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Research Article

On a Parametric Mulholland-Type Inequality and Applications

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In this paper, by the use of the weight functions, and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given. The equivalent statements of the best possible constant factor related to a few parameters are provided. As applications, the operator expressions and a few particular examples are considered.

1. Introduction

Assuming that $0 < \sum_{m=1}^{\infty} a_m^2 < \infty$ *and* $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, we have the following discrete Hilbert's inequality with the best possible constant factor π (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{m=1}^{\infty} a_m^2 \sum_{n=1}^{\infty} b_n^2\right)^{1/2}. \qquad (1)$$

We still have the following Mulholland's inequality with the same best possible constant π (cf. [1], Theorem 343):

$$\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\frac{a_m b_n}{\ln mn} < \pi\left(\sum_{m=2}^{\infty} m a_m^2 \sum_{n=2}^{\infty} n b_n^2\right)^{1/2}. \qquad (2)$$

If $0 < \int_0^{\infty} f^2(x)dx < \infty$ and $0 < \int_0^{\infty} g^2(y)dy < \infty$, then we have the following Hilbert's integral inequality:

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}dx\,dy < \pi\left(\int_0^{\infty} f^2(x)\,dx \int_0^{\infty} g^2(y)\,dy\right)^{1/2}, \qquad (3)$$

with the best possible constant factor π (cf. [1], Theorem 316).

Inequalities (1), (2), and (3) and their extensions with the conjugate exponents (p,q) $(p > 1, 1/p + 1/q = 1)$ and independent parameters are important in analysis and its applications (cf. [2–13]).

The following half-discrete Hilbert-type inequality was provided (cf. [1], Theorem 351). If $K(x)$ $(x > 0)$ is decreasing, $p > 1, 1/p + 1/q = 1, 0 < \phi(s) = \int_0^{\infty} K(x)x^{s-1}dx < \infty$, then

$$\int_0^{\infty} x^{p-2}\left(\sum_{n=1}^{\infty} K(nx)\,a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} a_n^p. \qquad (4)$$

Some new extensions of (4) were provided by [14–19].

In 2016, by the use of the technique of real analysis, Hong [20] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works about the extensions of (3) were provided by [21–25].

In this paper, according to the way given by [20], by the use of the weight functions and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given, which is an extension of (2). The equivalent statements of the best possible constant factor related to a few parameters are provided. As applications, the operator expressions and a few particular examples are considered.

2. Some Lemmas

In what follows, we suppose that $p > 1, 1/p + 1/q = 1$, $\lambda \in \mathrm{R}, \lambda_i, \lambda - \lambda_i \le 1$ $(i = 1, 2)$, $k_\lambda(x, y)$ is a positive homogeneous function of degree$-\lambda$, satisfying, for any $u, x, y > 0$,

$$k_\lambda (ux, uy) = u^{-\lambda} k_\lambda (x, y). \tag{5}$$

Also, $k_\lambda(x, y)$ is decreasing with respect to $x, y > 0$ (or $(\partial/\partial x)k_\lambda(x, y) \le 0, (\partial/\partial y)k_\lambda(x, y) \le 0$ $(x, y > 0)$), such that, for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda (\gamma) := \int_0^\infty k_\lambda (u, 1) u^{\gamma - 1} du \in \mathrm{R}_+ = (0, \infty). \tag{6}$$

We still assume that $a_m, b_n \ge 0$ $(m, n \in \mathrm{N} \setminus \{1\} = \{2, 3, \ldots\})$, satisfying

$$0 < \sum_{m=2}^\infty \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1} m}{m^{1-p}} a_m^p < \infty$$
$$\text{and } 0 < \sum_{n=2}^\infty \frac{\ln^{q[1-(\lambda_2/p+(\lambda-\lambda_1)/q)]-1} n}{n^{1-p}} b_n^q < \infty. \tag{7}$$

Definition 1. Define the following weight functions:

$$\omega_\lambda (\lambda_2, m) := \ln^{\lambda-\lambda_2} m \sum_{n=2}^\infty k_\lambda (\ln m, \ln n) \frac{\ln^{\lambda_2 - 1} n}{n} \quad (m \in \mathrm{N} \setminus \{1\}), \tag{8}$$

$$\varpi_\lambda (\lambda_1, n) := \ln^{\lambda-\lambda_1} n \sum_{m=2}^\infty k_\lambda (\ln m, \ln n) \frac{\ln^{\lambda_1 - 1} m}{m} \quad (n \in \mathrm{N} \setminus \{1\}). \tag{9}$$

Lemma 2. *We have the following inequalities:*

$$\omega_\lambda (\lambda_2, m) < k_\lambda (\lambda - \lambda_2) \quad (m \in \mathrm{N} \setminus \{1\}), \tag{10}$$

$$\varpi_\lambda (\lambda_1, n) < k_\lambda (\lambda_1) \quad (n \in \mathrm{N} \setminus \{1\}). \tag{11}$$

Proof. For $\lambda_2 - 1 \le 0$, it is evident that $k_\lambda(\ln m, \ln t)(\ln^{\lambda_2-1} t)/t$ is a strictly decreasing function with respect to $t > 1$. By the decreasing property, setting $u = \ln m / \ln t$, it follows that

$$\omega_\lambda (\lambda_2, m) < \ln^{\lambda-\lambda_2} m \int_1^\infty k_\lambda (\ln m, \ln t) \frac{\ln^{\lambda_2-1} t}{t} dt$$
$$= \int_0^\infty k_\lambda (u, 1) u^{(\lambda-\lambda_2)-1} du = k_\lambda (\lambda - \lambda_2). \tag{12}$$

Hence, we have (10). *For* $\lambda_1 - 1 \le 0$, it is evident that $k_\lambda(\ln t, \ln n)(\ln^{\lambda_1-1} t)/t$ is a strictly decreasing function with respect to $t > 1$. By the decreasing property, setting $u = \ln t / \ln n$, we find that

$$\varpi_\lambda (\lambda_1, n) < \ln^{\lambda-\lambda_1} n \int_1^\infty k_\lambda (\ln t, \ln n) \frac{\ln^{\lambda_1-1} t}{t} dt$$
$$= \int_0^\infty k_\lambda (u, 1) u^{\lambda_1-1} du = k_\lambda (\lambda_1). \tag{13}$$

Hence, we have (11). □

Lemma 3. *We have the following inequality:*

$$I := \sum_{n=2}^\infty \sum_{m=2}^\infty k_\lambda (\ln m, \ln n) a_m b_n < k_\lambda^{1/p} (\lambda - \lambda_2)$$
$$\cdot k_\lambda^{1/q} (\lambda_1) \left\{ \sum_{m=2}^\infty \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1} m}{m^{1-p}} a_m^p \right\}^{1/p}$$
$$\cdot \left\{ \sum_{n=2}^\infty \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1} n}{n^{1-q}} b_n^q \right\}^{1/q}. \tag{14}$$

Proof. By Hölder's inequality with weight (cf. [26]), we obtain

$$I := \sum_{n=2}^\infty \sum_{m=2}^\infty k_\lambda (\ln m, \ln n) \left[\frac{\ln^{(\lambda_2-1)p} n}{n^{1/p}} \frac{\ln^{(1-\lambda_1)/q} m}{m^{-1/q}} a_m \right]$$
$$\times \left[\frac{\ln^{(\lambda_1-1)/q} m}{\mathrm{m}^{1/q}} \frac{\ln^{(1-\lambda_2)/p} n}{n^{-1/p}} b_n \right]$$
$$\le \left[\sum_{m=2}^\infty \sum_{n=2}^\infty k_\lambda (\ln m, \ln n) \frac{\ln^{\lambda_2-1} n}{n} \frac{\ln^{(p-1)(1-\lambda_1)} m}{m^{1-p}} \right.$$
$$\left. \cdot a_m^p \right]^{1/p} \times \left[\sum_{n=2}^\infty \sum_{m=2}^\infty k_\lambda (\ln m, \ln n) \frac{\ln^{\lambda_1-1} m}{m} \right.$$
$$\left. \cdot \frac{\ln^{(q-1)(1-\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{1/q} = \left\{ \sum_{m=2}^\infty \omega_\lambda (\lambda_2, m) \right.$$
$$\left. \cdot \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1} m}{m^{1-p}} a_m^p \right\}^{1/p} \times \left\{ \sum_{n=2}^\infty \varpi_\lambda (\lambda_1, n) \right.$$
$$\left. \cdot \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1} n}{n^{1-q}} b_n^q \right\}^{1/q}. \tag{15}$$

Then by (10) and (11), we have (14). □

Remark 4. By (14), for $\lambda_1 + \lambda_2 = \lambda$, we find

$$0 < \sum_{m=2}^\infty \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p < \infty$$
$$\text{and } 0 < \sum_{n=2}^\infty \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-p}} b_n^q < \infty, \tag{16}$$

and the following inequality:

$$\sum_{n=2}^\infty \sum_{m=2}^\infty k_\lambda (\ln m, \ln n) a_m b_n < k_\lambda (\lambda_1)$$
$$\cdot \left[\sum_{m=2}^\infty \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{1/p} \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\lambda_2)-1} n}{n^{1-q}} b_n^q \right]^{1/q}. \tag{17}$$

In particular, for $p = q = 2$, we have

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n) a_m b_n < k_\lambda(\lambda_1)\left(\sum_{m=2}^{\infty}\frac{m}{\ln^{2\lambda_1-1}m}a_m^2 \sum_{n=2}^{\infty}\frac{n}{\ln^{2\lambda_2-1}n}b_n^2\right)^{1/2}. \tag{18}$$

For $\lambda = 1$, $k_1(x, y) = 1/(x + y)$, $\lambda_1 = \lambda_2 = 1/2$, (18) reduces to (2). Hence, (17) is an extension of (18) and (2).

Lemma 5. *The constant factor $k(\lambda_1)$ in (17) is the best possible.*

Proof. For any $\varepsilon > 0$, we set

$$\tilde{a}_m := \frac{\ln^{\lambda_1-\varepsilon/p-1}m}{m}, \quad \tilde{b}_n := \frac{\ln^{\lambda_2-\varepsilon/q-1}n}{n} \quad (m, n \in \mathrm{N} \setminus \{1\}). \tag{19}$$

If there exists a constant M ($M \le k_\lambda(\lambda_1)$), such that (17) is valid when replacing $k_\lambda(\lambda_1)$ by M, then, in particular, we have

$$\tilde{I} := \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n)\tilde{a}_m\tilde{b}_n < M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}\tilde{a}_m^p\right]^{1/p} \cdot \left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-p}}\tilde{b}_n^q\right]^{1/q}. \tag{20}$$

We obtain

$$\begin{aligned}\tilde{I} &< M\left[\sum_{m=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}\frac{\ln^{p\lambda_1-\varepsilon-p}m}{m^p}\right]^{1/p} \cdot \left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-q}}\frac{\ln^{q\lambda_2-\varepsilon-1}n}{n^q}\right]^{1/q} \\ &= M\left(\frac{\ln^{-\varepsilon-1}2}{2} + \sum_{m=3}^{\infty}\frac{\ln^{-\varepsilon-1}m}{m}\right) \\ &< M\left(\frac{\ln^{-\varepsilon-1}2}{2} + \int_2^{\infty}\frac{\ln^{-\varepsilon-1}t}{t}dt\right) \\ &= \frac{M}{\varepsilon\ln^{\varepsilon}2}\left(\frac{\varepsilon}{2\ln 2} + 1\right).\end{aligned} \tag{21}$$

By the decreasing property and Fubini theorem (cf. [27]), we find

$$\begin{aligned}\tilde{I} &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n)\frac{\ln^{\lambda_1-1}m}{m\ln^{\varepsilon/p}m}\cdot\frac{\ln^{\lambda_2-1}n}{n\ln^{\varepsilon/q}n} \\ &\ge \int_2^{\infty}\left(\int_2^{\infty} k_\lambda(\ln x, \ln y)\frac{\ln^{\lambda_1-\varepsilon/p-1}x}{x} \cdot \frac{\ln^{\lambda_2-\varepsilon/q-1}y}{y}dx\right)dy \left(u = \frac{\ln x}{\ln y}\right) \\ &= \int_2^{\infty}\frac{\ln^{-\varepsilon-1}y}{y}\left(\int_{\ln 2/\ln y}^{\infty} k_\lambda(u, 1) \cdot u^{\lambda_1-\varepsilon/p-1}du\right)dy \\ &= \int_2^{\infty}\frac{\ln^{-\varepsilon-1}y}{y}\left(\int_{\ln 2/\ln y}^{1} k_\lambda(u, 1) \cdot u^{\lambda_1-\varepsilon/p-1}du\right)dy \\ &\quad + \int_2^{\infty}\frac{\ln^{-\varepsilon-1}y}{y}\left(\int_1^{\infty} k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du\right)dy \\ &= \int_0^1\left(\int_{\eta+2^{1/u}}^{\infty}\frac{\ln^{-\varepsilon-1}y}{y}dy\right)k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du \\ &\quad + \frac{1}{\varepsilon\ln^{\varepsilon}2}\int_1^{\infty} k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du \\ &= \frac{1}{\varepsilon\ln^{\varepsilon}2}\left(\int_0^1 k_\lambda(u, 1) u^{\lambda_1+\varepsilon/q-1}du + \int_1^{\infty} k_\lambda(u, 1) \cdot u^{\lambda_1-\varepsilon/p-1}du\right).\end{aligned} \tag{22}$$

Then we have

$$\int_0^1 k_\lambda(u, 1) u^{\lambda_1+\varepsilon/q-1}du + \int_1^{\infty} k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du < M\left(\frac{\varepsilon}{2\ln 2} + 1\right). \tag{23}$$

For $\varepsilon \longrightarrow 0^+$, by Fatou lemma (cf. [27]), we find

$$\begin{aligned}k_\lambda(\lambda_1) &= \int_0^1 \varliminf_{\varepsilon\longrightarrow 0^+} k_\lambda(u, 1) u^{\lambda_1+\varepsilon/q-1}du + \int_1^{\infty}\varliminf_{\varepsilon\longrightarrow 0^+} k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du] \\ &\le \varliminf_{\varepsilon\longrightarrow 0^+}\left(\int_0^1 k_\lambda(u, 1) u^{\lambda_1+\varepsilon/q-1}du + \int_1^{\infty} k_\lambda(u, 1) u^{\lambda_1-\varepsilon/p-1}du\right) \le M.\end{aligned} \tag{24}$$

Hence, $M = k_\lambda(\lambda_1)$ is the best possible constant factor of (17). □

Remark 6. Setting $\widehat{\lambda}_1 := (\lambda-\lambda_2)/p+\lambda_1/q, \widehat{\lambda}_2 := (\lambda-\lambda_1)/q+\lambda_2/p$, we find

$$\widehat{\lambda}_1+\widehat{\lambda}_2 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p} = \frac{\lambda}{p}+\frac{\lambda}{q} = \lambda,$$
$$\widehat{\lambda}_1 \le \frac{1}{p}+\frac{1}{q} = 1, \quad (25)$$
$$\widehat{\lambda}_2 \le \frac{1}{q}+\frac{1}{p} = 1,$$

and by Hölder's inequality (cf. [26]), we have

$$\begin{aligned} 0 &< k_\lambda\left(\lambda-\widehat{\lambda}_2\right) = k_\lambda\left(\widehat{\lambda}_1\right) = k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right) \\ &= \int_0^\infty k_\lambda(u,1)\,u^{(\lambda-\lambda_2)/p+\lambda_1/q-1}du \\ &= \int_0^\infty k_\lambda(u,1)\left(u^{(\lambda-\lambda_2-1)/p}\right)\left(u^{(\lambda_1-1)/q}\right)du \\ &\le \left(\int_0^\infty k_\lambda(u,1)\,u^{\lambda-\lambda_2-1}du\right)^{1/p} \\ &\cdot \left(\int_0^\infty k_\lambda(u,1)\,u^{\lambda_1-1}du\right)^{1/q} = k_\lambda^{1/p}(\lambda-\lambda_2) \\ &\cdot k_\lambda^{1/q}(\lambda_1) < \infty. \end{aligned} \quad (26)$$

We can rewrite (14) as follows:

$$\begin{aligned} I &< k_\lambda^{1/p}(\lambda-\lambda_2)\,k_\lambda^{1/q}(\lambda_1)\left[\sum_{m=2}^\infty \frac{\ln^{p(1-\widehat{\lambda}_1)-1}m}{m^{1-p}}a_m^p\right]^{1/p} \\ &\cdot \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\widehat{\lambda}_2)-1}n}{n^{1-q}}b_n^q\right]^{1/q}. \end{aligned} \quad (27)$$

Lemma 7. *If the constant factor $k_\lambda^{1/p}(\lambda-\lambda_2)k_\lambda^{1/q}(\lambda_1)$ in (14) is the best possible, then $\lambda_1+\lambda_2=\lambda$.*

Proof. If the constant factor $k_\lambda^{1/p}(\lambda-\lambda_2)k_\lambda^{1/q}(\lambda_1)$ in (14) is the best possible, then, by (27) and (17), the unique best possible constant factor must be $k_\lambda(\widehat{\lambda}_1)(\in \mathrm{R}_+)$, namely,

$$k_\lambda\left(\widehat{\lambda}_1\right) = k_\lambda^{1/p}(\lambda-\lambda_2)\,k_\lambda^{1/q}(\lambda_1). \quad (28)$$

We observe that (26) keeps the form of equality if and only if there exist constants$A$and$B$, such that they are not all zero and (cf. [26])

$$Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1}\text{a.e.} \quad \text{in } \mathrm{R}_+ = (0,\infty). \quad (29)$$

Assuming that $A \ne 0$ (otherwise, $B = A = 0$), it follows that $u^{\lambda-\lambda_2-\lambda_1} = B/A$ *a.e.* in R_+, and then $\lambda-\lambda_2-\lambda_1 = 0$, namely, $\lambda_1+\lambda_2=\lambda$. □

3. Main Results

Theorem 8. *Inequality (14) is equivalent to*

$$\begin{aligned} J &:= \left[\sum_{n=2}^\infty \frac{\ln^{p((\lambda-\lambda_1)/q+\lambda_2/p)-1}n}{n}\left(\sum_{m=2}^\infty k_\lambda(\ln m,\ln n)\right.\right. \\ &\left.\left.\cdot a_m\right)^p\right]^{1/p} < k_\lambda^{1/p}(\lambda-\lambda_2)\,k_\lambda^{1/q}(\lambda_1) \\ &\cdot \left\{\sum_{m=2}^\infty \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1}m}{m^{1-p}}a_m^p\right\}^{1/p}. \end{aligned} \quad (30)$$

If the constant factor in (14) is the best possible, then so is the constant factor in (30).

Proof. Suppose that (30) is valid. By Hölder's inequality (cf. [26]), we find

$$\begin{aligned} I &= \sum_{n=2}^\infty \left[\frac{\ln^{-1/p+((\lambda-\lambda_1)/q+\lambda_2/p)}n}{n^{1/p}}\sum_{m=2}^\infty k_\lambda(\ln m,\ln n)\,a_m\right] \\ &\cdot \left[\frac{\ln^{1/p-((\lambda-\lambda_1)/q+\lambda_2/p)}n}{n^{-1/p}}b_n\right] \\ &\le J\left\{\sum_{n=2}^\infty \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1}n}{n^{1-q}}b_n^q\right\}^{1/q}. \end{aligned} \quad (31)$$

Then by (30), we obtain (14).

On the other hand, assuming that (14) is valid, we set

$$b_n := \frac{\ln^{p((\lambda-\lambda_1)/q+\lambda_2/p)-1}n}{n}\left(\sum_{m=2}^\infty k_\lambda(\ln m,\ln n)\,a_m\right)^{p-1}, \quad n \in \mathbf{N}\setminus\{1\}. \quad (32)$$

If $J = 0$, then (30) is naturally valid; if $J = \infty$, then it is impossible to make (30) valid, namely, $J < \infty$. Suppose that $0 < J < \infty$. By (14), it follows that

$$\begin{aligned} &\sum_{n=2}^\infty \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1}n}{n^{1-q}}b_n^q = J^p = I \\ &< k_\lambda^{1/p}(\lambda-\lambda_2)\,k_\lambda^{1/q}(\lambda_1) \\ &\cdot \left\{\sum_{m=2}^\infty \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1}m}{m^{1-p}}a_m^p\right\}^{1/p} \\ &\cdot \left\{\sum_{n=2}^\infty \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1}n}{n^{1-q}}b_n^q\right\}^{1/q}, \end{aligned}$$

$$J = \left\{ \sum_{n=2}^{\infty} \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1} n}{n^{1-q}} b_n^q \right\}^{1/p}$$
$$< k_\lambda^{1/p} (\lambda - \lambda_2) k_\lambda^{1/q} (\lambda_1)$$
$$\cdot \left\{ \sum_{m=2}^{\infty} \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1} m}{m^{1-p}} a_m^p \right\}^{1/p}, \tag{33}$$

namely, (30) follows, which is equivalent to (14).

If the constant factor in (14) is the best possible, then so is constant factor in (30). Otherwise, by (31), we would reach a contradiction that the constant factor in (14) is not the best possible. □

Theorem 9. *The statements (i), (ii), (iii), and (iv) are equivalent as follows:*

(i) $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ *is independent of* p, q

(ii) $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ *is expressible as a single integral*

(iii) $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ *is the best possible constant factor of (14)*

(iv) $\lambda_1 + \lambda_2 = \lambda$

If the statement (iv) follows, namely, $\lambda_1 + \lambda_2 = \lambda$, then we have (17) and the following equivalent inequality with the best possible constant factor $k_\lambda(\lambda_1)$:

$$\left[\sum_{n=2}^{\infty} \frac{\ln^{p\lambda_2-1} n}{n} \left(\sum_{m=2}^{\infty} k_\lambda (\ln m, \ln n) a_m \right)^p \right]^{1/p}$$
$$< k_\lambda (\lambda_1) \left[\sum_{m=2}^{\infty} \frac{\ln^{p(1-\lambda_1)-1} m}{m^{1-p}} a_m^p \right]^{1/p}. \tag{34}$$

Proof. (i)=>(ii). Since $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ is independent of p, q, we find

$$k_\lambda^{1/p} (\lambda - \lambda_2) k_\lambda^{1/q} (\lambda_1)$$
$$= \lim_{p\to\infty} \lim_{q\to 1^+} k_\lambda^{1/p} (\lambda - \lambda_2) k_\lambda^{1/q} (\lambda_1) = k_\lambda (\lambda_1), \tag{35}$$

namely, $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ is expressible as a single integral

$$k_\lambda (\lambda_1) = \int_0^\infty k_\lambda (u, 1) u^{\lambda_1 - 1} du. \tag{36}$$

(ii)=>(iv). In (26), if $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)$ is expressible as a single integral $k_\lambda((\lambda - \lambda_2)/p + \lambda_1/q)$, then (26) keeps the form of equality, which follows that $\lambda_1 + \lambda_2 = \lambda$.

(iv)=>(i). If $\lambda_1 + \lambda_2 = \lambda$, then $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1) = k_\lambda(\lambda_1)$, which is independent of p, q. Hence, we have (i)$\Longleftrightarrow$(ii)$\Longleftrightarrow$(iv).

(iii)=>(iv). By Lemma 7, we have $\lambda_1 + \lambda_2 = \lambda$.

(iv)=>(iii). By Lemma 5, for $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1)(= k_\lambda(\lambda_1))$ is the best possible constant factor of (14). Therefore, we have (iii) $\Longleftrightarrow$(iv).

Hence, the statements (i), (ii), (iii), and (iv) are equivalent. □

Remark 10. (i) For $\lambda = 1$, $\lambda_1 = 1/q$, $\lambda_2 = 1/p$ in (17) and (34), we have the following equivalent inequalities with the best possible constant factor $k_1(1/q)$:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m b_n$$
$$< k_1 \left(\frac{1}{q} \right) \left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}} \right)^{1/p} \left(\sum_{n=2}^{\infty} \frac{b_n^q}{n^{1-q}} \right)^{1/q}, \tag{37}$$

$$\left[\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m \right)^p \right]^{1/p}$$
$$< k_1 \left(\frac{1}{q} \right) \left(\sum_{m=2}^{\infty} \frac{a_m^p}{m^{1-p}} \right)^{1/p}. \tag{38}$$

(ii) For$\lambda = 1$, $\lambda_1 = 1/p$, $\lambda_2 = 1/q$ in (17) and (34), we have the following equivalent inequalities with the best possible constant factor $k_1(1/p)$:

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m b_n$$
$$< k_1 \left(\frac{1}{p} \right) \left(\sum_{m=2}^{\infty} \frac{\ln^{p-2} m}{m^{1-p}} a_m^p \right)^{1/p} \left(\sum_{n=2}^{\infty} \frac{\ln^{p-2} n}{n^{1-q}} b_n^q \right)^{1/q}, \tag{39}$$

$$\left[\sum_{n=2}^{\infty} \frac{\ln^{p-2} n}{n} \left(\sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m \right)^p \right]^{1/p}$$
$$< k_1 \left(\frac{1}{p} \right) \left(\sum_{m=2}^{\infty} \frac{\ln^{p-2} m}{m^{1-p}} a_m^p \right)^{1/p}. \tag{40}$$

(iii) For $p = q = 2$, both (37) and (39) reduce to

$$\sum_{n=2}^{\infty} \sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m b_n$$
$$< k_1 \left(\frac{1}{2} \right) \left(\sum_{m=2}^{\infty} m a_m^2 \sum_{n=2}^{\infty} n b_n^2 \right)^{1/2}, \tag{41}$$

and both (38) and (40) reduce to the equivalent form of (41) as follows:

$$\left[\sum_{n=2}^{\infty} \frac{1}{n} \left(\sum_{m=2}^{\infty} k_1 (\ln m, \ln n) a_m \right)^2 \right]^{1/2}$$
$$< k_1 \left(\frac{1}{2} \right) \left(\sum_{m=2}^{\infty} m a_m^2 \right)^{1/2}. \tag{42}$$

4. Operator Expressions and Some Particular Cases

We set functions

$$\varphi(m) := \frac{\ln^{p[1-((\lambda-\lambda_2)/p+\lambda_1/q)]-1} m}{m^{1-p}},$$
$$\psi(n) := \frac{\ln^{q[1-((\lambda-\lambda_1)/q+\lambda_2/p)]-1} n}{n^{1-q}}, \tag{43}$$

where

$$\psi^{1-p}(n) = \frac{\ln^{p((\lambda-\lambda_1)/q+\lambda_2/p)-1} n}{n} \quad (m, n \in \mathrm{N} \setminus \{1\}). \tag{44}$$

Define the following real normed spaces:

$$l_{p,\varphi} := \left\{ a = \{a_m\}_{m=2}^{\infty} ; \|a\|_{p,\varphi} := \left(\sum_{m=2}^{\infty} \varphi(m) |a_m|^p \right)^{1/p} < \infty \right\},$$
$$l_{q,\psi} := \left\{ b = \{b_n\}_{n=2}^{\infty} ; \|b\|_{q,\psi} := \left(\sum_{n=2}^{\infty} \psi(n) |b_n|^q \right)^{1/q} < \infty \right\}, \tag{45}$$
$$l_{p,\psi^{1-p}} := \left\{ c = \{c_n\}_{n=2}^{\infty} ; \|c\|_{p,\psi^{1-p}} := \left(\sum_{n=2}^{\infty} \psi^{1-p}(n) |c_n|^p \right)^{1/p} < \infty \right\}.$$

Assuming that $a \in l_{p,\varphi}$, setting

$$c = \{c_n\}_{n=2}^{\infty},$$
$$c_n := \sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n) a_m, \tag{46}$$
$$n \in \mathrm{N} \setminus \{1\},$$

we can rewrite (30) as follows:

$$\|c\|_{p,\psi^{1-p}} < k_\lambda^{1/p}(\lambda - \lambda_2) k_\lambda^{1/q}(\lambda_1) \|a\|_{p,\varphi} < \infty, \tag{47}$$

namely, $c \in l_{p,\psi^{1-p}}$.

Definition 11. Define a Mulholland-type operator $T : l_{p,\varphi} \longrightarrow l_{p,\psi^{1-p}}$ as follows: for any $a \in l_{p,\varphi}$, there exists a unique representation $c \in l_{p,\psi^{1-p}}$. Define the formal inner product of Ta and $b \in l_{q,\psi}$ and the norm of T as follows:

$$(Ta, b) := \sum_{n=2}^{\infty} \left(\sum_{m=2}^{\infty} k_\lambda(\ln m, \ln n) a_m \right) b_n,$$
$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\varphi}} \frac{\|Ta\|_{p,\psi^{1-p}}}{\|a\|_{p,\varphi}}. \tag{48}$$

By Theorems 8 and 9, we have the following.

Theorem 12. *If $a \in l_{p,\varphi}$, $b \in l_{q,\psi}$, $\|a\|_{p,\varphi}$, $\|b\|_{q,\psi} > 0$, then we have the following equivalent inequalities:*

$$(Ta, b) < k_\lambda^{1/p}(\lambda - \lambda_2) k_\lambda^{1/q}(\lambda_1) \|a\|_{p,\varphi} \|b\|_{q,\psi}, \tag{49}$$

$$\|Ta\|_{p,\psi^{1-p}} < k_\lambda^{1/p}(\lambda - \lambda_2) k_\lambda^{1/q}(\lambda_1) \|a\|_{p,\varphi}. \tag{50}$$

Moreover, $\lambda_1 + \lambda_2 = \lambda$ if and only if the constant factor $k_\lambda^{1/p}(\lambda - \lambda_2)k_\lambda^{1/q}(\lambda_1) = k_\lambda(\lambda_1)$ in (49) and (50) is the best possible, namely,

$$\|T\| = k_\lambda(\lambda_1). \tag{51}$$

Example 13. We set $k_\lambda(x, y) := 1/(cx + y)^\lambda$ $(c, \lambda > 0; x, y > 0)$. Then we find $k_\lambda(\ln m, \ln n) = 1/\ln^\lambda m^c n$. For $0 < \lambda_i, \lambda - \lambda_i \leq 1$ $(i = 1, 2)$, $k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x, y)$ is decreasing with respect to$x, y > 0$, and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1}}{(cu+1)^\lambda} du = \frac{1}{c^\gamma} \int_0^\infty \frac{v^{\gamma-1}}{(v+1)^\lambda} dv$$
$$= \frac{1}{c^\gamma} B(\gamma, \lambda - \gamma) \in \mathrm{R}_+. \tag{52}$$

In view of Theorem 12, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = k_\lambda(\lambda_1) = \frac{1}{c^{\lambda_1}} B(\lambda_1, \lambda_2). \tag{53}$$

Example 14. We set $k_\lambda(x, y) := \ln(cx/y)/(cx)^\lambda - y^\lambda$ $(c > 0, \lambda > 0; x, y > 0)$. Then we find $k_\lambda(\ln m, \ln n) = \ln(\ln m^c / \ln n)/(\ln^\lambda m^c - \ln^\lambda n)$. For $0 < \lambda_i, \lambda - \lambda_i \leq 1$ $(i = 1, 2)$, $k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x, y)$ is decreasing with respect to$x, y > 0$ (cf. [2], Example 2.2.1), and for $\gamma = \lambda_1, \lambda - \lambda_2$,

$$k_\lambda(\gamma) = \int_0^\infty \frac{u^{\gamma-1} \ln(cu)}{(cu)^\lambda - 1} du$$
$$= \frac{1}{c^\gamma \lambda^2} \int_0^\infty \frac{v^{(\gamma/\lambda)-1} \ln v}{v - 1} dv \tag{54}$$
$$= \frac{1}{c^\gamma} \left[\frac{\pi}{\lambda \sin(\pi\gamma/\lambda)} \right]^2 \in \mathrm{R}_+.$$

In view of Theorem 12, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = k_\lambda(\lambda_1) = \frac{1}{c^{\lambda_1}} \left[\frac{\pi}{\lambda \sin(\pi\lambda_1/\lambda)} \right]^2. \tag{55}$$

Example 15. We set $k_\lambda(x, y) := 1/\prod_{k=1}^s (x^{\lambda/s} + c_k y^{\lambda/s})$ $(0 < c_1 \leq \cdots \leq c_s, \lambda > 0; x, y > 0)$. Then we find

$$k_\lambda(\ln m, \ln n) = \frac{1}{\prod_{k=1}^s \left(\ln^{\lambda/s} m + c_k \ln^{\lambda/s} n \right)}. \tag{56}$$

For $0 < \lambda_i, \lambda - \lambda_i \leq 1$ $(i = 1, 2)$, $k_\lambda(x, y)$ is a positive homogeneous function of degree $-\lambda$, such that $k_\lambda(x, y)$ is decreasing with respect to $x, y > 0$, and for $\gamma = \lambda_1, \lambda - \lambda_2$, by *Example 1 of [28], it follows that*

$$k_\lambda^{(s)}(\gamma) = \int_0^\infty \frac{t^{\gamma-1}}{\prod_{k=1}^{s}(t^{\lambda/s} + c_k)} dt = \frac{\pi s}{\lambda \sin(\pi s\gamma/\lambda)} \sum_{k=1}^{s} c_k^{s\gamma/\lambda - 1} \prod_{j=1(j\neq k)}^{s} \frac{1}{c_j - c_k} \in \mathrm{R}_+. \quad (57)$$

In view of Theorem 12, it follows that $\lambda_1 + \lambda_2 = \lambda$ if and only if

$$\|T\| = k_\lambda^{(s)}(\lambda_1) = \frac{\pi s}{\lambda \sin(\pi s\lambda_1/\lambda)} \sum_{k=1}^{s} c_k^{s\lambda_1/\lambda - 1} \prod_{j=1(j\neq k)}^{s} \frac{1}{c_j - c_k}. \quad (58)$$

In particular, for $c_1 = \cdots = c_s = c$, we have $k_\lambda(x, y) = 1/(x^{\lambda/s} + cy^{\lambda/s})^s$ and

$$\|T\| = \widetilde{k}_\lambda^{(s)}(\lambda_1) := \int_0^\infty \frac{t^{\lambda_1-1}}{(t^{\lambda/s} + c)^s} dt = \frac{s}{\lambda c^{[1-(\lambda_1/\lambda)]s}} \int_0^\infty \frac{v^{s\lambda_1/\lambda-1}}{(v+1)^s} dv = \frac{s}{\lambda c^{[1-(\lambda_1/\lambda)]s}} B\left(\frac{s\lambda_1}{\lambda}, \frac{s\lambda_2}{\lambda}\right). \quad (59)$$

If $s = 1$, then we have $k_\lambda(x, y) = 1/(x^\lambda + cy^\lambda)$, $k_\lambda(\ln m, \ln n) = 1/(\ln^\lambda m + c\ln^\lambda n)$, and

$$\|T\| = \widetilde{k}_\lambda^{(1)}(\lambda_1) = \frac{1}{\lambda c^{1-(\lambda_1/\lambda)}} \frac{\pi}{\sin(\pi\lambda_1/\lambda)}. \quad (60)$$

5. Conclusions

In this paper, by the use of the weight functions and the idea of introducing parameters, a discrete Mulholland-type inequality with the general homogeneous kernel and the equivalent form are given in Lemma 3 and Theorem 8. The equivalent statements of the best possible constant factor related to a few parameters are considered in Theorem 9. As applications, the operator expressions and some particular examples are given in Theorem 12 and Examples 13–15. The lemmas and theorems provide an extensive account of this type of inequalities.

Data Availability

The study belongs to pure theory research. There are not any sharing data.

Conflicts of Interest

The authors declare that they have no competing interests.

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