INTERNAL STABILIZATION OF MAXWELL'S EQUATIONS IN HETEROGENEOUS MEDIA

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We consider the internal stabilization of Maxwell's equations with Ohm's law with space variable coefficients in a bounded region with a smooth boundary. Our result is mainly based on an observability estimate, obtained in some particular cases by the multiplier method, a duality argument and a weakening of norm argument, and arguments used in internal stabilization of scalar wave equations.

1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^3$ with a boundary Γ of class C^2. For the sake of simplicity we further assume that Ω is simply connected and that Γ is connected.

In this paper we study the stabilization of Maxwell's equations with Ohm's law:

$$D' - \operatorname{curl}(\mu B) + \sigma D = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.1}$$

$$B' + \operatorname{curl}(\lambda D) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.2}$$

$$\operatorname{div} B = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{1.3}$$

$$D(0) = D_0, \qquad B(0) = B_0 \quad \text{in } \Omega, \tag{1.4}$$

$$D \times \nu = 0, \qquad B \cdot \nu = 0 \quad \text{on } \Gamma \times (0,+\infty), \tag{1.5}$$

where D, B are three-dimensional vector-valued functions of t, $x = (x_1,x_2,x_3)$; $\mu = \mu(x)$, $\lambda = \lambda(x)$, $\sigma = \sigma(x)$ are scalar functions in $C^1(\overline{\Omega})$ such that $\sigma(x) \geq 0$ and λ and μ are uniformly bounded from below by a positive constant, that is,

$$\lambda(x) \geq \lambda_0 > 0, \quad \mu(x) \geq \mu_0 > 0, \quad \forall x \in \overline{\Omega}. \tag{1.6}$$

D_0, B_0 are the initial data in a suitable space and ν denotes the outward unit normal vector to Γ. We further assume that σ satisfies

$$\sigma(x) \geq \sigma_0 > 0, \quad \forall x \in \omega, \tag{1.7}$$

for some non empty open subset ω of Ω.

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In that paper we will give sufficient conditions on λ, μ and ω which guarantee the exponential decay of the energy

$$\mathscr{E}(t) := \frac{1}{2}\int_{\Omega}\left(\lambda(x)\,|\,D(x,t)\,|^2 + \mu(x)\,|\,B(x,t)\,|^2\right)dx \tag{1.8}$$

of our system.

The exact boundary controllability and stabilization of Maxwell's equations have been studied by many authors [4, 6, 7, 8, 10, 13, 15, 17, 18, 19, 21] and are usually based on an observability estimate obtained by different methods like the multiplier method, microlocal analysis, the frequency domain method. A similar strategy leads to the internal controllability of Maxwell's equations, see for instance [17, 18, 22, 23].

But to our knowledge the internal stabilization of Maxwell's equations with Ohm's law is only considered for constant coefficients λ and μ [17]. Therefore our goal is to consider the internal stabilization of Maxwell's equations with Ohm's law for space variable coefficients λ and μ. We then give sufficient conditions guaranteeing the exponential decay of the energy. Our method actually combines arguments used in the stabilization of scalar wave equation with locally distributed (internal) damping [24] with the use of an internal observability estimate for the standard Maxwell equations obtained for constant coefficients by Phung [17] using microlocal analysis and extended here to some subsets ω of Ω and space variable coefficients. This observability estimate is obtained using a vectorial multiplier method (see [11] in the scalar case and [22] for constant coefficients), a duality argument from [1, 12] and a weakening of norm argument (as in [11] in the scalar case).

The schedule of the paper is the following one: Well-posedness of the problem is analysed in Section 2 under appropriate conditions on Ω, λ, μ and σ using semigroup theory. Section 3 is devoted to the proof of the observability estimate when ω is a (small) neighbourhood of the boundary. Finally we conclude in Section 4 by the exponential stability of our system.

2. Well-posedness of the problem

Introduce the Hilbert spaces

$$\begin{gathered}\hat{J}(\Omega) := \{B \in L^2(\Omega)^3 : \operatorname{div} B = 0 \text{ in } \Omega;\ B\cdot\nu = 0 \text{ on } \Gamma\},\\ H := L^2(\Omega)^3 \times \hat{J}(\Omega),\end{gathered} \tag{2.1}$$

equipped with the inner product

$$\left(\begin{pmatrix} D \\ B\end{pmatrix}, \begin{pmatrix} D_1 \\ B_1\end{pmatrix}\right)_H = \int_{\Omega}\{\lambda D D_1 + \mu B B_1\}\,dx. \tag{2.2}$$

Now define the operator A as follows:

$$D(A) = H_0(\operatorname{curl},\Omega) \times (\hat{J}(\Omega) \cap H^1(\Omega)^3), \tag{2.3}$$

where, as usual,

$$H_0(\mathrm{curl},\Omega) = \{D \in L^2(\Omega)^3 : \mathrm{curl}\, D \in L^2(\Omega)^3,\ D \times \nu = 0 \text{ on } \Gamma\}. \tag{2.4}$$

For any $\binom{D}{B}$ in $D(A)$ we take

$$A\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} \mathrm{curl}(\mu B) - \sigma D \\ -\mathrm{curl}(\lambda D) \end{pmatrix}. \tag{2.5}$$

We then see that formally problem (1.1) to (1.5) is equivalent to

$$\begin{aligned} \frac{\partial \Phi}{\partial t} &= A\Phi, \\ \Phi(0) &= \Phi_0, \end{aligned} \tag{2.6}$$

when $\Phi = \binom{D}{B}$ and $\Phi_0 = \binom{D_0}{B_0}$.

We will prove that this problem (2.6) has a unique solution using Lumer-Phillips' theorem [16] by showing the following lemma.

Lemma 2.1. *A is a maximal dissipative operator.*

Proof. We start with the dissipativeness of A, in other words we need to show that

$$\Re(A\Phi,\Phi)_H \le 0, \quad \forall \Phi \in D(A). \tag{2.7}$$

With the above notation we have

$$(A\Phi,\Phi)_H = \int_\Omega \{\lambda(\mathrm{curl}(\mu B) - \sigma D) \cdot D - \mu\, \mathrm{curl}(\lambda D) B\}\, dx. \tag{2.8}$$

By Green's formula and the boundary condition $D \times \nu = 0$ on Γ, we get

$$(A\Phi,\Phi)_H = -\int_\Omega \lambda\sigma |D|^2 dx \le 0. \tag{2.9}$$

Let us now pass to the maximality. For that purpose it suffices to show that for all $\binom{f}{g}$ in H, there exists a unique $\binom{D}{B}$ in $D(A)$ such that

$$(I - A)\begin{pmatrix} D \\ B \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}. \tag{2.10}$$

Equivalently, we have

$$B = g - \mathrm{curl}(\lambda D), \tag{2.11}$$

$$D + \mathrm{curl}\left(\mu\, \mathrm{curl}(\lambda D)\right) + \sigma D = f + \mathrm{curl}(\mu g). \tag{2.12}$$

This last problem has a unique solution D in $H_0(\mathrm{curl},\Omega)$ because its variational formulation is

$$\begin{aligned}\int_\Omega \{\mu\,\mathrm{curl}(\lambda D)\cdot\mathrm{curl}(\lambda w)+\lambda(1+\sigma)D\cdot w\}\,dx \\ = \int_\Omega \{\lambda f\cdot w+\mu g\cdot\mathrm{curl}(\lambda w)\}\,dx, \quad \forall w\in H_0(\mathrm{curl},\Omega).\end{aligned} \tag{2.13}$$

This problem has a unique solution by the Lax-Milgram lemma because the bilinear form defined as the left-hand side is coercive on $H_0(\mathrm{curl},\Omega)$ because $\lambda(1+\sigma)\geq\lambda_0$.

It then remains to show that B given by (2.11) belongs to $\hat{J}(\Omega)\cap H^1(\Omega)^3$. Indeed by (2.11), we see that

$$\mathrm{curl}(\mu B)=(1+\sigma)D-f, \tag{2.14}$$

which shows that $\mathrm{curl}\,B\in L^2(\Omega)^3$. On the other hand $\mathrm{div}\,B=\mathrm{div}\,g=0$ since g belongs to $\hat{J}(\Omega)$. Finally $B\cdot\nu=0$ on Γ because the boundary condition $\lambda D\times\nu=0$ on Γ implies that $\mathrm{curl}(\lambda D)\cdot\nu=0$ on Γ and because $g\in\hat{J}(\Omega)$. Altogether we have that $B\in H_T(\mathrm{curl},\mathrm{div},\Omega)$, where

$$\begin{aligned}H_T(\mathrm{curl},\mathrm{div},\Omega):=\{B\in L^2(\Omega)^3:\mathrm{curl}\,B\in L^2(\Omega)^3,\\ \mathrm{div}\,B\in L^2(\Omega);\ B\cdot\nu=0\text{ on }\Gamma\}.\end{aligned} \tag{2.15}$$

Since the boundary Γ is supposed to be smooth we have the continuous embedding $H_T(\mathrm{curl},\mathrm{div},\Omega)\hookrightarrow(H^1(\Omega))^3$ (see, e.g., [5, Section I.3.4]), which leads to the requested regularity on B. □

Since it is well-known that $D(A)$ is dense in H (see [9, Section 7] or [10]), by Lumer-Phillips' theorem (see, e.g., [16, Theorem I.4.3]), we conclude that A generates a C_0-semigroup of contraction $T(t)$. Therefore we have the following existence result.

Theorem 2.2. *For all $\Phi_0\in H$, the problem (2.6) has a weak solution $\Phi\in C([0,\infty),H)$ given by $\Phi=T(t)\Phi_0$. If moreover $\Phi_0\in D(A)$, the problem (2.6) has a strong solution $\Phi\in C([0,\infty),D(A))\cap C^1([0,\infty),H)$.*

For our further use we also need the next result.

Theorem 2.3. *Fix $T>0$. Then for all $f\in L^2(0,T;L^2(\Omega)^3)$, the problem*

$$D'-\mathrm{curl}(\mu B)=f \quad \textit{in } Q_T:=\Omega\times(0,T), \tag{2.16}$$

$$B'+\mathrm{curl}(\lambda D)=0 \quad \textit{in } Q_T, \tag{2.17}$$

$$\mathrm{div}\,B=0 \quad \textit{in } Q_T, \tag{2.18}$$

$$D(0)=0,\quad B(0)=0 \quad \textit{in } \Omega, \tag{2.19}$$

$$D\times\nu=0,\quad B\cdot\nu=0 \quad \textit{on } \Sigma_T:=\Gamma\times(0,T), \tag{2.20}$$

has a unique mild solution $\binom{D}{B} \in C([0,T),H)$ *which satisfies the estimate*

$$\int_{Q_T} \left\{ |D(x,t)|^2 + |B(x,t)|^2 \right\} dx\,dt \leq CT^2 \int_{Q_T} |f(x,t)|^2 dx\,dt, \tag{2.21}$$

for some positive constant C *depending on* λ *and* μ.

Proof. Denoting by A_0 the above operator A corresponding to $\sigma = 0$, the above problem (2.16) to (2.20) is equivalent to

$$\begin{gathered} \frac{\partial \Phi}{\partial t} = A_0 \Phi + F, \\ \Phi(0) = 0, \end{gathered} \tag{2.22}$$

when $\Phi = \binom{D}{B}$ and $F = \binom{f}{0}$.

As A_0 generates a C_0-semigroup of contraction $T_0(t)$, problem (2.22) has a unique mild solution $\Phi \in C([0,\infty),H)$ given by (see [16, Section 4.4.2])

$$\Phi(t) = \int_0^t T_0(t-s) F(s) ds. \tag{2.23}$$

This identity implies that

$$||\Phi(t)||_H \leq \int_0^t ||F(s)||_H \, ds \leq \int_0^t \left(\int_\Omega \lambda(x) |f(x,s)|^2 \, dx \right)^{1/2} ds. \tag{2.24}$$

We conclude by integrating the square of this estimate in $t \in (0,T)$, using Cauchy-Schwarz's inequality and taking into account the assumption (1.6). □

We end this section by showing that the energy of our system is decreasing.

LEMMA 2.4. *Let* (D_0, B_0) *be an initial pair in* H *and let* (D,B) *be the solution of the system (1.1), (1.2), (1.3), (1.4), and (1.5). Then the derivative of the energy (defined by (1.8)) is*

$$\mathscr{E}'(t) = -\int_\Omega \lambda \sigma |D|^2 dx \leq 0, \quad \forall t > 0. \tag{2.25}$$

Proof. Deriving (1.8) we obtain

$$\mathscr{E}' = \int_\Omega \{\lambda D \cdot D' + \mu B \cdot B'\} dx, \tag{2.26}$$

then, by (1.1) and (1.2),

$$\mathscr{E}' = \int_\Omega \{\lambda D \cdot (\operatorname{curl} \mu B - \sigma D) - \mu B \cdot \operatorname{curl} \lambda D\} dx. \tag{2.27}$$

We conclude by integrating by parts in the first term of this right-hand side and using the boundary condition (1.5). □

From this lemma we directly conclude that the energy is non-increasing.

Corollary 2.5. *Let (D_0,B_0) be an initial pair in H and let (D,B) be the solution of the system (1.1), (1.2), (1.3), (1.4), and (1.5). Then, for all $0 \le S < T < +\infty$, we have*

$$\mathscr{E}(S) - \mathscr{E}(T) = \int_S^T \int_\Omega \lambda\sigma |D|^2 dx \ge 0. \tag{2.28}$$

3. An observability estimate

Let us consider the solution (D_h,B_h) of the standard Maxwell system:

$$D_h' - \operatorname{curl}(\mu B_h) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{3.1}$$

$$B_h' + \operatorname{curl}(\lambda D_h) = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{3.2}$$

$$\operatorname{div} D_h = \operatorname{div} B_h = 0 \quad \text{in } \Omega \times (0,+\infty), \tag{3.3}$$

$$D_h(0) = D_0, \quad B_h(0) = B_0 \quad \text{in } \Omega, \tag{3.4}$$

$$D_h \times \nu = 0, \quad B_h \cdot \nu = 0 \quad \text{on } \Gamma \times (0,+\infty). \tag{3.5}$$

For our next purposes, we need that the following internal observability estimate holds: The subset ω of Ω is such that there exist a time $T > 0$ and a constant $C > 0$ such that

$$\frac{1}{2}\int_\Omega \left(\lambda(x)\left|D_0(x)\right|^2 + \mu(x)\left|B_0(x)\right|^2\right)dx \le C\int_0^T \int_\omega \left|D_h(x,t)\right|^2 dx\,dt, \quad \forall (D_0,B_0) \in H_1, \tag{3.6}$$

where

$$H_1 = \{(D,B) \in H : \operatorname{div} D = 0 \text{ in } \Omega\}. \tag{3.7}$$

This estimate was proved by Phung [17, Theorem 3.4] using microlocal analysis, when μ and λ are constant and $\omega = \tilde{\omega} \cap \Omega$ such that $\tilde{\omega}$ controls geometrically Ω. We will extend such an estimate to variable coefficients and some open subsets ω using the multiplier method. For that purpose, we further require that there exist $x_0 \in \Omega$ and a positive constant c_0 such that

$$\begin{aligned} \lambda(x) - \nabla\lambda(x) \cdot (x - x_0) &\ge c_0 \lambda(x), \\ \mu(x) - \nabla\mu(x) \cdot (x - x_0) &\ge c_0 \mu(x), \end{aligned} \tag{3.8}$$

for all $x \in \Omega$.

We first reduce the estimate to the estimate of the electric field.

Lemma 3.1. *Fix $T > 0$. Let (D_h,B_h) be the solution of (3.1), (3.2), (3.3), (3.4), and (3.5) with initial datum $(D_0,B_0) \in H_1$. Then there exists $C > 0$ such that*

$$\frac{1}{2}\int_\Omega \left(\lambda(x)\left|D_0(x)\right|^2 + \mu(x)\left|B_0(x)\right|^2\right)dx \le C\int_0^T \int_\Omega \left|D_h(x,t)\right|^2 dx\,dt, \quad \forall (D_0,B_0) \in H_1. \tag{3.9}$$

Proof. We adapt step 1 of the proof of [17, Theorem 3.4] to our setting. Recall that the Hilbert space $H_T(\text{curl},\text{div},\Omega)$, defined in (2.15), equipped with its natural norm is compactly embedded into $(L^2(\Omega))^3$ [20]. Therefore there exists a unique $\psi \in H_T(\text{curl},\text{div},\Omega)$ solution of

$$\begin{aligned} \text{curl}(\lambda\,\text{curl}\,\psi) = B_h \quad \text{in } \Omega, \\ \text{div}\,\psi = 0 \quad \text{in } \Omega, \\ \psi\cdot\nu = 0, \quad \text{curl}\,\psi\times\nu = 0 \quad \text{on } \Gamma, \end{aligned} \tag{3.10}$$

in the sense that $\psi \in H_T(\text{curl},\text{div},\Omega)$ is the unique solution of

$$\int_\Omega \{\lambda\,\text{curl}\,\psi\cdot\text{curl}\,w + \text{div}\,\psi\,\text{div}\,w\}\,dx = \int_\Omega B_h\cdot w\,dx, \quad \forall w \in H_T(\text{curl},\text{div},\Omega). \tag{3.11}$$

Indeed the above compactness property and the hypotheses on Ω and Γ guarantee that the left-hand side of (3.11) is coercive on $H_T(\text{curl},\text{div},\Omega)$. On the other hand since $\text{div}\,B_h = 0$ in Ω we easily see that the solution ψ of (3.11) satisfies (3.10) (see [2, Theorem 1.1]). Setting $A = \text{curl}\,\psi$, we deduce that

$$B_h = \text{curl}(\lambda A) \quad \text{in } \Omega, \tag{3.12}$$

$$\text{div}\,A = 0 \quad \text{in } \Omega, \tag{3.13}$$

$$A\times\nu = 0 \quad \text{on } \Gamma. \tag{3.14}$$

Moreover taking $w = \psi$ in (3.11) we see that

$$\lambda_0\|A\|^2_{L^2(\Omega)^3} \le ||B_h||_{L^2(\Omega)^3}\|\psi\|_{L^2(\Omega)^3} \le C||B_h||_{L^2(\Omega)^3}\|A\|_{L^2(\Omega)^3}, \tag{3.15}$$

this last estimate following from the compact embedding of $H_T(\text{curl},\text{div},\Omega)$ into $(L^2(\Omega))^3$. In other words we have

$$\|A\|_{L^2(\Omega)^3} \le C||B_h||_{L^2(\Omega)^3}. \tag{3.16}$$

Using (3.2), (3.3), (3.5) and (3.12) to (3.14), we see that

$$\text{curl}\,\big(\lambda(A'+D_h)\big) = 0 \quad \text{in } \Omega, \tag{3.17}$$

$$\text{div}\,(A'+D_h) = 0 \quad \text{in } \Omega, \tag{3.18}$$

$$(A'+D_h)\times\nu = 0 \quad \text{on } \Gamma. \tag{3.19}$$

The first identity and the fact that Ω is simply connected imply that

$$\lambda(A'+D_h) = \nabla\varphi, \tag{3.20}$$

with $\varphi \in H^1(\Omega)$. The properties (3.18), (3.19) and the fact that Γ is connected imply that φ is constant and therefore we conclude that

$$A'+D_h = 0 \quad \text{in } \Omega. \tag{3.21}$$

Take $\Phi(t) = t(T-t)$ and consider

$$\int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt. \tag{3.22}$$

Then by (3.12) and Green's formula we get, owing to (3.14),

$$\int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt = \int_{Q_T} \Phi(t)^2 \operatorname{curl}(\mu B_h) \cdot \lambda A\,dx\,dt. \tag{3.23}$$

Therefore by (3.1) we obtain

$$\int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt = \int_{Q_T} \lambda \Phi(t)^2 D_h' \cdot A\,dx\,dt. \tag{3.24}$$

Now by integration by parts in t, we get

$$\int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt = -\int_{Q_T} \lambda \left(2\Phi\Phi' A + \Phi^2 A'\right) \cdot D_h\,dx\,dt. \tag{3.25}$$

The identity (3.21) then yields

$$\int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt = -2\int_{Q_T} \lambda \Phi\Phi' A \cdot D_h dx\,dt + \int_{Q_T} \lambda \Phi^2 \left| D_h \right|^2 dx\,dt. \tag{3.26}$$

Using Young's inequality we arrive at

$$\begin{aligned} \int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt \le{}& \epsilon \int_{Q_T} \lambda \Phi^2 |A|^2 dx\,dt \\ &+ \frac{1}{\epsilon}\int_{Q_T} \lambda (\Phi')^2 \left| D_h \right|^2 dx\,dt + \int_{Q_T} \lambda \Phi^2 \left| D_h \right|^2 dx\,dt, \end{aligned} \tag{3.27}$$

for any $\epsilon > 0$. Using finally the estimate (3.16) we have proved that

$$\begin{aligned} \int_{Q_T} \mu(x)\Phi(t)^2 \left| B_h(x,t) \right|^2 dx\,dt \le{}& \frac{C\epsilon}{\mu_0} \int_{Q_T} \Phi^2 \mu \left| B_h \right|^2 dx\,dt \\ &+ \frac{1}{\epsilon}\int_{Q_T} \lambda (\Phi')^2 \left| D_h \right|^2 dx\,dt + \int_{Q_T} \lambda \Phi^2 \left| D_h \right|^2 dx\,dt, \end{aligned} \tag{3.28}$$

for any $\epsilon > 0$. Choosing ϵ small enough we arrive at

$$\int_{Q_T} \mu \Phi^2 \left| B_h \right|^2 dx\,dt \le C \int_{Q_T} \lambda \left| D_h \right|^2 dx\,dt. \tag{3.29}$$

Using the conservation of energy (identity (2.28) with $\sigma = 0$) we may write

$$\int_{Q_T} \left(\mu \left| B_h \right|^2 + \lambda \left| D_h \right|^2\right) dx\,dt = 3\int_{T/3}^{2T/3} \int_{\Omega} \left(\mu \left| B_h \right|^2 + \lambda \left| D_h \right|^2\right) dx\,dt. \tag{3.30}$$

As $\Phi(t) \geq 2T^2/9$ on $[T/3, 2T/3]$ we get

$$\int_{Q_T} \left(\mu |B_h|^2 + \lambda |D_h|^2\right) dx\,dt \leq \frac{243}{4T^4} \int_{T/3}^{2T/3} \mu\Phi^2 |B_h|^2 dx\,dt + 3\int_{Q_T} \lambda |D_h|^2 dx\,dt. \tag{3.31}$$

The conclusion follows from (3.29). □

Since it remains to estimate $\int_0^T \int_\Omega |D_h(x,t)|^2 dx\,dt$ we are looking at D_h as solution of the following second order system:

$$D_h'' + \operatorname{curl}(\mu \operatorname{curl}(\lambda D_h)) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{3.32}$$

$$\operatorname{div} D_h = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{3.33}$$

$$D_h(0) = D_0, \quad D_h'(0) = D_1 = \operatorname{curl}(\mu B_0) \quad \text{in } \Omega, \tag{3.34}$$

$$D_h \times \nu = 0, \quad \operatorname{curl}(\lambda D_h) \cdot \nu = 0 \quad \text{on } \Gamma \times (0, +\infty). \tag{3.35}$$

Consider the set

$$\begin{aligned} &H_N(\operatorname{curl}, \operatorname{div}, \Omega) \\ &\quad := \{D \in L^2(\Omega)^3 : \operatorname{curl} D \in L^2(\Omega)^3,\ \operatorname{div} D \in L^2(\Omega);\ D \times \nu = 0 \text{ on } \Gamma\}, \end{aligned} \tag{3.36}$$

continuously embedded into $H^1(\Omega)^3$ (see, e.g., [5, Section I.3.4]) and compactly embedded into $L^2(\Omega)^3$ [20]. Let us set

$$\begin{aligned} \mathcal{H} &:= \{D \in L^2(\Omega)^3 : \operatorname{div} D = 0 \text{ in } \Omega\}, \\ \mathcal{V} &:= \{D \in H_N(\operatorname{curl}, \operatorname{div}, \Omega) : \operatorname{div} D = 0 \text{ in } \Omega\}, \\ a(D, D_1) &:= \int_\Omega \mu \operatorname{curl}(\lambda D) \cdot \operatorname{curl}(\lambda D_1)\, dx, \quad \forall D, D_1 \in \mathcal{V}. \end{aligned} \tag{3.37}$$

The bilinear form a is symmetric and strongly coercive on $\mathcal{V}$, moreover $\mathcal{V}$ is compactly embedded into $\mathcal{H}$ (see [10]). By spectral analysis, the above problem has a unique solution $D_h \in C([0,T], \mathcal{V}) \cap C^1([0,T], \mathcal{H})$ if (D_0, D_1) belongs to $\mathcal{V} \times \mathcal{H}$. Obviously D_h is the same as the one from problem (3.1), (3.2), (3.3), (3.4), and (3.5) if $(D_0, B_0) \in \mathcal{V} \times (\hat{J}(\Omega) \cap H^1(\Omega)^3)$, because then $(D_0, D_1 = \operatorname{curl}(\mu B_0))$ belongs to $\mathcal{V} \times \mathcal{H}$.

The energy of the solution of that system is given by

$$E_D(t) := \frac{1}{2} \int_\Omega \left(\lambda(x) |D_h'(x,t)|^2 + \mu(x) |\operatorname{curl}(\lambda(x) D_h(x,t))|^2\right) dx. \tag{3.38}$$

A simple application of Green's formula shows that

$$E_D'(t) = 0, \tag{3.39}$$

and therefore the energy E_D is constant.

Using a vectorial multiplier method we first prove the following lemma. An analogous lemma was proved in [22] in the case of constant coefficients.

Lemma 3.2. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$, and let $q : \overline{\Omega} \to \mathbb{R}^3$ a C^1 vector field. Then for any time $T > 0$ the following identity holds:*

$$\begin{aligned}
&\left[\int_\Omega 2(D_h', q, \operatorname{curl}(\lambda D_h))\,dx\right]_0^T \\
&\quad = \int_0^T \int_\Gamma \left[\lambda(q\cdot\nu)|D_h'|^2 - \mu(q\cdot\nu)|\operatorname{curl}(\lambda D_h)|^2\right] d\Gamma\,dt \\
&\qquad + \int_0^T \int_\Omega \Bigg[\left(\lambda|D_h'|^2 + \mu|\operatorname{curl}(\lambda D_h)|^2\right)\operatorname{div} q - 2\lambda \sum_{i,j=1}^3 (D_h')_i (D_h')_j \partial_i q_j \\
&\qquad\qquad - 2\mu \sum_{i,j=1}^3 (\operatorname{curl}(\lambda D_h))_i (\operatorname{curl}(\lambda D_h))_j \partial_i q_j\Bigg] dx\,dt \\
&\qquad - \int_0^T \int_\Omega \left[|D_h'|^2 q\cdot\nabla\lambda + |\operatorname{curl}(\lambda D_h)|^2 q\cdot\nabla\mu\right] dx\,dt,
\end{aligned} \tag{3.40}$$

where the notation $(a,b,c) = a\cdot(b\times c)$ means the mixed product of the vectors a, b, c.

Proof. By (3.32)

$$\begin{aligned}
0 &= \int_0^T \int_\Omega 2(D_h'' + \operatorname{curl}(\mu \operatorname{curl}(\lambda D_h)), q, \operatorname{curl}(\lambda D_h))\,dx\,dt \\
&= \left[\int_\Omega 2(D_h', q, \operatorname{curl}(\lambda D_h))\,dx\right]_0^T + \int_0^T \int_\Gamma 2\mu(\nu, \operatorname{curl}(\lambda D_h), q\times\operatorname{curl}(\lambda D_h))\,d\Gamma\,dt \\
&\quad + \int_0^T \int_\Omega 2\left[\mu \operatorname{curl}(\lambda D_h)\cdot\operatorname{curl}(q\times\operatorname{curl}(\lambda D_h)) - (D_h', q, \operatorname{curl}(\lambda D_h'))\right] dx\,dt.
\end{aligned} \tag{3.41}$$

Integrating by parts we obtain

$$\begin{aligned}
&\int_0^T \int_\Omega -2(D_h', q, \operatorname{curl}(\lambda D_h'))\,dx\,dt = \int_0^T \int_\Omega 2\lambda D_h' \cdot \operatorname{curl}(q\times D_h')\,dx\,dt \\
&\quad = \int_0^T \int_\Omega 2\lambda \Bigg[D_h' \cdot (q \operatorname{div} D_h' - D_h' \operatorname{div} q) + \sum_{i,j=1}^3 (D_h')_i (D_h')_j \partial_i q_j \\
&\qquad\qquad - \sum_{i,j=1}^3 (D_h')_j q_i \partial_i (D_h')_j\Bigg] dx\,dt \\
&\quad = \int_0^T \int_\Omega \left[2\lambda \sum_{i,j=1}^3 (D_h')_i (D_h')_j \partial_i q_j - 2\lambda |D_h'|^2 \operatorname{div} q - \lambda q\cdot\nabla\left(|D_h'|^2\right)\right] dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_\Omega \left[2\lambda \sum_{i,j=1}^{3} (D_h')_i (D_h')_j \partial_i q_j - 2\lambda \,|\, D_h' \,|^2 \operatorname{div} q + |\, D_h' \,|^2 \operatorname{div}(\lambda q) \right] dx\,dt \\
&\quad - \int_0^T \int_\Gamma \lambda (q \cdot \nu) \,|\, D_h' \,|^2 d\Gamma\,dt,
\end{aligned}
\tag{3.42}
$$

and then

$$
\begin{aligned}
&\int_0^T \int_\Omega -2 \left(D_h', q, \operatorname{curl} (\lambda D_h') \right) dx\,dt \\
&\quad = -\int_0^T \int_\Gamma \lambda (q \cdot \nu) \,|\, D_h' \,|^2 d\Gamma\,dt \\
&\quad\quad + \int_0^T \int_\Omega \left[2\lambda \sum_{i,j=1}^{3} (D_h')_i (D_h')_j \partial_i q_j - \lambda \,|\, D_h' \,|^2 \operatorname{div} q + |\, D_h' \,|^2 q \cdot \nabla \lambda \right] dx\,dt.
\end{aligned}
\tag{3.43}
$$

Analogously, we can rewrite

$$
\begin{aligned}
&\int_0^T \int_\Omega 2\mu \operatorname{curl} (\lambda D_h) \cdot \operatorname{curl} (q \times \operatorname{curl} (\lambda D_h))\, dx\,dt \\
&\quad = \int_0^T \int_\Omega 2\mu \Bigg\{ \operatorname{curl} (\lambda D_h) \cdot \left[q \operatorname{div} \operatorname{curl} (\lambda D_h) - \operatorname{curl} (\lambda D_h) \operatorname{div} q \right] \\
&\qquad\qquad + \sum_{i,j=1}^{3} (\operatorname{curl} (\lambda D_h))_i (\operatorname{curl} (\lambda D_h))_j \partial_i q_j \\
&\qquad\qquad - \sum_{i,j=1}^{3} (\operatorname{curl} (\lambda D_h))_j q_i \partial_i (\operatorname{curl} (\lambda D_h))_j \Bigg\} dx\,dt \\
&\quad = \int_0^T \int_\Omega \Bigg\{ 2\mu \Bigg[\sum_{i,j=1}^{3} (\operatorname{curl} (\lambda D_h))_i (\operatorname{curl} (\lambda D_h))_j \partial_i q_j - |\operatorname{curl} (\lambda D_h)|^2 \operatorname{div} q \Bigg] \\
&\qquad\qquad - \mu q \cdot \nabla \left(|\operatorname{curl} (\lambda D_h)|^2 \right) \Bigg\} dx\,dt \\
&\quad = \int_0^T \int_\Omega \Bigg\{ 2\mu \Bigg[\sum_{i,j=1}^{3} (\operatorname{curl} (\lambda D_h))_i (\operatorname{curl} (\lambda D_h))_j \partial_i q_j - |\operatorname{curl} (\lambda D_h)|^2 \operatorname{div} q \Bigg] \\
&\qquad\qquad + |\operatorname{curl} (\lambda D_h)|^2 \operatorname{div}(\mu q) \Bigg\} dx\,dt - \int_0^T \int_\Gamma \mu (q \cdot \nu) |\operatorname{curl} (\lambda D_h)|^2 d\Gamma\,dt,
\end{aligned}
\tag{3.44}
$$

and then

$$\begin{aligned}\int_0^T\int_\Omega 2\mu\,\mathrm{curl}\,(\lambda D_h)\cdot\mathrm{curl}\,(q\times\mathrm{curl}\,(\lambda D_h))\,dx\,dt\\ = -\int_0^T\int_\Gamma \mu(q\cdot\nu)\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2\,d\Gamma\,dt\\ +\int_0^T\int_\Omega\bigg\{2\mu\bigg[\sum_{i,j=1}^3(\mathrm{curl}\,(\lambda D_h))_i(\mathrm{curl}\,(\lambda D_h))_j\partial_i q_j-\mu\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2\,\mathrm{div}\,q\bigg]\\ +\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2 q\cdot\nabla\mu\bigg\}\,dx\,dt.\end{aligned}\tag{3.45}$$

Putting (3.43) and (3.45) in the first identity, we obtain

$$\begin{aligned}0=\bigg[\int_\Omega 2(D_h',q,\mathrm{curl}\,(\lambda D_h))\,dx\bigg]_0^T+\int_0^T\int_\Omega\Big[\,|\,D_h'\,|^2 q\cdot\nabla\lambda+|\,\mathrm{curl}\,(\lambda D_h)\,|^2 q\cdot\nabla\mu\Big]\,dx\,dt\\ +\int_0^T\int_\Gamma\Big[2\mu(\nu,\mathrm{curl}\,(\lambda D_h),\ q\times\mathrm{curl}\,(\lambda D_h))-\lambda(q\cdot\nu)\,|\,D_h'\,|^2\\ -\mu(q\cdot\nu)\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2\Big]\,d\Gamma\,dt\\ +\int_0^T\int_\Omega\bigg[2\lambda\sum_{i,j=1}^3(D_h')_i(D_h')_j\partial_i q_j+2\mu\sum_{i,j=1}^3(\mathrm{curl}\,(\lambda D_h))_i(\mathrm{curl}\,(\lambda D_h))_j\partial_i q_j\\ -\Big(\lambda\,|\,D_h'\,|^2+\mu\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2\Big)\,\mathrm{div}\,q\bigg]\,dx\,dt.\end{aligned}\tag{3.46}$$

Therefore (3.40) follows observing that the boundary term can be rewritten using

$$\begin{aligned}2\mu(\nu,\mathrm{curl}\,(\lambda D_h),\ q\times\mathrm{curl}\,(\lambda D_h))\\ =2\mu(q\cdot\nu)\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2\\ -2\mu(\nu\cdot\mathrm{curl}\,(\lambda D_h))(q\cdot\mathrm{curl}\,(\lambda D_h))\\ =2\mu(q\cdot\nu)\,|\,\mathrm{curl}\,(\lambda D_h)\,|^2,\end{aligned}\tag{3.47}$$

recalling that $\mathrm{curl}(\lambda D_h)\cdot\nu=0$ on $\Gamma\times(0,\infty)$. □

For any $\varepsilon>0$ let us denote by $\mathcal{N}_\varepsilon(\Gamma)$ the neighborhood of Γ of radius ε, that is,

$$\mathcal{N}_\varepsilon(\Gamma)=\Big\{x\in\Omega:\inf_{y\in\Gamma}|x-y|<\varepsilon\Big\}.\tag{3.48}$$

Using the previous identity we prove the following lemma:

Lemma 3.3. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$. If $\tilde{\omega} = \mathcal{N}_{\epsilon/2}(\Gamma)$, for some $\epsilon > 0$ and λ, μ satisfy (1.6), (3.8), then there exist $T_0 > 0$ and $C > 0$ such that for $T > T_0$ we have*

$$(T - T_0) E_D(0) \leq C \int_0^T \int_{\tilde{\omega}} \left(|D_h'(x,t)|^2 + |D_h(x,t)|^2 \right) dx\, dt. \tag{3.49}$$

Proof. From (3.40), using the standard multiplier $q(x) = m(x) = x - x_0$, we obtain for any $T > 0$

$$\begin{aligned} \int_0^T \int_\Gamma (m \cdot \nu) \left[\lambda |D_h'|^2 - \mu |\operatorname{curl}(\lambda D_h)|^2 \right] d\Gamma\, dt & \\ = \left[\int_\Omega 2 (D_h', m, \operatorname{curl}(\lambda D_h))\, dx \right]_0^T - \int_0^T \int_\Omega \left[\lambda |D_h'|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2 \right] dx\, dt & \\ + \int_0^T \int_\Omega \left[|D_h'|^2 m \cdot \nabla \lambda + |\operatorname{curl}(\lambda D_h)|^2 m \cdot \nabla \mu \right] dx\, dt. & \end{aligned} \tag{3.50}$$

Using the assumption (3.8), the above identity implies

$$\begin{aligned} c_0 T \int_\Omega \left[\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2 \right] dx - 2 \left[\int_\Omega (D_h', m, \operatorname{curl}(\lambda D_h))\, dx \right]_0^T & \\ \leq \int_0^T \int_\Gamma (m \cdot \nu) \left[\mu |\operatorname{curl}(\lambda D_h)|^2 - \lambda |D_h'|^2 \right] d\Gamma dt. & \end{aligned} \tag{3.51}$$

Note that by (1.6)

$$\left| \left[2 \int_\Omega (D_h', m, \operatorname{curl}(\lambda D_h))\, dx \right]_0^T \right| \leq \frac{2 \max_{\overline{\Omega}} |m|}{\sqrt{\lambda_0 \mu_0}} \int_\Omega \left[\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2 \right] dx. \tag{3.52}$$

So, setting

$$\tilde{T} = \frac{2 \max_{\overline{\Omega}} |m|}{c_0 \sqrt{\lambda_0 \mu_0}}, \tag{3.53}$$

we obtain

$$\begin{aligned} c_0 (T - \tilde{T}) \int_\Omega \left(\lambda |D_1|^2 + \mu |\operatorname{curl}(\lambda D_0)|^2 \right) dx & \\ \leq \int_0^T \int_\Gamma (m \cdot \nu) \left[\mu |\operatorname{curl}(\lambda D_h)|^2 - \lambda |D_h'|^2 \right] d\Gamma\, dt. & \end{aligned} \tag{3.54}$$

Now, set $\omega_0 = \mathcal{N}_{\varepsilon/4}(\Gamma)$ and apply (3.40) using as multiplier $q(x) = \varphi(x) m(x)$ with $\varphi \in C^1(\overline{\Omega})$, $0 \leq \varphi(x) \leq 1$,

$$\varphi(x) \equiv 1, \quad x \in \mathcal{N}_{\varepsilon/8}(\Gamma), \qquad \varphi(x) \equiv 0, \quad x \in \Omega \setminus \omega_0. \tag{3.55}$$

We obtain

$$\begin{aligned}\int_0^T\int_\Gamma (m\cdot\nu)\Big[\mu|\operatorname{curl}(\lambda D_h)|^2-\lambda|D_h'|^2\Big]d\Gamma\,dt&\\ \leq C\int_0^T\int_{\omega_0}\Big(|D_h'|^2+|\operatorname{curl}(\lambda D_h)|^2\Big)dx\,dt&\\ +c_0\tilde{T}\int_\Omega\Big(\lambda|D_1|^2+\mu|\operatorname{curl}(\lambda D_0)|^2\Big)dx,&\end{aligned}\tag{3.56}$$

for a suitable constant $C>0$. Then, from (3.54) and (3.56),

$$c_0(T-2\tilde{T})\int_\Omega\Big(\lambda|D_1|^2+\mu|\operatorname{curl}(\lambda D_0)|^2\Big)dx\leq C\int_0^T\int_{\omega_0}\Big(|D_h'|^2+|\operatorname{curl}(\lambda D_h)|^2\Big)dx\,dt.\tag{3.57}$$

Now, let $g:\overline{\Omega}\to\mathbb{R}$ be a C^1 function with $0\leq g(x)\leq 1$, and

$$g(x)\equiv 1,\quad x\in\omega_0,\qquad g(x)\equiv 0,\quad x\in\Omega\setminus\tilde{\omega}.\tag{3.58}$$

By (3.32), for any positive time T, by integration by parts, we have

$$\begin{aligned}0=\int_0^T\int_\Omega\Big[D_h''+\operatorname{curl}(\mu\operatorname{curl}(\lambda D_h))\Big]\cdot(g\lambda D_h)\,dx\,dt=\Big[\int_\Omega\lambda g D_h'\cdot D_h dx\Big]_0^T&\\ -\int_0^T\int_\Omega\lambda g|D_h'|^2dx\,dt+\int_0^T\int_\Omega\mu\operatorname{curl}(\lambda D_h)\cdot\Big[-\lambda D_h\times\nabla g+g\operatorname{curl}(\lambda D_h)\Big]dx\,dt.&\end{aligned}\tag{3.59}$$

Then,

$$\begin{aligned}\int_0^T\int_\Omega\mu g|\operatorname{curl}(\lambda D_h)|^2dx\,dt=\int_0^T\int_\Omega\lambda g|D_h'|^2dx\,dt-\Big[\int_\Omega\lambda g D_h'\cdot D_h\,dx\Big]_0^T&\\ +2\int_0^T\int_\Omega\mu\sqrt{g}\operatorname{curl}(\lambda D_h)\cdot(\lambda D_h\times\nabla\sqrt{g})\,dx\,dt.&\end{aligned}\tag{3.60}$$

By Young's inequality we can estimate

$$\begin{aligned}\left|2\int_0^T\int_\Omega\mu\sqrt{g}\operatorname{curl}(\lambda D_h)\cdot(\lambda D_h\times\nabla\sqrt{g})\,dx\,dt\right|&\\ \leq\frac{1}{2}\int_0^T\int_{\tilde{\omega}}\mu g|\operatorname{curl}(\lambda D_h)|^2dx\,dt+C\int_0^T\int_{\tilde{\omega}}|D_h|^2dx\,dt.&\end{aligned}\tag{3.61}$$

Moreover, using the inequality

$$\int_{\Omega} |D_h|^2 dx \leq C \int_{\Omega} |\operatorname{curl}(\lambda D_h)|^2 dx, \tag{3.62}$$

consequence of the compact embedding of $H_N(\operatorname{curl}, \operatorname{div}, \Omega)$ into $L^2(\Omega)^3$, we have

$$\left| \left[\int_{\Omega} \lambda g D_h' \cdot D_h dx \right]_0^T \right| \leq C \int_{\Omega} \left(\lambda |D_h'|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2 \right) dx. \tag{3.63}$$

Therefore, using (3.61) and (3.63) in (3.60), we obtain

$$\begin{aligned} \int_0^T \int_{\omega_0} |\operatorname{curl}(\lambda D_h)|^2 dx\, dt &\leq \int_0^T \int_{\tilde{\omega}} g |\operatorname{curl}(\lambda D_h)|^2 dx\, dt \\ &\leq C \int_{\Omega} \left(\lambda |D_h'|^2 + \mu |\operatorname{curl}(\lambda D_h)|^2 \right) dx + C' \int_0^T \int_{\tilde{\omega}} \left(|D_h|^2 + |D_h'|^2 \right) dx\, dt, \end{aligned} \tag{3.64}$$

for suitable positive constants C, C'. Finally, by (3.57) and (3.64) we have

$$(T - 2\tilde{T}) E_D(0) \leq C \int_0^T \int_{\tilde{\omega}} \left(|D_h'|^2 + |D_h|^2 \right) dx\, dt + C E_D(0), \tag{3.65}$$

for some constant $C > 0$. So, we can deduce the existence of a time T_0 such that for $T > T_0$

$$(T - T_0) E_D(0) \leq \int_0^T \int_{\tilde{\omega}} \left(|D_h'|^2 + |D_h|^2 \right) dx\, dt. \tag{3.66}$$

□

In a second step using a duality argument as in [1] (see also [12, Lemma 10]) we prove the following estimate.

Lemma 3.4. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0, D_1) \in \mathcal{V} \times \mathcal{H}$. If $\omega = \mathcal{N}_\epsilon(\Gamma)$ and $\tilde{\omega} = \mathcal{N}_{\epsilon/2}(\Gamma)$, for some $\epsilon > 0$, then there exists $C > 0$ such that for any $\eta > 0$ we have*

$$\int_0^T \int_{\tilde{\omega}} |D_h(x,t)|^2 dx\, dt \leq \frac{C}{\eta} \int_0^T \int_{\omega} |D_h'(x,t)|^2 dx\, dt + \eta \int_0^T E_D(t) dt + C E_D(0). \tag{3.67}$$

Proof. Fix $\beta \in \mathcal{D}(\mathbb{R}^3)$ such that $\beta \equiv 1$ on $\tilde{\omega}$ with a support included into ω.
Consider $z \in H_N(\text{curl}, \text{div}, \Omega)$ the unique solution of

$$\int_\Omega \mu \,\text{curl}(\lambda z) \cdot \text{curl}(\lambda w)\,dx + \int_\Omega \text{div}\, z \,\text{div}\, w\,dx = \int_\Omega \beta\lambda D_h(x,t) \cdot w(x)\,dx, \tag{3.68}$$

for all $w \in H_N(\text{curl}, \text{div}, \Omega)$. This solution z satisfies (due to the compact embedding of $H_N(\text{curl}, \text{div}, \Omega)$ in $L^2(\Omega)^3$ and to the properties of Ω and Γ)

$$\|z\|_{L^2(\Omega)^3} \le C\|\beta D_h\|_{L^2(\Omega)^3}, \tag{3.69}$$

for some $C > 0$.
Multiplying (3.32) by λz and integrating in Q_T we get

$$0 = \int_{Q_T} \lambda\left(D_h'' + \text{curl}\left(\mu \,\text{curl}\left(\lambda D_h\right)\right)\right) \cdot z\,dx\,dt. \tag{3.70}$$

Applying Green's formula (in space and time) and taking into account the boundary condition $z \times \nu = 0$ on Γ we obtain

$$0 = -\int_{Q_T} \lambda D_h' z'\,dx\,dt + \left[\int_\Omega \lambda D_h' z\,dx\right]_0^T + \int_{Q_T} \mu \,\text{curl}\left(\lambda D_h\right) \cdot \text{curl}(\lambda z)\,dx\,dt. \tag{3.71}$$

Now taking into account (3.33) and using (3.68) with $w = D_h$ we arrive at

$$0 = -\int_{Q_T} \lambda D_h' z'\,dx\,dt + \left[\int_\Omega \lambda D_h' z\,dx\right]_0^T + \int_{Q_T} \beta\lambda |D_h|^2\,dx\,dt. \tag{3.72}$$

By Cauchy-Schwarz's inequality and the fact that $\beta \equiv 1$ on $\tilde{\omega}$, we get

$$\begin{aligned}
\int_0^T \int_{\tilde{\omega}} \lambda |D_h|^2\,dx\,dt &\le \int_{Q_T} \beta\lambda |D_h|^2\,dx\,dt = \int_{Q_T} \lambda D_h' z'\,dx\,dt - \left[\int_\Omega \lambda D_h' z\,dx\right]_0^T \\
&\le \left(\int_{Q_T} \lambda |D_h'|^2\,dx\,dt\right)^{1/2} \left(\int_{Q_T} \lambda |z'|^2\,dx\,dt\right)^{1/2} \\
&\quad + \left(\int_\Omega \lambda |D_h'(x,t)|^2\,dx\right)^{1/2} \left(\int_\Omega \lambda |z(x,t)|^2\,dx\right)^{1/2} \Big|_{t=0,T}.
\end{aligned} \tag{3.73}$$

Using the estimates (3.69), (3.62) and the definition of the energy we get

$$\begin{aligned}\int_0^T\int_{\tilde{\omega}}\lambda|D_h|^2dx\,dt &\leq C\left(\int_{Q_T}\lambda|D_h'|^2dx\,dt\right)^{1/2}\left(\int_{Q_T}\beta|D_h'|^2dx\,dt\right)^{1/2}+CE_D(0)\\ &\leq C\left(\int_0^T E_D(t)dt\right)^{1/2}\left(\int_0^T\int_\omega|D_h'|^2dx\,dt\right)^{1/2}+CE_D(0).\end{aligned} \tag{3.74}$$

We conclude by Young's inequality. □

Corollary 3.5. *Let D_h be the solution of the system (3.32), (3.33), (3.34), and (3.35) with $(D_0,D_1)\in\mathcal{V}\times\mathcal{H}$. If $\omega=\mathcal{N}_\epsilon(\Gamma)$, for some $\epsilon>0$ and λ, μ satisfy (1.6), (3.8), then there exist $T_1>0$ and $C>0$ such that for $T>T_1$ we have*

$$(T-T_1)E_D(0)\leq C\int_0^T\int_\omega|D_h'(x,t)|^2dx\,dt. \tag{3.75}$$

Proof. By (3.49) and (3.67) we may write

$$\begin{aligned}(T-T_0)E_D(0)\leq{}& C\int_0^T\int_{\tilde{\omega}}|D_h'(x,t)|^2dx\,dt\\ &+\frac{C}{\eta}\int_0^T\int_\omega|D_h'(x,t)|^2dx\,dt+C\eta\int_0^T E_D(t)dt+CE_D(0),\end{aligned} \tag{3.76}$$

for any $\eta>0$. By the conservation of energy, this yields

$$(T-T_0)E_D(0)\leq C\int_0^T\int_{\tilde{\omega}}|D_h'(x,t)|^2dx\,dt+\frac{C}{\eta}\int_0^T\int_\omega|D_h'(x,t)|^2dx\,dt+C(\eta T+1)E_D(0). \tag{3.77}$$

The conclusion follows by choosing η small enough. □

We now finish by adapting a weakening of norm argument from [11, Section VII.2.4].

Lemma 3.6. *Fix $T>T_1$. Let (D_h,B_h) be the solution of (3.1), (3.2), (3.3), (3.4), and (3.5) with initial datum $(D_0,B_0)\in H_1$. If $\omega=\mathcal{N}_\epsilon(\Gamma)$, for some $\epsilon>0$, then there exists $C>0$ (depending on T) such that*

$$\int_0^T\int_\Omega|D_h(x,t)|^2dx\,dt\leq C\int_0^T\int_\omega|D_h(x,t)|^2dx\,dt. \tag{3.78}$$

Proof. We only need to prove (3.78) for $(D_0,B_0)\in\mathcal{V}\times(\hat{J}(\Omega)\cap H^1(\Omega)^3)$ since this space is dense in H_1 ([9, 10]).

Consider $\chi\in H_N(\text{curl},\text{div},\Omega)$ the unique solution of (with $D_1=\text{curl}(\mu B_0)$)

$$\begin{gathered}\text{curl}\,(\mu\,\text{curl}(\lambda\chi))=D_1\quad\text{in }\Omega,\\ \text{div}\,\chi=0\quad\text{in }\Omega,\\ \chi\times\nu=0,\quad \text{curl}(\lambda\chi)\cdot\nu=0\quad\text{on }\Gamma,\end{gathered}\tag{3.79}$$

in the sense that $\chi\in H_N(\text{curl},\text{div},\Omega)$ is the unique solution of

$$\int_\Omega\{\mu\,\text{curl}(\lambda\chi)\cdot\text{curl}(\lambda w)+\text{div}\,\chi\,\text{div}\,w\}\,dx=\int_\Omega\lambda D_1\cdot w\,dx,\quad\forall w\in H_N(\text{curl},\text{div},\Omega).\tag{3.80}$$

Set

$$w(t)=\int_0^t D_h(s)\,ds+\chi.\tag{3.81}$$

Then from (3.32), (3.33), (3.34), and (3.35) and (3.79), we see that w satisfies (3.32), (3.33), (3.35) and the initial conditions

$$w(0)=\chi\in\mathcal{V},\qquad w'(0)=D_0\in\mathcal{H}.\tag{3.82}$$

Therefore by Corollary 3.5 we have

$$\frac{T-T_1}{2T}\int_0^T\int_\Omega\left(\lambda(x)\,|\,w'(x,t)\,|^2+\mu(x)\,|\,\text{curl}\,(\lambda w(x,t))\,|^2\right)dx\,dt\le C\int_0^T\int_\omega|\,w'(x,t)\,|^2dx\,dt.\tag{3.83}$$

This estimate directly leads to the conclusion as $w'=D_h$. □

By Lemmas 3.1 and 3.6 we directly conclude the following theorem.

Theorem 3.7. *If $\omega=\mathcal{N}_\epsilon(\Gamma)$, for some $\epsilon>0$, and λ, μ satisfy (1.6), (3.8), then (3.6) holds for T large enough.*

4. The stability result

Based on the stability estimate of the previous section, we deduce our main result.

Theorem 4.1. *Let ω be a subset of Ω such that (3.6) holds. Assume that σ satisfies (1.7). Then there exist $C\ge 1$ and $\gamma>0$ such that*

$$\mathcal{E}(t)\le Ce^{-\gamma t}\mathcal{E}(0),\tag{4.1}$$

for every solution (D,B) of the system (1.1), (1.2), (1.3), (1.4), and (1.5) with initial datum in H_1.

Proof. As in [24, Theorem 1.1], we split up (D,B), solution of (1.1), (1.2), (1.3), (1.4), and (1.5) as follows:

$$(D,B) = (D_h,B_h) + (D_{nh},B_{nh}), \tag{4.2}$$

where (D_h,B_h) is solution of (3.1), (3.2), (3.3), (3.4), and (3.5) and (D_{nh},B_{nh}) is the remainder which then satisfies

$$\begin{gathered} D'_{nh} - \operatorname{curl}(\mu B_{nh}) = -\sigma D \quad \text{in } \Omega\times(0,+\infty), \\ B'_{nh} + \operatorname{curl}(\lambda D_{nh}) = 0 \quad \text{in } \Omega\times(0,+\infty), \\ \operatorname{div} B_{nh} = 0 \quad \text{in } \Omega\times(0,+\infty), \\ D_{nh}(0) = 0, \quad B_{nh}(0) = 0 \quad \text{in } \Omega, \\ D_{nh}\times\nu = 0, \quad B_{nh}\cdot\nu = 0 \quad \text{on } \Gamma\times(0,+\infty). \end{gathered} \tag{4.3}$$

Equivalently (D_{nh},B_{nh}) satisfies (2.16), (2.17), (2.18), (2.19), and (2.20) with $f = -\sigma D$. Therefore by Theorem 2.3, it holds

$$\int_{Q_T} \left\{ |D_{nh}(x,t)|^2 + |B_{nh}(x,t)|^2 \right\} dx\,dt \le CT^2 \int_{Q_T} |\sigma D(x,t)|^2 dx\,dt, \tag{4.4}$$

and since σ is bounded we get

$$\int_{Q_T} \left\{ |D_{nh}(x,t)|^2 + |B_{nh}(x,t)|^2 \right\} dx\,dt \le CT^2 \max_{x\in\bar\Omega}\sigma(x) \int_{Q_T} \sigma |D(x,t)|^2 dx\,dt. \tag{4.5}$$

On the other hand by (3.6) we have

$$\begin{aligned} \mathscr{E}(T) \le \mathscr{E}(0) &= \frac{1}{2}\int_\Omega \left(\lambda(x)|D_0(x)|^2 + \mu(x)|B_0(x)|^2\right) dx \\ &\le C\int_0^T\int_\omega |D_h(x,t)|^2 dx\,dt \\ &\le C\int_0^T\int_\omega \left\{ |D(x,t)|^2 + |D_{nh}(x,t)|^2 \right\} dx\,dt \\ &\le \frac{C}{\sigma_0}\int_0^T\int_\omega \sigma|D(x,t)|^2 dx\,dt + C\int_0^T\int_\omega |D_{nh}(x,t)|^2 dx\,dt. \end{aligned} \tag{4.6}$$

By (4.5) we conclude that

$$\mathscr{E}(T) \le C\int_{Q_T} \sigma|D(x,t)|^2 dx\,dt, \tag{4.7}$$

which leads to the conclusion due to (2.25), using a standard argument (see, e.g., [3, Theoremm 3.3] or [14, Section 3]). □

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