PROXIMINALITY IN GEODESIC SPACES

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Let X be a complete CAT(0) space with the geodesic extension property and Alexandrov curvature bounded below. It is shown that if C is a closed subset of X, then the set of points of X which have a unique nearest point in C is G_δ and of the second Baire category in X. If, in addition, C is bounded, then the set of points of X which have a unique farthest point in C is dense in X. A proximity result for set-valued mappings is also included.

1. Introduction

This paper is primarily motivated by a recent paper of Zamfirescu [17], in which it is shown that for any compact set C in a complete length space X without bifurcating geodesics, the nearest point projection P_C of X onto C is properly single valued at most points of X, that is, on a set of second Baire category. We show here that the same is true for a closed subset of a complete CAT(0) space X, provided X has the geodesic extension property and has Alexandrov curvature bounded below. We also show that if C is bounded and closed, then the set of points of X which have a unique farthest point in C is dense in X. These are extensions pioneering results, see Edelstein [7] and Stečkin [16]. Also see [15] for other generic results.

A metric space is a CAT(0) *space* (the term is due to Gromov—see, e.g., [1, page 159]) if it is geodesically connected, and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane. Precise definitions are given below. For a detailed discussion of the properties of such spaces, see Bridson and Haefliger [1] or Burago et al. [3]. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples include the

classical hyperbolic spaces, Euclidean buildings (see [2]), the complex Hilbert ball with the hyperbolic metric (see [9]; also [14, inequality (4.3)] and subsequent comments), and many others.

2. Preliminaries

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic from x to y*) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image α of c is called a *geodesic* (or *metric*) *segment* joining x and y. When unique, this geodesic is denoted $[x,y]$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$. A subset $Y \subseteq X$ is said to be *convex* if Y includes every geodesic segment joining any two of its points.

For complete details and further discussion, see, for example, [1] or [3].

For $\kappa \in (-\infty, 0]$, let M_κ^2 denote the classical surface of curvature κ. Thus M_κ^2 is just the Euclidean plane $\mathbb{E}^2$ if $\kappa = 0$, and M_κ^2 is obtained from the classical hyperbolic plane by multiplying the distance function by $1/\sqrt{-\kappa}$ if $\kappa < 0$.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the *vertices* of Δ) and a geodesic segment between each pair of vertices (the *edges* of Δ). A *comparison triangle* for geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\Delta}(x_1, x_2,x_3) := \Delta(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in M_κ^2 such that $d_{M_\kappa^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Comparison triangles always exist; see [1, Lemma 2.14].

A geodesic metric space is said to be a CAT(0) space if all distances between points on the sides of a geodesic triangle Δ are no *larger* than the distances between the corresponding points on any comparison triangle $\overline{\Delta}$ in $\mathbb{E}^2$. A geodesic space is said to have Alexandrov curvature bounded below if there exists $\kappa \in (-\infty, 0]$ for which the distances between all points on any geodesic triangle Δ are no *smaller* than distances between the corresponding points on any comparison triangle $\overline{\Delta}$ in M_κ^2.

Now let X be a CAT(0) space, let $a,x,y \in X$, let $m[x,y]$ denote the midpoint of the segment $[x,y]$ (which is necessarily unique), and let $\varepsilon \in [0,2]$. It is possible to define a modulus δ_a by setting

$$\delta_a(\varepsilon) = \inf \{1 - d(a,m[x,y])\}, \tag{2.1}$$

where the infimum is taken over all points $x,y \in X$ for which $d(a,x) \le 1$, $d(a,y) \le 1$, and $d(x,y) \ge \varepsilon$. By using comparison triangles it is easy to see that $\delta_a(\varepsilon) \ge \delta(\varepsilon)$ for all $a \in X$, where δ is the usual modulus of convexity in $\mathbb{E}^2$. In particular, $\delta_a(\varepsilon) > 0$ if $\varepsilon > 0$.

3. Nearest and farthest points

In this section we assume that the space X has the *geodesic extension property*. Although this property is usually defined locally, for a complete CAT(0) space it is equivalent to the assertion that each nonconstant geodesic $c : [a,b] \to X$ can be extended to a line $c : \mathbb{R} \to X$. For a technical discussion of this property along with examples, see [1, page 207].

Let C be a subset of a CAT(0) space X. For $x \in X$, set

$$P_C(x) = \{y \in C : d(x,y) = \inf\{(d(x,u) : u \in C)\}\},$$
$$F_C(x) = \{y \in C : d(x,y) = \sup\{(d(x,u) : u \in C)\}\}. \tag{3.1}$$

We now state our main results.

THEOREM 3.1. *Suppose X is a complete CAT(0) space with the geodesic extension property and with Alexandrov curvature bounded below. Let C be a nonempty closed subset of X. Then the set of points of X at which the mapping P_C is well defined and single valued is a set of the second Baire category.*

THEOREM 3.2. *Suppose X is a complete CAT(0) space with the geodesic extension property and with Alexandrov curvature bounded below. Let C be a nonempty bounded closed subset of X. Then F_C is well defined and single valued on a dense subset of X.*

The proofs of both theorems depend on the fact proved below, which uses the assumption that the curvature is bounded below. This is an analog of a property of uniformly convex Banach space used by Stečkin [16, Lemma 4].

PROPOSITION 3.3. *Suppose X is a complete CAT(0) space satisfying the geodesic extension property and with Alexandrov curvature bounded below by $\kappa \in (-\infty, 0]$. Then if ε, d, and r are fixed positive numbers, there exist $\xi = \xi(\varepsilon, d, r) > 0$ and $\xi' = \xi'(\varepsilon, d, r) > 0$ such that if $c, c' \in X$ satisfy $d(c, c') = r$, then*

$$\operatorname{diam}\left[B(c;d) \cap (X \backslash B(c';d+r-\xi))\right] < \varepsilon,$$
$$\operatorname{diam}\left[B(c;d-r+\xi) \cap (X \backslash B(c';d))\right] < \varepsilon. \tag{3.2}$$

Proof. We only prove the first conclusion since the formulations are equivalent to one another. If the first conclusion is false then there exist positive numbers ε, d, r, a sequence $\{\xi_i\}$ of positive numbers, and sequences $\{c_i\}$, $\{c_i'\}$ in X for which

$$\operatorname{diam}(S_i) \geq \varepsilon, \tag{3.3}$$

where

$$S_i = B(c_i;d) \cap (X \backslash B(c_i';d+r-\xi_i)). \tag{3.4}$$

For each i let $u_i \in X$ so that c_i is on the segment $[c_i', u_i]$ and so that $d(c_i, u_i) = d$. (Here we are using the geodesic extension property.) Then $u_i \in S_i$, so by assumption there exists $w_i \in S_i$ for which $d(u_i, w_i) \geq \varepsilon/2$. Now choose h_i on the segment $[c_i', w_i]$ so that $d(c_i', h_i) = r$. Fix $a, b \in M_\kappa^2$ with $d(a,b) = d+r$ and let $\Delta(b, \overline{w}_i, a)$ be the comparison triangle in M_κ^2 corresponding to $\Delta(u_i, w_i, c_i')$ (with b corresponding to u_i and a corresponding to c_i'). See the proof of [1, Lemma 2.14]). The point $\overline{c}_i$ of the segment $[a,b]$ corresponding to c_i is always the same, so we can label it c. Since $d(u_i, w_i) \geq \varepsilon/2$, and since X has Alexandrov curvature bounded below by κ, it follows that

$$d(h_i, c_i) \geq d(\overline{h}_i, c). \tag{3.5}$$

Since b, $\overline{w}_i$ are in the ball $B(a;d+r)$, with $d(b,\overline{w}_i) \geq \varepsilon/2$, it follows that $d(\overline{h}_i,c)$ is uniformly bounded away from 0. Thus there exists $\varepsilon' \in (0,2]$ such that $d(h_i,c_i) \geq \varepsilon' r$. Since

$$\lim_{i\to\infty}[r + d(h_i,w_i)] = \lim_{i\to\infty} d(c'_i,w_i) \leq r + d, \tag{3.6}$$

it follows that $\lim_{i\to\infty} d(h_i,w_i) \leq d$. Also $\lim_{i\to\infty} d(c_i,w_i) \leq d$. Let z_i be the midpoint of the segment $[h_i,c_i]$, $i = 1,2,\dots$. Since closed balls are convex,

$$\limsup_{i\to\infty} d(w_i,z_i) \leq d. \tag{3.7}$$

Since the modulus $\delta := \delta_{c'_i}(\varepsilon') > 0$, it is possible to choose $\mu > 0$ so small that

$$(1-\delta)r + d + \mu \leq k < r + d. \tag{3.8}$$

Also, since $d(c'_i,c_i) = d(c'_i,h_i) \equiv r$, it must be the case that

$$d(c'_i,z_i) \leq (1-\delta)r. \tag{3.9}$$

Thus for i sufficiently large,

$$d(c'_i,w_i) \leq d(c'_i,z_i) + d(w_i,z_i) \leq (1-\delta)r + d + \mu \leq k. \tag{3.10}$$

Since $\lim_{i\to\infty} d(c'_i,w_i) = r + d$, we have a contradiction. $\square$

For the proof of Theorem 3.1 we introduce some notation. For $x \in X$ and $d = \mathrm{dist}(x, C) > 0$, put

$$Y_\varepsilon(x) = B(x;d+\varepsilon) \cap C \quad (\varepsilon > 0),$$

$$d_\varepsilon(x) = \mathrm{diam}\,(Y_\varepsilon(x)), \tag{3.11}$$

$$d_0(x) = \lim_{\varepsilon\to 0^+} d_\varepsilon(x).$$

Let $\alpha > 0$ and denote by $F_\alpha = F_\alpha(C)$ the set of all points $x \in X$ such that $d_0(x) \geq 1/\alpha$. The proof of the following lemma is a straightforward adaptation of the proof of [16, Lemma 2], or the proof of [4, Proposition 2.3]

LEMMA 3.4. *Let C be a subset of a complete* CAT(0) *space X. Then F_α is closed for every* $\alpha > 0$.

Theorem 3.1 is a direct consequence of the preceding lemma and the following theorem.

THEOREM 3.5. *Suppose X is a complete* CAT(0) *space with the geodesic extension property and with Alexandrov curvature bounded below. Let C be a nonempty subset of X. Then the set $K(C)$ of all points $x \in X$ for which $d_0(x) = 0$ is a G_δ set and of the second Baire category.*

Proof (this is a slight modification of Stečkin's original proof [16] in a uniformly convex Banach space setting; see also the proof of [4, Theorem 2.14]). It is clear that

$$K(C) = X \backslash H(C), \tag{3.12}$$

where

$$H(C) = \bigcup_{n=1}^{\infty} F_n. \tag{3.13}$$

By Lemma 3.4, F_n is closed for all n, and we now prove that F_n is nowhere dense for all n.

Let $x \in F_n$ and let $\mathrm{dist}(x, C) = d > 0$. Suppose that N is a neighborhood of x, and choose $r > 0$ so that $B(x; r) \subset N$. If $\varepsilon \in (0, 1/n)$ is given, then by Proposition 3.3 there exists $\xi_0 = \xi_0(\varepsilon, d, r) > 0$ such that if $c, c' \in X$ satisfy $d(c, c') = r$, then

$$\mathrm{diam}\left[B(c; d - r + \xi)\right] \cap (X \backslash B(c'; d)) < \varepsilon < \frac{1}{n} \tag{3.14}$$

for all $\xi \in (0, \xi_0]$. Since $\mathrm{dist}(x, C) = d$, there exists $y \in C$ such that $d(x, y) < d + \xi_0$. Choose $c \in X$ so that $c \in [x, y]$ and $d(x, c) = r$. Then $c \in N$ and

$$Y_{\xi_0}(c) \subset B(c; d - r + \xi_0) \cap (X \backslash B(x; d)). \tag{3.15}$$

This implies that $\mathrm{diam}(Y_{\xi_0}(c)) < 1/n$. Hence $d_0(c) < 1/n$ and therefore c is not in F_n. It follows that F_n has empty interior. □

Proof of Theorem 3.2. Once Proposition 3.3 has been established, the details of proving that the mapping F_C is well defined on a dense subset of X can be found in [10], or deduced by a straightforward modification the Banach space proof given in [11].

To see that F_C is *single-valued* on a dense subset of X, suppose $F_C(x)$ is nonempty for $x \in X$ and choose $y \in F_C(x)$. Now choose z so that x is on the segment $[z, y]$. Choose $u \in C$ with $u \neq y$, and suppose $d(z, u) \geq d(z, y)$. Then

$$d(z, y) = d(z, x) + d(x, y) \geq d(z, x) + d(x, u)$$
$$\geq d(z, u) \geq d(z, y). \tag{3.16}$$

Thus

$$d(z, x) + d(x, u) = d(z, u) \tag{3.17}$$

and hence x is on the segment $[z, u]$. Again this contradicts the fact that segments do not bifurcate in X. Therefore if $u \in X$ with $u \neq y$, $d(z, u) < d(z, y)$. It follows that $F_C(z) = \{y\}$. Since z can be chosen arbitrarily near x, the conclusion follows. □

Remark 3.6. One implication of Proposition 3.3 is that in such spaces the intersection of two spheres $S(p; d) \cap S(q; r)$ with $d = d(p, q) + r$ consists of exactly one point. This fact and Theorem 3.1 lead to the following result. The proof is an obvious modification of the proof of [13, Theorem 2.2].

THEOREM 3.7. *Suppose X is a complete* CAT(0) *space with the geodesic extension property and with Alexandrov curvature bounded below. Let C be a closed subset of X, suppose $T : C \to C$ is nonexpansive, and suppose the fixed point set of T has nonempty interior. Then for each $x \in C$ the Picard sequence $\{T^n(x)\}$ converges to a fixed point of T.*

The following example is relevant to Theorems 3.1 and 3.2.

Definition 3.8. An $\mathbb{R}$-tree is a metric space T such that:
   (i) there is a unique geodesic segment (denoted by $[x,y]$) joining each pair of points $x, y \in T$;
   (ii) if $[y,x] \cap [x,z] = \{x\}$, then $[y,x] \cup [x,z] = [y,z]$.

It is known that an $\mathbb{R}$-tree is a CAT(0) space (see [1, page 167]). The following example shows that Theorem 3.2 is false if the assumption that the Alexandrov curvature is bounded below is dropped.

Example 3.9. Consider the complete $\mathbb{R}$-tree X obtained by taking all lines L_i in ℓ_2 of the form

$$L_i = \{te_i : -\infty < t < \infty\}, \tag{3.18}$$

where e_i is the standard ith unit basis vector, and give X the shortest path metric. Let C_1 be the set

$$\left\{te_i : t \geq 1 + \frac{1}{i},\ i = 1, 2, \dots\right\}. \tag{3.19}$$

Then clearly C_1 is a closed set in X since it is the union of disjoint closed rays with the distance between any two distinct rays at least 2. However no point in the open set $U := \{(u,0,0,\dots) : u < 0\}$ has a nearest point in C_1. Similarly, if C_2 is the set

$$\left\{te_i : 0 \leq t \leq 1 - \frac{1}{i},\ i = 1, 2, \dots\right\}, \tag{3.20}$$

then no point in U has a farthest point in C_2. Clearly U has nonempty interior.

A metric space is said to be *without bifurcating geodesics* if any two segments having another common point are either identical, or one contains the other. In view of Example 3.9, the condition that geodesics do not bifurcate is a necessary condition for Theorems 3.1 and 3.2 to hold. Is it sufficient?

Problem 3.10. Can the assumption that the Alexandrov curvature is bounded below in Theorems 3.1 and 3.2 be replaced with the assumption that geodesics do not bifurcate?

We leave another question open.

Problem 3.11. Do Theorems 3.1 and 3.2 remain true without assuming the geodesic extension property?

4. Proximity pairs

We conclude with an observation about set-valued mappings. Here we use d_H to denote the usual Hausdorff metric defined on the family of nonempty closed subsets of a metric space X.

It is known that if A is a nonempty closed convex subset of a CAT(0) space, then the nearest point projection P_A of X onto A is nonexpansive (see [1, page 176]). The following lemma is a straightforward consequence of this fact (see [12]).

LEMMA 4.1. *Let A be a nonempty closed convex subset of a* CAT(0) *space X. If C and D are nonempty closed and bounded subsets of X, then*

$$d_H\left(P_A(C), P_A(D)\right) \leq d_H(C, D). \tag{4.1}$$

We will also need the following fact.

PROPOSITION 4.2 [1, page 176]. *If X is a* CAT(0) *space, then the distance function $d :$ $X \times X \to \mathbb{R}$ is convex.*

This means that given any pair of geodesics $c : [0,1] \to X$ and $c' : [0,1] \to X$ parametrized proportional to arc length, the following inequality holds for all $t \in [0,1]$:

$$d\left(c(t), c'(t)\right) \leq (1-t)d\left(c(0), c'(0)\right) + td\left(c(1), c'(1)\right). \tag{4.2}$$

We now give a consequence of Proposition 4.2. Here

$$\text{dist}(A, B) = \inf\left\{d(x, y) : x \in A, \ y \in B\right\}. \tag{4.3}$$

The analog of this result holds in reflexive Banach spaces and the proof usually given invokes weak compactness of A. Our proof circumvents this approach, using instead the fact that descending sequences of nonempty bounded closed convex sets in a complete CAT(0) space have nonempty intersection (see [1]).

PROPOSITION 4.3. *Let A and B be disjoint closed convex subsets of a complete* CAT(0) *space X with A bounded. Then the sets*

$$A_0 := \left\{a \in A : d(a, b) = \text{dist}(a, B) \text{ for some } b \in B\right\},$$
$$B_0 := \left\{b \in B : d(a, b) = \text{dist}(b, A) \text{ for some } a \in A\right\} \tag{4.4}$$

are nonempty and satisfy

$$P_B(A_0) \subseteq B_0, \qquad P_A(B_0) \subseteq A_0. \tag{4.5}$$

Proof. First we observe that if $u, v \in A$ satisfy

$$\text{dist}(u, B) \le r, \qquad \text{dist}(v, B) \le r, \tag{4.6}$$

then for each $\varepsilon > 0$ there exist $w, z \in B$ such that

$$d(u, w) \le r + \varepsilon, \qquad d(v, z) \le r + \varepsilon. \tag{4.7}$$

Let $t \in (0, 1)$ and let m_t be the point of the segment $[u, v]$ for which

$$d(u, m_t) = (1 - t)d(u, v), \qquad d(v, m_t) = td(u, v). \tag{4.8}$$

Then if n_t is the point of the segment $[w, z]$ for which

$$d(w, n_t) = (1 - t)d(w, z), \qquad d(z, n_t) = td(w, z), \tag{4.9}$$

Proposition 4.2 implies $d(m_t, n_t) \le (1 - t)(r + \varepsilon) + t(r + \varepsilon) = r + \varepsilon$. Therefore

$$\text{dist}\,(m_t, B) \le r + \varepsilon \tag{4.10}$$

and since $\varepsilon > 0$ is arbitrary, $\text{dist}(m_t, B) \le r$. This proves that the set

$$\{x \in A : \text{dist}(x, B) \le r\} \tag{4.11}$$

is convex.

Now let $d = \text{dist}(A, B)$ and for each $n = 1, 2, \dots$, let

$$A_n = \left\{ x \in A : \text{dist}(x, B) \le d + \frac{1}{n} \right\}. \tag{4.12}$$

Then $\{A_n\}$ is a sequence of closed convex subsets of A, so $\bigcap_{n=1}^{\infty} A_n$ is nonempty, closed, and convex. Moreover, if $u \in \bigcap_{n=1}^{\infty} A_n$, then there exists $w \in B$ (since B is proximinal) such that $d(u, w) = \text{dist}(u, B) = \text{dist}(A, B)$. This proves that A_0 and B_0 are nonempty. If $y \in P_B(A_0)$, then $y = P_B(x)$ for some $x \in A_0$, so $d(x, y) = \text{dist}(A, B)$. Hence $y \in B_0$. Similarly, $P_A(B_0) \subseteq A_0$.   □

THEOREM 4.4. *Suppose (X, d) is a complete CAT(0) space, let K be a bounded closed convex subset of X, and suppose T is a nonexpansive set-valued mapping of K into 2^K whose values are nonempty compact subsets of K. The T has a fixed point.*

Proof. Since asymptotic centers of bounded sequences are unique in CAT(0) spaces, it is possible to follow standard proof of the analog of the theorem in uniformly convex Banach spaces (cf., [8, page 165]). (For extensions of this result, see [5, 6].)   □

Given the preceding facts, the following result can be obtained by repeating the proof of [12, Theorem 3.3], changing the norm to distance, and applying Theorem 4.4 instead of Lim's theorem at the relevant step.

THEOREM 4.5. *Let X be a complete* CAT(0) *space and let A, B be nonempty closed convex subsets of X with A bounded. Let $T : A \to 2^B$ be a multivalued mapping such that*

   (i) *for each $x \in A$, $T(x)$ is a nonempty compact subset of B,*

   (ii) *$d_H(T(x), T(y)) \le d(x, y)$ for all $x, y \in A$,*

   (iii) *$T(A_0) \subseteq B_0$.*

Then there exists $x_0 \in A$ such that

$$\mathrm{dist}\left(x_0, T(x_0)\right) = \mathrm{dist}(A, B) = \inf\left\{\mathrm{dist}\left(x, T(x)\right) : x \in A\right\}. \qquad (4.13)$$

Acknowledgments

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