THE MAPPINGS OF DEGREE 1

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The maps of the form $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$, called 1-degree maps, are introduced and investigated. For noncommutative algebras and modules over them 1-degree maps give an analogy of linear maps and differentials. Under some conditions on the algebra $\mathcal{A}$, contractibility of the group of 1-degree isomorphisms is proved for the module $l_2(\mathcal{A})$. It is shown that these conditions are fulfilled for the algebra of linear maps of a finite-dimensional linear space. The notion of 1-degree map gives a possibility to define a nonlinear Fredholm map of $l_2(\mathcal{A})$ and a Fredholm manifold modelled by $l_2(\mathcal{A})$. 1-degree maps are also applied to some problems of Markov chains.

1. Motivation

For the last several decades various algebras and algebra modules have been intensively investigated. Module maps are traditionally called linear if they preserve addition and multiplication by elements of the algebra (for noncommutative algebra—left or right multiplication). For nonlinear maps of modules it is easy to give a definition of the derivative similar to Freche derivative in linear spaces. In the case of noncommutative algebra, even simplest nonlinear maps, for example, power maps of the algebra (i.e., here a "one-dimensional" module) do not have such derivatives, however there exists analogue of the differential, containing only first power of the "argument increment." Namely,

$$(x + \Delta x)^2 - x^2 = x \cdot \Delta x + \Delta x \cdot x + (\Delta x)^2,$$
$$(x + \Delta x)^3 - x^3 = x^2 \cdot \Delta x + x \cdot \Delta x \cdot x + \Delta x \cdot x^2 + (\Delta x)^2 \cdot x + \Delta x \cdot x \cdot \Delta x + x \cdot (\Delta x)^2 + (\Delta x)^3,$$

$$(1.1)$$

and so forth.

2 The mappings of degree 1

This was the reason for the author to introduce and investigate the maps as $f_1(\Delta x) = x \cdot \Delta x + \Delta x \cdot x$ and $f_2(\Delta x) = x^2 \cdot \Delta x + x \cdot \Delta x \cdot x + \Delta x \cdot x^2$.

2. 1-degree maps of algebras

Let $\mathscr{A}$ be an algebra.

Definition 2.1 [2–10]. The map $f : \mathscr{A} \to \mathscr{A}$ is called a mapping of degree 1 (1-degree map for short) if $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$ for some $n \in N$ and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in \mathscr{A}$.

By $d_1(\mathscr{A})$ denote the set of 1-degree maps.

Obviously, for a commutative algebra $\mathscr{A}$ a 1-degree map is a trivial multiplication by a specified element. That is why $\mathscr{A}$ will denote further a noncommutative algebra.

Let Map($\mathscr{A}$) be the algebra of all maps from $\mathscr{A}$ to $\mathscr{A}$ with usual addition and multiplication by number and with map composition as element's multiplication.

Let A be $\mathscr{A}$ without the element's multiplication. A is a linear space over some number field F. Let $L(A)$ denote the algebra of its linear operators. It is obvious that $d_1(\mathscr{A}) \subset L(A) \subset \text{Map}(\mathscr{A})$.

Definition 2.2. We say that $\mathscr{A}$ is *1-algebra* if $d_1(\mathscr{A}) = L(A)$.

THEOREM 2.3. *If $\mathscr{A} = L(F^n)$ for some field F, then $\mathscr{A}$ is a 1-algebra.*

Proof. It is sufficient to prove that $L(A) \subset d_1(\mathscr{A})$. Let $X \in \mathscr{A}$. Since $\mathscr{A} = L(F^n)$, then $\mathscr{A}$ is isomorphic to the algebra of square F-matrices, that is, A is n^2-dimensional linear space with matrices

$$
P_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \tag{2.1}
$$

as a basis. (Matrix P_{ij} consists of zeroes except 1 at the intersection of ith row and jth column.) If

$$
X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \tag{2.2}
$$

$X = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} P_{ij}$ and $f(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} f(P_{ij})$ for $f \in L(A)$. If $x_{ij} \neq 0$, then $P_{ij} = (1/x_{ij})P_{ii}XP_{jj} = ((1/x_{ij})P_{ii})XP_{jj} = P_{ii}X((1/x_{ij})P_{jj})$. ($P_{ij}$ can be obtained from X in another way as well.)

Let

$$f(P_{ij}) = \begin{pmatrix} a_{11}^{ij} & \cdots & a_{1n}^{ij} \\ \vdots & & \vdots \\ a_{n1}^{ij} & \cdots & a_{nn}^{ij} \end{pmatrix} = \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}^{ij} P_{kl}. \tag{2.3}$$

For any $i, j \in \{1, 2, \ldots, n\}$ we have $P_{kl} = P_{ki} P_{ij} P_{jl}$. Hence

$$\begin{aligned} f(X) &= \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}^{ij} P_{ki} P_{ij} P_{jl} \\ &= \sum_{x_{ij} \neq 0} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}^{ij} P_{ki} x_{ij} \frac{1}{x_{ij}} P_{ii} X P_{jj} P_{jl} \\ &= \sum_{x_{ij} \neq 0} \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}^{ij} P_{ki} P_{ii} X P_{jj} P_{jl}. \end{aligned} \tag{2.4}$$

If $x_{ij} = 0$, then $P_{ii} X P_{jj}$ is a nil-matrix. So,

$$f(X) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} (a_{kl}^{ij} P_{ki} P_{ii}) X (P_{jj} P_{jl}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} (a_{kl}^{ij} P_{ki}) X (P_{jl}), \tag{2.5}$$

that is, $f \in d_1(\mathscr{A})$. Therefore $d_1(\mathscr{A}) = L(A)$. The theorem follows. $\qquad\square$

Replace the multiplication xy in $\mathscr{A}$ by $x \odot y = yx$ and denote the new algebra by $\widetilde{\mathscr{A}}$. There is an algebra homomorphism (epimorphism) $h : \mathscr{A} \otimes \widetilde{\mathscr{A}} \to d_1(\mathscr{A}), h(\sum_{i=1}^{n} a_i \otimes b_i) = f$, where $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$. $\operatorname{Ker} h$ is not trivial. Some conditions, under which $\sum_{i=1}^{n} a_i \otimes b_i \in \operatorname{Ker} h$, are stated below.

THEOREM 2.4. *If $\mathscr{A} = L(E)$ for some linear space E over any field F, then*
 (1) *$a \otimes b \in \operatorname{Ker} h$ if and only if $a = 0$ or $b = 0$;*
 (2) *for $a, b, c, d \neq 0$ one gets $a \otimes b - c \otimes d \in \operatorname{Ker} h$ if and only if $\exists \lambda \in F$ is such that $a = \lambda c$ and $b = (1/\lambda)d$.*

Proof. (1) If $a = 0$ or $b = 0$, then, obviously, for any $x \in \mathscr{A}$ one gets $axb = 0$, that is, $a \otimes b \in \operatorname{Ker} h$. If $a \neq 0$ and $b \neq 0$, then there exist $e_1, e_2 \in E$ such that $b(e_1) \neq 0$ and $a(e_2) \neq 0$. Since $\mathscr{A}$ is the algebra of *all* linear maps from E to E, there is $x \in \mathscr{A}$ such that $x(b(e_1)) = e_2$. So, $(axb)(e_1) = (ax)(b(e_1)) = a(x(b(e_1))) = a(e_2) \neq 0$. Therefore, $axb \neq 0 \in \mathscr{A}$, thus $a \otimes b \notin \operatorname{Ker} h$.

(2) Suppose that $a, b, c, d \neq 0$ but there is no $\lambda \in F$ with properties formulated in this theorem. (If such λ exists, then, evidently, $a \otimes b - c \otimes d \in \operatorname{Ker} h$.) Consider the following cases.

(2.1) There exists $\alpha \in F$ such that $\alpha \neq 0$, $a = \alpha c$, $b \neq (1/\alpha)d$. As $b \neq (1/\alpha)d$, $b, d \neq 0$, we can find $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) \neq (1/\alpha)d(e_1)$, and at least one element $b(e_1)$ or $d(e_1)$ is not equal to zero. As $a, c \neq 0$, $a = \alpha c$, we can find $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) = \alpha c(e_2) \neq 0$.

If $b(e_1) = 0$, then for any $x \in \mathcal{A}(axb)(e_1) = 0$. But again as far as $\mathcal{A} = L(E)$ is the algebra of *all* linear maps from E to E, there exists $x \in \mathcal{A}$ such that $x(d(e_1)) = e_2$ and $(cxd)(e_1) = c(x(d(e_1))) = c(e_2) \neq 0$. So, $(axb)(e_1) \neq (cxd)(e_1)$ and $axb \neq cxd$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $d(e_1) = 0$, then for any $x \in \mathcal{A}(cxd)(e_1) = 0$ and we take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$. We obtain $(axb)(e_1) = a(x(b(e_1))) = a(e_2) \neq 0$, that is, $(axb)(e_1) \neq (cxd)(e_1)$, $axb \neq cxd$, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $b(e_1) \neq 0$, $d(e_1) \neq 0$, we take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$, $x(d(e_1)) = \beta e_2$, where $\beta \neq 0$, $\alpha \neq \beta$. Therefore $(axb)(e_1) = a(x(b(e_1))) = a(e_2) = \alpha c(e_2) = \alpha c((1/\beta)x(d(e_1))) = (\alpha/\beta)c(x(d(e_1))) = (\alpha/\beta)(cxd)(e_1) \neq (cxd)(e_1)$ and $axb \neq cxd$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

(2.2) There exists $\alpha \in F$ such that $\alpha \neq 0$, $a \neq \alpha c$, $b = (1/\alpha)d$. As $b = (1/\alpha)d \neq 0$, there exists $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) = (1/\alpha)d(e_1) \neq 0$. As $a, c \neq 0$, $a \neq \alpha c$, there exists $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) \neq \alpha c(e_2)$, and at least one element $a(e_2)$ or $c(e_2)$ is not equal to zero. Let us take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$.

If one element, either $a(e_2)$, or $c(e_2)$ is equal to zero, then, obviously, $(axb)(e_1) \neq (cxd)(e_1)$ because only one of these elements is equal to zero.

If $a(e_2) \neq 0$, $c(e_2) \neq 0$, then $(axb)(e_1) = a(x(b(e_1))) = a(e_2) \neq \alpha c(e_2) = \alpha c(x(b(e_1))) = \alpha c(x((1/\alpha)d(e_1))) = \alpha(1/\alpha)c(x(d(e_1))) = (cxd)(e_1)$, hence $axb \neq cxd$. Thus $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

(2.3) $a \neq \alpha c$, $b \neq (1/\alpha)d$ for each $\alpha \in F$, $\alpha \neq 0$. As $b \neq 0$, there exists $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) \neq 0$.

If $d(e_1) = 0$, then take $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) \neq 0$ that exists because $a \neq 0$ and $x \in \mathcal{A}$ is such that $x(b(e_1)) = e_2$. Then $(axb)(e_1) = a(e_2) \neq 0 = (bxd)(e_1)$, therefore $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $d(e_1) \neq 0$ and there exists $\gamma \in F$ such that $\gamma \neq 0$, $b(e_1) = \gamma d(e_1)$, then take $e_2 \in E$ and $x \in \mathcal{A}$ such that $e_2 \neq 0$, $a(e_2) \neq (1/\gamma)c(e_2)$, $x(b(e_1)) = e_2$. We obtain $(axb)(e_1) = a(e_2) \neq (1/\gamma)c(e_2) = (1/\gamma)c(x(b(e_1))) = (1/\gamma)c(x(\gamma d(e_1))) = (1/\gamma)\gamma c(x(d(e_1))) = (cxd)(e_1)$. Thus $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $b(e_1)$ and $d(e_1)$ are linearly independent (therefore $b(e_1) \neq 0$ and $d(e_1) \neq 0$), then take $e_2 \in E$, $x \in \mathcal{A}$ such that $e_2 \neq 0$, $x(b(e_1)) = e_2$. If $a(x(b(e_1))) \neq c(x(d(e_1)))$, then $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$. If it appears to be $a(x(b(e_1))) = c(x(d(e_1)))$, then replace x by $\tilde{x}$ such that $\tilde{x}(b(e_1)) = x(b(e_1))$, $\tilde{x}(d(e_1)) = \delta x(d(e_1))$ where $\delta \neq 0$, $\delta \neq 1$. Thus we obtain $(a\tilde{x}b)(e_1) = a(\tilde{x}(b(e_1))) = a(x(b(e_1))) = c(x(d(e_1))) = c((1/\delta)\tilde{x}(d(e_1))) = (1/\delta)c(\tilde{x}(d(e_1))) \neq c(\tilde{x}(d(e_1)))$. This yields $a\tilde{x}b \neq c\tilde{x}d$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$. The theorem is proved.   □

Let $\mathcal{A}$ be an involutive algebra.

Definition 2.5 [8, 9]. The elements $f, f^* \in d_1(\mathcal{A})$ are called *conjugate* if

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i, \qquad f^*(x) = \sum_{i=1}^{n} b_i^* \cdot x \cdot a_i^*. \qquad (2.6)$$

Operation $f \leftrightarrow f^*$ is an involution for $d_1(\mathcal{A})$. It is clear that $f^*(x^*) = (f(x))^*$ for all $x \in \mathcal{A}$.

Definition 2.6 [8, 9]. An element $f \in d_1(\mathcal{A})$ is called *self-conjugate* if $f = f^*$.

Obviously, for self-conjugate $f \in d_1(\mathcal{A})$ and for all $x \in \mathcal{A}$ one gets $f(x^*) = (f(x))^*$. In addition $f \in d_1(\mathcal{A})$ is self-conjugate if and only if for all $i \in \{1,2,\ldots,n\}$ either $b_i = a_i^*$ or there exists $j \in \{1,2,\ldots,n\}$, $j \neq i$ such that $a_j = b_i^*$, $b_j = a_i^*$.

Definition 2.7 [8, 9]. Self-conjugate $f \in d_1(\mathcal{A})$ is called positive if

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot a_i^*, \quad \text{for some } a_1, a_2, \ldots, a_n \in \mathcal{A}. \tag{2.7}$$

This notation is chosen since for C^*-algebra $\mathcal{A}$ a positive map sends positive elements of algebra $\mathcal{A}$ to positive ones. Indeed, for C^*-algebra $\mathcal{A}$ one of definitions of positive elements is $x = yy^*$ for a certain $y \in \mathcal{A}$. If $f \in d_1(\mathcal{A})$ is positive and $x \in \mathcal{A}$ is positive, then

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot a_i^* = \sum_{i=1}^{n} a_i \cdot (yy^*) \cdot a_i^* = \sum_{i=1}^{n} (a_i y) \cdot (y^* a_i^*) = \sum_{i=1}^{n} (a_i y) \cdot (a_i y)^*. \tag{2.8}$$

Since for C^*-algebra the cone P of positive elements is convex, then $f(x) \in P$.

Now consider some properties of eigenvalues and eigenelements for self-conjugate and positive 1-degree maps. It is useful to note that characteristic values of self-conjugate maps are real. The next three theorems were announced in [9].

THEOREM 2.8. *If $\mathcal{A}$ is a C^*-algebra, $f \in d_1(\mathcal{A})$ is a positive map, $\lambda \in R$ is a characteristic value of f, and there exists a positive characteristic element $x \in \mathcal{A}$, corresponding to the characteristic value λ, then $\lambda \geq 0$.*

Proof. Suppose that $\mathcal{A}$ is a C^*-algebra, $f \in d_1(\mathcal{A})$ is a positive map, $x \in \mathcal{A}$ is a positive element $(x \neq 0)$, $\lambda \in R$, $f(x) = \lambda x$. If $\lambda < 0$, then $-\lambda > 0$ and $-\lambda x$ is positive, that is, $\lambda x \in (-P)$, where P is the cone of positive elements. But $\lambda x = f(x) \in P$. Hence $\lambda x \in P \cap (-P) = \{0\}$. It is impossible since for $\lambda \neq 0$, $x \neq 0$, $\lambda x \neq 0$. Therefore the assumption $\lambda < 0$ is false and $\lambda \geq 0$. □

THEOREM 2.9. *If $\mathcal{A}$ is an involutive algebra, $f \in d_1(\mathcal{A})$ is a self-conjugate map, $\lambda \in R$ is a characteristic value of f, then there exists a self-conjugate characteristic element $x \in \mathcal{A}(x \neq 0)$ corresponding to the characteristic value λ.*

Proof. For any $x \in \mathcal{A}$ there exists a unique representation as the sum of "real" and "imaginary" parts $x = x_1 + ix_2$, where $x_1, x_2 \in \mathcal{A}$ are self-conjugate elements $(x_1 = (1/2)(x + x^*), x_2 = (1/2i)(x - x^*))$. Therefore $f(x) = \lambda x \Leftrightarrow f(x_1 + ix_2) = \lambda(x_1 + ix_2) \Leftrightarrow f(x_1) + if(x_2) = \lambda x_1 + i\lambda x_2$.

The elements λx_1, λx_2 are self-conjugate. As f, x_1, x_2 are self-conjugate, $f(x_1)$, $f(x_2)$ are self-conjugate as well. Since the representation as the sum of "real" and "imaginary"

parts is unique, then $f(x_1) = \lambda x_1$, $f(x_2) = \lambda x_2$. For $x \neq 0$ at least one element x_1 or x_2 is not equal to zero. This completes the proof.   □

THEOREM 2.10. *If $\mathscr{A}$ is a C^*-algebra, $f \in d_1(\mathscr{A})$ is a positive map, and there is a characteristic value $\lambda < 0$ of f, then there exist positive elements x_1, x_2, $a \in \mathscr{A}$ such that $f(x_1) - \lambda x_1 = f(x_2) - \lambda x_2 = a$.*

Proof. A positive map is self-conjugate. From Theorem 2.9 it follows that there is a self-conjugate element $x \in \mathscr{A}$ such that $x \neq 0$ and $f(x) = \lambda x$. If $\mathscr{A}$ is C^*-algebra and $x \in \mathscr{A}$ is a self-conjugate element, then there exists unique couple of positive or equal to zero elements x_1, $x_2 \in \mathscr{A}$ such that $x = x_1 - x_2$, $x_1 x_2 = x_2 x_1 = 0$.

We have $f(x) = f(x_1 - x_2) = \lambda(x_1 - x_2) = \lambda x_1 - \lambda x_2$. Therefore $f(x_1) - \lambda x_1 = f(x_2) - \lambda x_2$. Denote this element by a. It follows from Theorem 2.8 that $x_1 \neq 0$, $x_2 \neq 0$. Hence both x_1 and x_2 are positive. As x_1, x_2, f, $(-\lambda)$ are positive, then $f(x_1)$, $f(x_2)$, $(-\lambda x_1)$, $(-\lambda x_2)$, and a are positive. This completes the proof.   □

If $\mathscr{A}$ is a normed algebra, then the standard norm on $d_1(\mathscr{A})$ is also given, since $d_1(\mathscr{A}) \subset L(A)$, that is, for $f \in d_1(\mathscr{A})$,

$$\|f\| = \sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|}. \tag{2.9}$$

Obviously, $\|f^*\| = \|f\|$ and if $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$, then $\|f\| \leq \sum_{i=1}^{n} \|a_i\| \cdot \|b_i\|$.

As far as $L(A)$ is closed, $\overline{d_1(\mathscr{A})} \subset L(A)$, where $\overline{d_1(\mathscr{A})}$ is the closure of $d_1(\mathscr{A})$.

Definition 2.11. A normed algebra $\mathscr{A}$ is *$\bar{1}$-algebra* if $\overline{d_1(\mathscr{A})} = L(A)$. Obviously, if $\mathscr{A}$ is a normed 1-algebra, then $\mathscr{A}$ is a $\bar{1}$-algebra.

Definition 2.12. If $\mathscr{A}$ is an involutive algebra, then the maps $f, f^* \in \overline{d_1(\mathscr{A})}$ are called *conjugate* if $f(x) = \lim_{n \to \infty} f_n(x)$, $f^*(x) = \lim_{n \to \infty} f_n^*(x)$ for some $f_1, f_2, \ldots, f_n, \ldots \in d_1(\mathscr{A})$.

3. 1-degree maps of modules

(A) Let $\mathscr{M}_1$, $\mathscr{M}_2$ be free finitely generated $\mathscr{A}$-modules,

$$\mathscr{M}_1 = \mathscr{A}^n, \qquad \mathscr{M}_2 = \mathscr{A}^m, \qquad F : \mathscr{M}_1 \longrightarrow \mathscr{M}_2, \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}, \qquad f_i = P_i \circ F : \mathscr{M}_1 \longrightarrow \mathscr{A}, \tag{3.1}$$

where $P_i : \mathscr{M}_2 \to \mathscr{A}$ is the projection of $\mathscr{M}_2 = \mathscr{A} \oplus \mathscr{A} \oplus \cdots \oplus \mathscr{A}$ onto ith term.

Definition 3.1. $F : \mathscr{M}_1 \to \mathscr{M}_2$ is called *1-degree ($\bar{1}$-degree) map* if for each $i \in \{1, 2, \ldots, m\}$ the relation $f_i(x_1, x_2, \ldots, x_n) = f_{i1}(x_1) + f_{i2}(x_2) + \cdots + f_{in}(x_n)$ holds, where $f_{i1}, f_{i2}, \ldots, f_{in} \in d_1(\mathscr{A})(\in \overline{d_1(\mathscr{A})})$.

Assign the following matrix to a 1-degree or a $\overline{1}$-degree map:

$$
F \longrightarrow \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix}.
\tag{3.2}
$$

Definition 3.2. If $\mathscr{A}$ is an involutive algebra, then the 1-degree or $\overline{1}$-degree maps $F = (f_{ij})_{m \times n}$, $F^* = (f_{ij}^*)_{m \times n}$ are called *conjugate*.

As well as it is for algebra maps, $F^*(x^*) = (F(x))^*$.

If the modules $\mathcal{M}_1, \mathcal{M}_2$ are normed (e.g., for a normed algebra $\mathscr{A}$ the norm can be defined as $\|x\| = \sqrt{\sum_i \|x_i\|^2}$), then $\|F\|$ is defined in the standard way

$$
\|F\| = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|}.
\tag{3.3}
$$

The straightforward computation gives

$$
\|F\| \leq \sqrt{\sum_{i=1}^{m} \left(\sum_{j=1}^{n} \|f_{ij}\| \right)^2} \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \|f_{ij}\|^2}
\tag{3.4}
$$

and $\|F\| = \|F^*\|$.

(B) Let $\mathscr{A}$ be a C^*-algebra and let $l_2(\mathscr{A})$ be a Hilbert $\mathscr{A}$-module,

$$
F : l_2(\mathscr{A}) \longrightarrow l_2(\mathscr{A}), \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix},
\tag{3.5}
$$

where $f_i = P_i \circ F$, as above. Since $F(x) \in l_2(\mathscr{A})$ it has to be $\|F(x)\|^2 = \sum_{i=1}^{\infty} \|f_i(x)\|^2 < \infty$ for any $x \in l_2(\mathscr{A})$. This is fulfilled, in particular, if $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$. In this case

$$
\|F(x)\|^2 \leq \sum_{i=1}^{\infty} \|f_i\|^2 \cdot \|x\|^2 ,
\tag{3.6}
$$

therefore

$$
\|F\| \leq \sqrt{\sum_{i=1}^{\infty} \|f_i\|^2} < \infty \quad \text{for } \|F\| = \sup_{x \neq 0} \frac{\|F(x)\|}{\|x\|}.
\tag{3.7}
$$

Definition 3.3. A bounded map $F : l_2(\mathscr{A}) \to l_2(\mathscr{A})$ is called *1-degree map* if for each $i \in N$ $f_i(x) = f_i(x_1, x_2, \ldots) = \sum_{j=1}^{\infty} f_{ij}(x_j)$, where $f_{ij} \in d_1(\mathscr{A})$ for any $j \in N$,

$$
F \longrightarrow \begin{pmatrix} f_{11} & f_{12} & \cdots \\ f_{21} & f_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.
\tag{3.8}
$$

Since $\lim_{n\to\infty}\sum_{j=1}^{n}f_{ij}(x_j)$ must exist for all $x=(x_1,x_2,\dots)\in l_2(\mathscr{A})$, we assume that for each $i\in N$ there exists a constant $M_{F,i}>0$ such that $\|f_{ij}\|\le M_{F,i}$ for each $j\in N$. Then

$$
\begin{aligned}
\|f_i(x)\|^2 = \left\|\sum_{j=1}^{\infty}f_{ij}(x_j)\right\|^2 &\le \sum_{j=1}^{\infty}\|f_{ij}\|^2\cdot\|x_j\|^2\\
&\le \sum_{j=1}^{\infty}M_{F,i}^2\cdot\|x_j\|^2 = M_{F,i}^2\sum_{j=1}^{\infty}\|x_j\|^2
\end{aligned}
\tag{3.9}
$$

and $\|f_i\|\le M_{F,i}$. Assume that $\sum_{i=1}^{\infty}M_{F,i}^2<\infty$ for each 1-degree map F. Then for each 1-degree map $F\|F\|<\infty$.

By $d_1(l_2(\mathscr{A}))$ denote the set of all 1-degree maps from $l_2(\mathscr{A})$ to $l_2(\mathscr{A})$ and by $\overline{d_1(l_2(\mathscr{A}))}$ denote its closure. We say that $F:l_2(\mathscr{A})\to l_2(\mathscr{A})$ is a $\overline{1}$-degree map if $F\in\overline{d_1(l_2(\mathscr{A}))}$. Obviously, $F\in\overline{d_1(l_2(\mathscr{A}))}\leftrightarrow f_{ij}\in\overline{d_1(\mathscr{A})}$ for all $i,j\in N$.

Let $e_1=(1,0,0,\dots)$, $e_2=(0,1,0,\dots)$, $e_3=(0,0,1,\dots)$, and so forth be basic elements in $l_2(\mathscr{A})$ and let $L_{n,\mathscr{A}}^{\perp}$ be a submodule of the module $l_2(\mathscr{A})$ generated by elements $e_{n+1},e_{n+2},\dots$.

Definition 3.4. Following [14], we say that a map $F\in\overline{d_1(l_2(\mathscr{A}))}$ is *compact* if $\lim_{n\to\infty}\|F|_{L_{n,\mathscr{A}}^{\perp}}\|=0$.

Definition 3.5. A map $F\in\overline{d_1(l_2(\mathscr{A}))}$ is called *Fredholm $\overline{1}$-map* if $F=H+C$, where $H,C\in\overline{d_1(l_2(\mathscr{A}))}$, H is an isomorphism, and C is a compact map.

The set $C_{\overline{1}}(l_2(\mathscr{A}))$ of all compact $\overline{1}$-degree maps from $l_2(\mathscr{A})$ to $l_2(\mathscr{A})$ is an ideal in the algebra $\overline{d_1(l_2(\mathscr{A}))}$.

Definition 3.6. A manifold M modelled by $l_2(\mathscr{A})$ is called *Fredholm $\overline{1}$-manifold* if M has an atlas with transformation functions having the "derivatives" in the form $I+C$, where I is the identity map, $C\in C_{\overline{1}}(l_2(\mathscr{A}))$.

By $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ denote the group of $\overline{1}$-degree isomorphisms of $l_2(\mathscr{A})$.

THEOREM 3.7. *If $\mathscr{A}$ is C^*-$\overline{1}$-algebra, then $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ is contractible.*

Proof. Let E be the Banach space obtained from $l_2(\mathscr{A})$ by ignoring the element multiplication in $\mathscr{A}$, where $\|\cdot\|_E=\|\cdot\|_{l_2(\mathscr{A})}$.

For C^*-$\overline{1}$-algebra $\mathscr{A}$ we get $\overline{d_1(l_2(\mathscr{A}))}=B(E)$ and $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))=\mathrm{GL}(E)$. Hence $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ is an open subset of Banach space $B(E)$. Therefore, by Milnor's theorem [13] $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ has the homotopy type of a CW-complex and by Whitehead's theorem [16] the strong and weak homotopy trivialities of $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ are equivalent. So, it is sufficient to prove the weak homotopy triviality.

The proof technique repeats that for $\mathrm{GL}(H)$ in [11] or for $\mathrm{GL}(l_2(\mathscr{A}))$ in [12]. First we use Atiyah's theorem on small balls: if f is a continuous map from n-dimensional sphere S^n to $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$, then f is homotopic to f' so that $f'(S^n)$ is a finite polyhedron lying in $\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ together with the homotopy.

Then we use from [11, Lemma 3]: there exist a decomposition $l_2(\mathscr{A})$ in $H'\oplus H_1$, where $H'\cong H_1\cong l_2(\mathscr{A})$, and a continuous map $f'':S^n\to\mathrm{GL}_{\overline{1}}(l_2(\mathscr{A}))$ homotopic to f' such that $f''(s)(x)=x$ for all $s\in S^n$, $x\in H'$.

Next we prove the statement similar to [11, Lemma 7], and [12, Lemma 7.1.4]: $V = \{g \in \mathrm{GL}_{\bar{1}}(l_2(\mathscr{A})) \mid g|_{H'} = Id_{H'}, g(H_1) = H_1\}$ is contractible to 1 in $\mathrm{GL}_{\bar{1}}(l_2(\mathscr{A}))$. Here we use the condition from the hypothesis of our theorem that $\mathscr{A}$ is C^*-$\bar{1}$-algebra. If so, then for any $u \in \mathrm{GL}_{\bar{1}}(l_2(\mathscr{A}))$, u^{-1} and all linear combinations of u and u^{-1} belong to $\mathrm{GL}_{\bar{1}}(l_2(\mathscr{A}))$ as well. The construction of homotopy is the same as in [11, 12].

Since $H' \cong l_2(\mathscr{A})$, we can decompose H' into the sum $H' = H_2 \oplus H_3 \oplus \cdots H_i \cong l_2(\mathscr{A})$, then we get $l_2(\mathscr{A}) = H_1 \oplus H_2 \oplus H_3 \oplus \cdots$.

Let $g \in V$ on, then the matrix of g related to the above decomposition is $g = \mathrm{diag}(u, 1, 1, 1, \ldots) = \mathrm{diag}(u, u^{-1}u, 1, u^{-1}u, 1, \ldots)$, where $u = g|_{H_1}$.

For $t \in [0, \pi/2]$ we get $g_t|_{H_1} = g|_{H_1} = u$ and for each $i \in N$,

$$
g_t|_{H_{2i} \oplus H_{2i+1}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}
$$
$$
= \begin{pmatrix} 1 \cdot \cos^2 t + u^{-1} \sin^2 t & (u-1)\sin t \cos t \\ (1 - u^{-1})\sin t \cos t & u\sin^2 t + 1 \cdot \cos^2 t \end{pmatrix}.
$$
(3.10)

For $t \in [\pi/2, \pi]$ and for each $i \in N$

$$
g_t|_{H_{2i-1} \oplus H_{2i}} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}
$$
$$
= \begin{pmatrix} 1 \cdot \cos^2 t + u\sin^2 t & (u^{-1}-1)\sin t \cos t \\ (1-u)\sin t \cos t & u^{-1}\sin^2 t + 1 \cdot \cos^2 t \end{pmatrix},
$$
(3.11)

$g_0 = g, g_{\pi/2} = \mathrm{diag}(u, u^{-1}, u, u^{-1}, u, \ldots)$ by both formulas of homotopies, and $g_\pi = \mathrm{diag}(1, 1, 1, 1, \ldots)$.

Since all these maps are linear isomorphisms in $L(E)$, then by the hypothesis of the theorem they are $\bar{1}$-degree isomorphisms.

At last we repeat [12, Lemma 7.1.5]: the set

$$
W = \{g \in \mathrm{GL}_{\bar{1}}(l_2(\mathscr{A})) \mid g|_{H'} = Id_{H'}\}
$$
(3.12)

is contractible to V. (In [12] this is proved for $\mathrm{GL}(l_2(\mathscr{A}))$.) This concludes the proof. □

4. Polynomials over a noncommutative algebra and nonlinear Fredholm maps

Let $\mathscr{A}$ be a noncommutative algebra, $x \notin \mathscr{A}$, $x^1 = x$, $x^k \cdot x^l = x^{k+l}$ for all $k, l \in N$. Let us give the recurrent definition of a monomial in x.

Definition 4.1. (1) Every element of $\mathscr{A}$ is a *0-degree monomial*.

(2) x is a *1-degree monomial*.

(3) If m_1 is a k-degree monomial for any $k \in N$ and m_2 is an l-degree monomial for any $l \in N$, then $m_1 \cdot m_2$ is a $(k+l)$-*degree monomial*.

(4) The other monomials do not exist.

Definition 4.2. A finite sum of n-degree monomials is called a *homogeneous n-degree polynomial*.

Definition 4.3. A finite sum of homogeneous polynomials is called an all linear*n-degree polynomial*, where n is the maximal degree of the terms.

The set P of all polynomials forms a graded algebra $P = \oplus_{n=1}^{\infty} P_n$, where P_n is the set of homogeneous n-degree polynomials. Obviously, $P_0 \cong \mathscr{A}, P_1 \cong \mathscr{A} \otimes \widetilde{\mathscr{A}}$.

Definition 4.4. The map corresponding to the homogeneous n-degree polynomial is called an *n-degree map* and the map corresponding to any polynomial is called a *polynomial map*.

The set $d(\mathscr{A})$ of all polynomial maps forms a graded algebra as well: $d(\mathscr{A}) = \oplus_{n=1}^{\infty} d_n(\mathscr{A})$, where $d_n(\mathscr{A})$ is the set of n-degree maps. There is a natural map of graded algebras $h : P \to d(\mathscr{A})$ assigning to the polynomial its corresponding map. For $n = 1$ it was shown in Section 2 that $h|_{P_1} : P_1 \cong \mathscr{A} \otimes \widetilde{\mathscr{A}} \to d_1(\mathscr{A})$ is not injective. Polynomial maps defined above are maps that have 1-degree "derivatives."

We will obtain the definition of a monomial in several variables $x_1, x_2, \ldots, x_n, \ldots$ if in Definition 4.1 we replace (2) by "x_i is a 1-degree monomial for every $i \in N$." Then Definitions 4.2 and 4.3 give the notions of polynomials in $x_1, x_2, \ldots, x_n, \ldots$ and of corresponding polynomial maps from $l_2(\mathscr{A})$ to $\mathscr{A}$.

Definition 4.5. The map $F : U \to l_2(\mathscr{A})$ is called *polynomial* if f_i are polynomial maps for all $i \in N$, where $U \subset l_2(\mathscr{A})$ is an open domain and

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}. \tag{4.1}$$

The "derivative"

$$F'(x) = \begin{pmatrix} df_1 \\ df_2 \\ \vdots \end{pmatrix} \tag{4.2}$$

of a polynomial map F is a 1-degree map for every $x \in U$.

Definition 4.6. The polynomial map $F : U \to l_2(\mathscr{A})$ is called a *Fredholm map* if $F'(x)$ is a Fredholm 1-degree map for each $x \in U$.

5. A certain application of 1-degree maps

In probability theory a discrete Markov chain is described by the transition matrix consisting of elements from $[0, 1]$. Markov's theorem proclaims that if some power of this matrix does not contain zeroes, then limit transition probabilities exist. We assume that kth power of the transition matrix does not contain zeroes and so the question is arising: how the elements of transition matrix that can be changed for kth power of matrix remain without zeroes.

Assume that the transition matrix is an $n \times n$-matrix and consider the set of the real $n \times n$-matrices as a normed involutive algebra with transposition as the involution and

the norm $\|A\| = n \cdot \max_{1 \le i,j \le n} |a_{ij}|$, (equivalent to the norm $\|A\| = \sup_{x \ne 0} \|Ax\|/\|x\|$ but more suitable for the calculation). The inequality $\|AB\| \le \|A\| \cdot \|B\|$ for the normed algebra is fulfilled.

Let $P = (p_{ij})_{n \times n}$ be a transition matrix, let $P^k = \left(p_{ij}^{(k)}\right)_{n \times n}$ be the kth power of P and $f(P) = P^k$. Using the equality $f(P + \Delta P) - f(P) = (P + \Delta P)^k - P^k = df + o(\Delta P)$ we approximate the difference at the left-hand side by the "differential" df and instead of the inequality $\|\Delta f\| = \|f(P + \Delta P) - f(P)\| < \varepsilon$ we solve the inequality $\|df\| < \varepsilon$. Here

$$df = P^{k-1} \cdot \Delta P + P^{k-2} \cdot \Delta P \cdot P + \cdots + P \cdot \Delta P \cdot P^{k-2} + \Delta P \cdot P^{k-1} \tag{5.1}$$

is a d_1-map of ΔP,

$$
\begin{aligned}
o(\Delta P) = {} & \left[P^{k-2} \cdot (\Delta P)^2 + P^{k-3} \cdot (\Delta P)^2 \cdot P + \cdots + P \cdot (\Delta P)^2 \cdot P^{k-3} \right. \\
& \left. + P^{k-3} \cdot \Delta P \cdot P \cdot \Delta P + \cdots + \Delta P \cdot P \cdot \Delta P \cdot P^{k-3} \right] \\
& + \left[P^{k-3} \cdot (\Delta P)^3 + \cdots \right] + \cdots + (\Delta P)^k.
\end{aligned}
\tag{5.2}
$$

Taking into account the estimate

$$
\begin{aligned}
\|df\| \le {} & \left\| P^{k-1} \right\| \cdot \|\Delta P\| + \left\| P^{k-2} \right\| \cdot \|\Delta P\| \cdot \|P\| \\
& + \cdots + \|P\| \cdot \|\Delta P\| \cdot \left\| P^{k-2} \right\| + \|\Delta P\| \cdot \left\| P^{k-1} \right\| \\
\le {} & \|P\|^{k-1} \cdot \|\Delta P\| + \|P\|^{k-2} \cdot \|\Delta P\| \cdot \|P\| \\
& + \cdots + \|P\| \cdot \|\Delta P\| \cdot \left\| P^{k-2} \right\| + \|\Delta P\| \cdot \left\| P^{k-1} \right\| = k \cdot \|P\|^{k-1} \cdot \|\Delta P\|,
\end{aligned}
\tag{5.3}
$$

it is sufficient to solve the inequality $k \cdot \|P\|^{k-1} \cdot \|\Delta P\| < \varepsilon$. Thus we get

$$\|\Delta P\| < \delta_1(\varepsilon) = \frac{\varepsilon}{k \cdot \|P\|^{k-1}}, \tag{5.4}$$

therefore for all $i, j \in \{1, 2, \ldots, n\}$,

$$|\Delta p_{ij}| < \frac{\varepsilon}{n \cdot k \cdot \|P\|^{k-1}}. \tag{5.5}$$

Further we will not reject $o(\Delta P)$ and obtain more exact estimate:

$$
\begin{aligned}
\|o(\Delta P)\| \le {} & \left[\|P\|^{k-2} \cdot \|\Delta P\|^2 + \|P\|^{k-3} \cdot \|\Delta P\|^2 \cdot \|P\| + \cdots \right. \\
& + \|P\| \cdot \|\Delta P\|^2 \cdot \|P\|^{k-3} + \|\Delta P\|^2 \cdot \|P\|^{k-2} \\
& \left. + \|P\|^{k-3} \cdot \|\Delta P\| \cdot \|P\| \cdot \|\Delta P\| + \cdots + \|\Delta P\| \cdot \|P\| \cdot \|\Delta P\| \cdot \|P\|^{k-3} \right] \\
& + \left[\|P\|^{k-3} \cdot \|\Delta P\|^3 + \cdots \right] + \cdots + \|\Delta P\|^k \\
= {} & C_k^2 \cdot \|P^{k-2}\| \cdot \|\Delta P\|^2 + C_k^3 \cdot \|P\|^{k-3} \cdot \|\Delta P\|^3 + \cdots + C_k^s \cdot \|P\|^{k-s} \cdot \|\Delta P\|^s \\
& + \cdots + \|\Delta P\|^k = \left(\|P\| + \|\Delta P\| \right)^k - \|P\|^k - k \cdot \|P\|^{k-1} \cdot \|\Delta P\|.
\end{aligned}
\tag{5.6}
$$

Let $\varphi(x) = x^k$ be a real function. Then for $x = \|P\|$, $\Delta x = \|\Delta P\|$ we get

$$\Delta \varphi = \varphi(x + \Delta x) - \varphi(x) = \varphi(\|P\| + \|\Delta P\|) - \varphi(\|P\|) = (\|P\| + \|\Delta P\|)^k - \|P\|^k,$$
$$d\varphi = \varphi'(x)\Delta x = kx^{k-1}\Delta x = k \cdot \|P\|^{k-1} \cdot \|\Delta P\|,$$
$$(\|P\| + \|\Delta P\|)^k - \|P\|^k - k \cdot \|P\|^{k-1} \cdot \|\Delta P\| = \Delta \varphi - d\varphi = o(\Delta x) = o(\|\Delta P\|).$$
$$(5.7)$$

Consider $o(\Delta x)$ in the form $(\varphi''(x + t \cdot \Delta x)/2)(\Delta x)^2$, where $0 < t < 1$. For $\varphi''(x) = k(k-1)x^{k-2}$, we obtain $\varphi''(x + t \cdot \Delta x) = k(k-1)(x + t \cdot \Delta x)^{k-2}$.

Since P and $P + \Delta P$ are transition matrices, that is, their elements are probabilities, then $|\Delta p_{ij}| \leq 1$ for all $i, j \in \{1, 2, \ldots, n\}$ and $\|\Delta P\| \leq n$. Thus for $x = \|P\|$, $\Delta x = \|\Delta P\|$ we get $x \geq 0$, $0 < \Delta x \leq n$, $x + t \cdot \Delta x \leq x + \Delta x \leq x + n$, and $(x + t \cdot \Delta x)^{k-2} \leq (x + n)^{k-2}$. Therefore $\|o(\Delta P)\| \leq (k(k-1)/2)(\|P\| + n)^{k-2}\|\Delta P\|^2$ and

$$\|(P + \Delta P)^k - P^k\| \leq \|df\| + \|o(\Delta P)\| \leq k \cdot \|P\|^{k-1} \cdot \|\Delta P\| + \frac{k(k-1)}{2}(\|P\| + n)^{k-2}\|\Delta P\|^2.$$
$$(5.8)$$

Now we have to solve the square inequality

$$k \cdot \|P\|^{k-1} \cdot \|\Delta P\| + \frac{k(k-1)}{2}(\|P\| + n)^{k-2}\|\Delta P\|^2 < \varepsilon \tag{5.9}$$

for $\|\Delta P\|$. Taking into account $\|\Delta P\| \geq 0$ we obtain

$$\|\Delta P\| < \delta_2(\varepsilon) = \frac{\sqrt{(k\|P\|^{k-1})^2 + 2\varepsilon k(k-1)(\|P\| + n)^{k-2}} - k\|P\|^{k-1}}{k(k-1)(\|P\| + n)^{k-2}}. \tag{5.10}$$

Since $\|\Delta P\| = n \cdot \max_{1 \leq i, j \leq n} |\Delta p_{ij}|$, we get for all $i, j \in \{1, 2, \ldots, n\}$,

$$|\Delta p_{ij}| < \frac{\sqrt{(k\|P\|^{k-1})^2 + 2\varepsilon k(k-1)(\|P\| + n)^{k-2}} - k\|P\|^{k-1}}{nk(k-1)(\|P\| + n)^{k-2}}. \tag{5.11}$$

This is a sufficient condition for $\|\Delta f\| < \varepsilon$.

The number ε must be chosen so that all elements of the matrix $(P + \Delta P)^k$ are strictly positive, that is, they differ from the corresponding elements of P^k less than by $m = \min_{1 \leq i, j \leq n} p_{ij}^{(k)} > 0$. Hence it must be $\|df\| = \|(P + \Delta P)^k - P^k\| < nm$, therefore ε has to be chosen so that $\varepsilon \leq nm$, for example, $\varepsilon = nm$.

In this case the first formula yields $|\Delta p_{ij}| < m/k\|P\|^{k-1}$ and the second formula yields

$$|\Delta p_{ij}| < \frac{\sqrt{(k\|P\|^{k-1})^2 + 2nmk(k-1)(\|P\| + n)^{k-2}} - k\|P\|^{k-1}}{nk(k-1)(\|P\| + n)^{k-2}} \tag{5.12}$$

for all $i, j \in \{1, 2, \ldots, n\}$. The formulae (5.5) and (5.12) were announced in [10].

6. Conclusion

Let us ascertain the interdependence between the notions defined in this paper and some other ones.

(1) Let f be a polynomial in x with coefficients from $\mathcal{A}$, then df is a 1-degree polynomial in Δx, where the coefficients are also polynomials in x. So, $d : P \to P \otimes \widetilde{P}$ (see Section 2 for the description of what $\widetilde{\mathcal{A}}$ with respect to $\mathcal{A}$ is). Let $g : P \otimes \widetilde{P} \to P$ be an algebra homomorphism defined as $g(f_1 \otimes f_2) = f_1 \cdot f_2$. Then $\delta = g \circ d : P \to P$ is a derivation of the algebra P, that is, $\delta(f_1 \cdot f_2) = \delta(f_1) \cdot f_2 + f_1 \cdot \delta(f_2)$ for all $f_1, f_2 \in P$. If f is an n-degree polynomial, then $\delta(f)$ is also an $(n-1)$-degree polynomial.

(2) For a noncommutative operator algebra $\mathcal{A}$ consider the algebra homomorphisms $\mu_A : \mathcal{F}_1 \to \mathcal{A}$ and $\mu_{\overset{1}{A_1},\ldots,\overset{n}{A_n}} : \mathcal{F}_n \to \mathcal{A}$ that are defined, for example, in [15]. Here $A, \overset{1}{A_1}, \ldots, \overset{n}{A_n} \in \mathcal{A}$ are called generators, $\mathcal{F}_1$ is a space of one-place symbols, $\mathcal{F}_n$ is a space of n-place symbols, numbers over letters are called Feynman numbers, and the set of letters with numbers over them is called Feynman set. The one-place or n-place symbols are real or complex functions of one or n variables, respectively. These homomorphisms μ_A and $\mu_{\overset{1}{A_1},\ldots,\overset{n}{A_n}}$ may be called "evaluation homomorphisms" because they give a possibility to find in $\mathcal{A}$ the values $f(A)$ or $f(\overset{1}{A_1},\ldots,\overset{n}{A_n})$ for different f and specified A or $\overset{1}{A_1},\ldots,\overset{n}{A_n}$. For example, if f is a polynomial in one or several variables, it is necessary to substitute A or $\overset{1}{A_1},\ldots,\overset{n}{A_n}$ for variables of f.

If f is a polynomial in $2n+1$ variables, $f = f(y_1,\ldots,y_n,z,x_1,\ldots,x_n) = \sum_{i=1}^{n} y_i \cdot z \cdot x_i$, then for $a_1,\ldots,a_n,c,b_1,\ldots,b_n \in \mathcal{A}$ we can evaluate $f(\overset{1}{b_1},\ldots,\overset{n}{b_n},\overset{n+1}{c},\overset{n+2}{a}_1,\ldots,\overset{2n+1}{a}_n)$. Specifying $a_1,\ldots,a_n,b_1,\ldots,b_n$ and replacing c by $\mathcal{A}$, we obtain the 1-degree map corresponding to the 1-degree polynomial in x of the form $\sum_{i=1}^{n} a_i \cdot x \cdot b_i$. Obviously, for another polynomial it is necessary to use another homomorphism and the quantity of generators depends on the quantity of polynomial terms.

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