \tag{3.17}$$

The last inequality follows since the function $z \mapsto \log(2+z)/\log z$ is decreasing.

Thus, we have seen that for some points, the ratio j_G/α_G is bounded from above by $L + \sqrt{L^2-1}$ and for all others by $\log(2+d)/\log d$. This means that G has the comparison property with constant less than or equal to

$$\max\left\{L + \sqrt{L^2-1}, \frac{\log\left(2\sqrt{(L+1)/(L-1)}+1\right)}{\log\left(2\sqrt{(L+1)/(L-1)}-1\right)}\right\}. \tag{3.18}$$

Next, we prove that the first term in the maximum is always greater than the second one. We introduce a new variable, $u^2 := (L+1)/(L-1)$, which satisfies $u > 1$.

We have to prove that

$$\frac{\log(2u+1)}{\log(2u-1)} \le \frac{u^2+1}{u^2-1} + \left[\left(\frac{u^2+1}{u^2-1}\right)^2 - 1\right]^{1/2} = \frac{u+1}{u-1}. \tag{3.19}$$

Since $\log(2u-1) > 0$, this is equivalent to

$$g(u) := \frac{u+1}{u-1}\log(2u-1) - \log(2u+1) \ge 0. \tag{3.20}$$

We will show that g is decreasing; we differentiate g:

$$\frac{dg}{du} = -\frac{2}{(u-1)^2}\log(2u-1) + \frac{u+1}{u-1}\frac{2}{2u-1} - \frac{2}{2u+1}. \tag{3.21}$$

Then we multiply the inequality $dg/du \le 0$ by $-(u-1)^2/2$ to get the equivalent inequality

$$\begin{aligned} h(u) &:= \log(2u-1) + \frac{(u-1)^2}{2u+1} - \frac{u^2-1}{2u-1} \\ &= \log(2u-1) - 6u\frac{u-1}{4u^2-1} \ge 0. \end{aligned} \tag{3.22}$$

We find that

$$\frac{dh}{du} = \frac{2}{2u-1} - 6\frac{4u^2-2u+1}{(4u^2-1)^2} = 8\frac{(u-1)(2u^2+1)}{(4u^2-1)^2}, \tag{3.23}$$

and so it is clear that h is increasing. It follows that $h(u) \geq h(1) = 0$, which means that $dg/du \leq 0$. Since g is decreasing, it follows that

$$g(u) \geq \lim_{u\to\infty} g(u) \geq \lim_{u\to\infty} \log(2u-1) - \log(2u+1) = 0, \tag{3.24}$$

which means that the first term in the maximum is greater than the second one and completes the proof that $L+\sqrt{L^2-1}$ is an upper bound for the comparison constant.

To show that this constant is the best possible, recall from Section 2.5 that the quasi-isotropy constant is always less than one half of the comparison constant. It was proven in [9, Theorem 1.4(1)] that there exists an L-EB domain with quasi-isotropy constant $2(L+\sqrt{L^2-1})$ and so the comparison constant of this domain is at least $L+\sqrt{L^2-1}$, which concludes the proof. □

4. The spiral mapping

In this section, we will define a bilipschitz mapping that has large rotational distortion even with small bilipschitz constant. This mapping is a generalization of a mapping of $\mathbb{R}^2$ onto itself considered by Freedman and He in [4]. Note that the difficulty in extending it to $\mathbb{R}^n$ lies therein, that we wish to preserve the property $f(x) = x$ for $x \notin B^n$. The following quite lengthy proof is based on a series of elementary estimates.

Lemma 4.1. *Let $P \geq 1$ and let $x = (r\cos\theta, r\sin\theta, \hat{x}) \in \mathbb{R}^n$, where $r \in [0,\infty)$, $\theta \in [0,2\pi)$, and $\hat{x} \in \mathbb{R}^{n-2}$. Let $\theta' := \theta + (P - 1/P)\log(r/(1-|\hat{x}|))$. The mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ given by*

$$f(x) = f(r\cos\theta, r\sin\theta, \hat{x}) := (r\cos\theta', r\sin\theta', \hat{x}) \tag{4.1}$$

for $r + |\hat{x}| < 1$ and $f(x) = x$ for $r + |\hat{x}| \geq 1$ is P^2-bilipschitz.

Proof. Suppose that x and y are two points in $\mathbb{R}^n$ with $x = (r\cos\theta, r\sin\theta, \hat{x})$ and $y = (s\cos\phi, s\sin\phi, \hat{y})$ where $\hat{x}, \hat{y} \in \mathbb{R}^{n-2}$. We define

$$H := \left\{ z \in \mathbb{R}^n : \sqrt{z_1^2+z_2^2} + \sqrt{z_3^2+\cdots+z_n^2} < 1 \right\}. \tag{4.2}$$

Observe that H is precisely the domain in which f is not by definition equal to the identity. We first assume that $r + |\hat{x}| < 1$ and that $s + |\hat{y}| < 1$, that is, $x, y \in H$.

We need to show that

$$\begin{aligned} P^4|x-y|^2 &= P^4\big[(r\cos\theta - s\cos\phi)^2 + (r\sin\theta - s\sin\phi)^2 + |\hat{x}-\hat{y}|^2\big] \\ &\geq (r\cos\theta' - s\cos\phi')^2 + (r\sin\theta' - s\sin\phi')^2 + |\hat{x}-\hat{y}|^2 \\ &= \big|f(x)-f(y)\big|^2, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} \theta' &:= \theta + \left(P - \frac{1}{P}\right)\log\left(\frac{r}{1-|\hat{x}|}\right), \\ \phi' &:= \phi + \left(P - \frac{1}{P}\right)\log\left(\frac{s}{1-|\hat{y}|}\right). \end{aligned} \tag{4.4}$$

This inequality can be reexpressed as

$$(P^4-1)\big[r^2+s^2+|\hat{x}-\hat{y}|^2\big] \geq 2rs\big[P^4\cos\gamma - \cos(\gamma+\lambda)\big], \tag{4.5}$$

where $\gamma := \theta - \phi$ and $\lambda := (P-1/P)[\log(r/(1-|\hat{x}|)) - \log(s/(1-|\hat{y}|))]$. We will use the elementary estimate

$$\begin{aligned} &P^4\cos\gamma - \cos(\gamma+\lambda) \\ &\quad = (P^4-\cos\lambda)\cos\gamma + \sin\lambda\sin\gamma \leq \sqrt{(P^4-\cos\lambda)^2+\sin^2\lambda} \\ &\quad = \sqrt{P^8+1-2P^4\cos\lambda} \leq \sqrt{P^8+1-2P^4+2P^4\lambda^2} \\ &\quad = (P^2-1)\sqrt{(1+P^2)^2+2P^2\left[\log\left(\frac{r}{1-|\hat{x}|}\right)-\log\left(\frac{s}{1-|\hat{y}|}\right)\right]^2}. \end{aligned} \tag{4.6}$$

We use (4.6) in (4.5) and see that it is sufficient to prove that

$$\begin{aligned} &(P^2+1)\big[r^2+s^2+|\hat{x}-\hat{y}|^2\big] \\ &\qquad \geq (P^2+1)\big[r^2+s^2+(|\hat{x}|-|\hat{y}|)^2\big] \\ &\qquad \geq 2rs\sqrt{(P^2+1)^2+2P^2\left[\log\left(\frac{r}{s}\right)+\log\left(\frac{1-|\hat{y}|}{1-|\hat{x}|}\right)\right]^2}. \end{aligned} \tag{4.7}$$

We divide through by (P^2+1). Since $2P^2/(P^2+1)^2 \leq 1/2$, it suffices to prove that

$$\frac{\big[r^2+s^2+(|\hat{x}|-|\hat{y}|)^2\big]^2}{(2rs)^2} \geq 1 + \left(\frac{1}{2}\right)\left[\log\left(\frac{r}{s}\right)+\log\left(\frac{1-|\hat{y}|}{1-|\hat{x}|}\right)\right]^2. \tag{4.8}$$

We assume, without loss of generality, that $r \geq s$ and denote that $|\hat{x}| - |\hat{y}| = c$. We see that $(1-|\hat{y}|)/(1-|\hat{x}|) = 1 + c/(1-|\hat{x}|)$ is maximized by maximizing $|\hat{x}|$ for $c > 0$, and that the ratio is smaller than 1 for $c < 0$. Since $|\hat{x}| < 1 - r$, we see that the right-hand side is less than or equal to

$$1+\frac{\left[\log(rs)+\log\left((r+c)/r\right)\right]^2}{2}=1+\frac{\log^2\left((r+c)s\right)}{2}. \tag{4.9}$$

We introduce the variables $u := r/s \geq 1$ and $v := c^2/(rs)$. Then we have to prove that

$$\begin{aligned}\frac{\left[r^2+s^2+c^2\right]^2}{(2rs)^2}=\frac{\left[u+1/u+v\right]^2}{4}&\geq 1+\frac{\log^2\left(u+\sqrt{uv}\right)}{2}\\ &=1+\frac{\log^2\left((r+c)/s\right)}{2}.\end{aligned} \tag{4.10}$$

We define yet another variable $w := u+\sqrt{uv}$. We will consider how $u+1/u+v$ varies for fixed $w \geq 1$. Since $v=(w-u)^2/u$, this amounts to considering the function

$$g(u) := u+\frac{1}{u}+\frac{(w-u)^2}{u}=2u-2w+\frac{1+w^2}{u}. \tag{4.11}$$

Now, $g'(u)=2-(1+w^2)/u^2$ has one zero at $u=\sqrt{(1+w^2)/2}$, which is a minimum of g. Hence,

$$g(u)\geq\sqrt{2(1+w^2)}-2w+\frac{1+w^2}{\sqrt{(1+w^2)/2}}=2\sqrt{2}\sqrt{1+w^2}-2w, \tag{4.12}$$

and we see that it suffices to prove that

$$\begin{aligned}\frac{\left[2\sqrt{2}\sqrt{1+w^2}-2w\right]^2}{4}&=\left[\sqrt{2}\sqrt{1+w^2}-w\right]^2\\ &=2+3w^2-w\sqrt{8(1+w^2)}\geq 1+\frac{\log^2(w)}{2}\end{aligned} \tag{4.13}$$

for $w \geq 1$. Clearly, this inequality holds for $w=1$ and so it suffices to show that the left-hand side grows faster than the right-hand side. In terms of derivatives, this means that

$$6w-\sqrt{8}\frac{1+2w^2}{\sqrt{1+w^2}}\geq\frac{\log w}{w}. \tag{4.14}$$

Since $\log w \leq w-1$, it suffices to show that

$$(6w^2-w+1)\sqrt{1+w^2}\geq\sqrt{8}(2w^2+1)w. \tag{4.15}$$

Squaring both sides and collecting all terms on one side, we see that this inequality is equivalent to

$$\left[(2w-1)^2w^2+4w^2+1\right](w-1)^2 \geq 0, \tag{4.16}$$

which is obvious. We have now proved that f is P^2-lipschitz in H.

Next, assume that $x \in H$ and $y \notin H$. Let z be the point on ∂H such that $|x-z|+|z-y| = |x-y|$. Since $f(z) = z$ and $f(y) = y$, we have

$$\begin{aligned} \left|f(x)-f(y)\right| &\leq \left|f(x)-f(z)\right| + \left|f(z)-f(y)\right| \\ &\leq P^2\left|x-z\right| + |z-y| \leq P^2|x-y|. \end{aligned} \tag{4.17}$$

Finally, the case $x, y \notin H$ is trivial since f is the identity for these points.

The inverse of f is of the same form as f; only the direction of rotation is changed. It is therefore clear that f^{-1} is P^2-lipschitz too, and so f is P^2-bilipschitz. □

5. The limiting behavior of the Apollonian bilipschitz modulus

In this section, we study how the quantity $\alpha_L(G)$ behaves when $L \to 1$. The results derived on the behavior of $\alpha_L(G)$ in [10] are useful only for large L and thus these two approaches are complementary. We prove in Theorem 1.4 that $\alpha_L(G) \to 1$ as $L \to 1$ if and only if G is a ball or half-space. To prove this result, we need to show two things: if G is a ball or half-space, then $\alpha_L(G) \to 1$ and if $\alpha_L(G) \to 1$, then G is a ball or half-space. The comparison results that have been derived so far in this paper are good for a lower bound on α_G and the following result provides the upper bound. This result will suffice for the first implication.

The next lemma uses Seittenranta's metric δ_G in an intermediate step; since this is the only use for the metric here, the reader is referred to [14] for the definition.

Lemma 5.1. *Let $f : H^n \to \mathbb{R}^n$ be L-bilipschitz and denote $G := f(H^n)$. Then f is L^4-lipschitz with respect to the Apollonian metric, that is,*

$$\alpha_G\big(f(x), f(y)\big) \leq L^4\alpha_{H^n}(x,y). \tag{5.1}$$

Proof. By [14, Theorem 3.11], we know that the inequality $\alpha_G \leq \delta_G$ is valid in every domain $G \subsetneq \mathbb{R}^n$. Also, we have $\alpha_{H^n} = \delta_{H^n}$ since both metrics are equal to the hyperbolic metric in the half-space (by [2, Lemma 3.1] and [17, Lemma 8.39] for α_{H^n} and δ_{H^n}, respectively). It follows that

$$\alpha_G\big(f(x), f(y)\big) \leq \delta_G\big(f(x), f(y)\big) \leq L^4\delta_{H^n}(x,y) = L^4\alpha_{H^n}(x,y), \tag{5.2}$$

where the second inequality is stated in [14, Theorem 3.18]. □

As usual, the lower bound for the Apollonian metric is harder to come by. We need a series of lemmas. The next lemma follows easily from an extension result of Väisälä [15].

Lemma 5.2. *Let $f : B^n \to \mathbb{R}^n$ be an L-bilipschitz mapping. There exists an $L_0 > 1$ such that $f(B^n)$ has the $K(L)$-EB property for $L < L_0$ with $K(L) \to 1$ as $L \to 1$.*

Proof. It follows from [15, Example 6.13] that there exists an L'-bilipschitz mapping $f' : \mathbb{R}^n \to \mathbb{R}^n$ such that $f'|_{B^n} = f$. Moreover, $L' \to 1$ as $L \to 1$. Using this extended mapping, it is easy to see that the claim of the lemma holds. □

We need the previous result in the half-space. The following lemma will be used to transfer it to this setting.

Lemma 5.3. *Let $G \subsetneq \mathbb{R}^n$ and let $f : G \to \mathbb{R}^n$ be an L-bilipschitz mapping. Let π be the inversion in a sphere with radius $r > 0$ whose center is not in G. Then $\pi \circ f \circ \pi$ is L^3-bilipschitz in $\pi(G)$.*

Proof. Denote the center of inversion by w. It is well known, and follows easily from the definition of an inversion, that

$$\left|\pi(x) - \pi(y)\right| = \frac{r^2|x-y|}{|x-w||y-w|} \tag{5.3}$$

for $x, y \in G$. We denote that $x' := \pi(x)$ and $y' := \pi(y)$. It follows from the inequality

$$\begin{aligned} \left|g(x) - g(y)\right| &= \frac{r^2\left|f(x') - f(y')\right|}{\left|f(x') - w\right|\left|f(y') - w\right|} \\ &\le \frac{r^2L|x'-y'|}{|x'-w||y'-w|/L^2} = L^3|x-y|, \end{aligned} \tag{5.4}$$

(and similarly for the lower bound) that g is bilipschitz in $\pi(G)$ with constant L^3. In this inequality, we used $|f(x') + e_n| = |f(x') + f(e_n)| \ge |x' + e_n|/L$ and so forth. □

Corollary 5.4. *Let $f : H^n \to \mathbb{R}^n$ be an L-bilipschitz mapping. If $L < L_0$, then $f(H^n)$ has the $K(L)$-EB property with $K(L) \to 1$ as $L \to 1$.*

Proof. Let π be the inversion in $S^{n-1}(-e_n, \sqrt{2})$. Then $\pi \circ f \circ \pi$ satisfies the assumptions of Lemma 5.2 and is thus extendable. If $\tilde{g}$ denotes the extension, we define $\tilde{f} = \pi \circ \tilde{g} \circ \pi$. By Lemma 5.3, this mapping is a bilipschitz extension of f. Using this extension, we easily see that the claim holds. □

To prove the converse implication of the main theorem, we use the concept of quasi-isotropy. The idea behind the next lemma is that the bilipschitz condition does not constrain rotation very much, provided that it happens in a sufficiently small ball.

Lemma 5.5. *If* $G \subsetneq \mathbb{R}^n$*, then* $\alpha_L(G) \geq qi(G)$ *for every* $L > 1$.

Proof. Let $G \subsetneq \mathbb{R}^n$ be a domain with $1 < qi(G) < \infty$ (the claim is trivial if $qi(G) = 1$, the case $qi(G) = \infty$ is considered below). Fix $\epsilon > 0$ and let $\theta, \phi \in S^{n-1}$, $x \in G$, and $0 < \delta \leq \delta(x)/2$ be such that $\alpha_G(x, x+t\theta)/\alpha(x, x+t\phi) > qi(G) - \epsilon$ for $|t| \leq \delta$.

We know from Section 4 (from [4] for $n = 2$) that there exists an L-bilipschitz mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(x) = x$, $f(x+\delta''\phi) = x+\delta''\theta$, and $f(z) = z$ for $|x - z| > \delta'$, where $0 < \delta'' < \delta' < \delta$ and δ'' depends on L and δ'. Note that for this mapping, we have $f(G) = G$. It follows that

$$\alpha_L(G) \geq \frac{\alpha_{f(G)}\big(f(x), f(x+\delta''\phi)\big)}{\alpha_G(x, x+\delta''\phi)} = \frac{\alpha_G(x, x+\delta''\theta)}{\alpha_G(x, x+\delta''\phi)} > qi(G) - \epsilon. \tag{5.5}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\alpha_L(G) \geq qi(G)$.

If $qi(G) = \infty$, then for every $\epsilon > 0$, we find $\theta, \phi \in S^{n-1}$, $x \in G$, and $0 < \delta \leq \delta(x)/2$ such that $\alpha_G(x, x+t\theta)/\alpha(x+t\phi) > 1/\epsilon$ for $|t| \leq \delta$. We then argue as above that $\alpha_L(G) > 1/\epsilon$, and so we find that $\alpha_L(G) = \infty$. □

We are now ready for the proof of the main result.

Proof of Theorem 1.4. Suppose first that $\alpha_L(G) \to 1$ as $L \to 1$. It follows from Lemma 5.5 that $qi(G) \leq 1$, hence, $qi(G) = 1$, that is, G is isotropic. It then follows from [9, Theorem 1.10] that G is a ball or half-space.

Next, suppose that G is a half-space and assume, without loss of generality, that $G = H^n$. Let $f : H^n \to \mathbb{R}^n$ be an L-bilipschitz mapping. Then by Lemma 5.1 we have $\alpha_{f(H^n)} \leq L^4 \alpha_{H^n}$ and so only the lower bound remains to be established (the situation is asymmetric, since H^n is a half-space, but $f(H^n)$ need not be).

It follows from Corollary 5.4 that $f(H^n)$ is an L'-EB domain. Hence, it follows from Theorem 1.3 that

$$\begin{aligned} \alpha_{f(H^n)}\big(f(x), f(y)\big) &\geq \Big(L' - \sqrt{L'^2 - 1}\Big) j_{f(H^n)}\big(f(x), f(y)\big) \\ &\geq \Big(1 - \sqrt{1 - L'^{-2}}\Big) j_{H^n}(x, y), \end{aligned} \tag{5.6}$$

since f is L'-bilipschitz with respect to j_{H^n}, which follows directly from the Bernoulli inequalities. We see that it suffices to find a lower bound for $j_{H^n}(x, y)/\alpha_{H^n}(x, y)$.

In the half-space, we have an explicit formula for α_{H^n}, which equals the hyperbolic metric in this case [1, page 35], namely,

$$\begin{aligned} \alpha_{H^n}(x, y) &= \operatorname{arcosh}\left(1 + \frac{|x-y|^2}{2x_n y_n}\right) \\ &= \log\left(1 + \frac{|x-y|^2}{2x_n y_n} + \frac{|x-y|}{2x_n y_n}\sqrt{|x-y|^2 + 2x_n y_n}\right), \end{aligned} \tag{5.7}$$

where x_n denotes the nth coordinate of x. We use the inequality $\log(1+a)/\log(1+b) \geq a/b$, which is valid for $a \leq b$, to estimate j_{H^n}/α_{H^n}. We find that

$$\begin{aligned}\frac{j_{H^n}(x,y)}{\alpha_{H^n}(x,y)} &\geq \frac{|x-y|/\min\{x_n,y_n\}}{|x-y|^2/(2x_ny_n)+|x-y|\sqrt{|x-y|^2+4x_ny_n}/(2x_ny_n)} \\ &= \frac{2x_ny_n}{\min\{x_n,y_n\}}\frac{1}{|x-y|+\sqrt{|x-y|^2+4x_ny_n}} \\ &\geq \frac{x_ny_n}{\min\{x_n,y_n\}}\frac{1}{|x-y|+\sqrt{x_ny_n}} \\ &\geq \frac{\sqrt{x_ny_n}}{|x-y|+\sqrt{x_ny_n}} \geq \frac{1}{1+|x-y|/\min\{x_n,y_n\}},\end{aligned} \tag{5.8}$$

where we used $\min\{x_n,y_n\} \leq \sqrt{x_ny_n}$ in the last two estimates.

Combining this estimate with inequality (5.6), we find that

$$\begin{aligned}\alpha_{f(H^n)}(f(x),f(y)) &\geq \left(1-\sqrt{1-L'^{-2}}\right)j_{H^n}(x,y) \\ &\geq \left(1-\sqrt{1-L'^{-2}}\right)\frac{\alpha_{H^n}(x,y)}{1+|x-y|/\min\{x_n,y_n\}}.\end{aligned} \tag{5.9}$$

This estimate is good only when $|x-y|/\min\{x_n,y_n\}$ is small. Thus, we need another approach when $|x-y|/\min\{x_n,y_n\}$ is large.

We always have $\alpha_{f(H^n)}(f(x),f(y)) \geq \alpha_{H^n}(x,y) - 4\log L$, directly from the definition of the Apollonian metric. Hence, we find that if $\alpha_{H^n}(x,y) \geq 4\log(L)/\sqrt{L-1}$, then

$$\frac{\alpha_{f(H^n)}(f(x),f(y))}{\alpha_{H^n}(x,y)} \geq 1 - \frac{4\log L}{\alpha_{H^n}(x,y)} \geq 1-\sqrt{L-1}. \tag{5.10}$$

On the other hand, if $\alpha_{H^n}(x,y) \leq 4\log(L)/\sqrt{L-1}$, then we have (the first inequality follows from [14, Theorem 4.2] since H^n is convex)

$$\begin{aligned}1+\frac{|x-y|}{\min\{x_n,y_n\}} &= \exp\{j_{H^n}(x,y)\} \\ &\leq \exp\{\alpha_{H^n}(x,y)\} \leq \exp\left\{\frac{4\log(L)}{\sqrt{L-1}}\right\}.\end{aligned} \tag{5.11}$$

We combine the estimates from (5.9) and (5.10). This gives

$$\frac{\alpha_{f(H^n)}(f(x),f(y))}{\alpha_{H^n}(x,y)} \geq \min\left\{\left(1-\sqrt{1-L'^{-2}}\right)L^{-4/\sqrt{L-1}}, 1-\sqrt{L-1}\right\}, \tag{5.12}$$

irrespective of the value of $\alpha_{H^n}(x,y)$. Thus, we have a lower bound that approaches 1 as $L \to 1$.

Now, we are done with the case when H^n equals a half-space. But using Lemma 5.3, we easily deduce the conclusion for B^n from this. □

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