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Research Article

Determinantal Representations of General and (Skew-)Hermitian Solutions to the Generalized Sylvester-Type Quaternion Matrix Equation

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In this paper, we derive explicit determinantal representation formulas of general, Hermitian, and skew-Hermitian solutions to the generalized Sylvester matrix equation involving $*$-Hermicity $\mathbf{AXA}^* + \mathbf{BYB}^* = \mathbf{C}$ over the quaternion skew field within the framework of the theory of noncommutative column-row determinants.

1. Introduction

Let $\mathbb{H}^{m\times n}$ and $\mathbb{H}_r^{m\times n}$ stand for the set of all $m \times n$ matrices and for its subset of matrices with rank r, respectively, over the quaternion skew field

$$\mathbb{H} = \left\{a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \mid \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ a_0, a_1, a_2, a_3 \in \mathbb{R}\right\}, \quad (1)$$

where $\mathbb{R}$ is the real number field. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, the symbol $\mathbf{A}^*$ stands for conjugate transpose (Hermitian adjoint) of $\mathbf{A}$. A matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m\times n}$ is called the unique matrix $\mathbf{X} \in \mathbb{H}^{n\times m}$ satisfying the following four equations

$$\begin{aligned} &1.\ \mathbf{AXA} = \mathbf{A},\\ &2.\ \mathbf{XAX} = \mathbf{X},\\ &3.\ (\mathbf{AX})^* = \mathbf{AX},\\ &4.\ (\mathbf{XA})^* = \mathbf{XA}. \end{aligned} \quad (2)$$

It is denoted by $\mathbf{A}^\dagger$.

The two-sided generalized Sylvester matrix equation

$$\mathbf{AXB} + \mathbf{CYD} = \mathbf{E} \quad (3)$$

has been well studied in matrix theory. For instance, Huang [1] obtained necessary and sufficient conditions for the existence of solutions to (3) with $\mathbf{X} = \mathbf{Y}$ over the quaternion skew field. Baksalary and Kala [2] derived the general solution to (3) expressed in terms of generalized inverses which has been extended to an arbitrary division ring and on any regular ring with identity in [3, 4]. Ranks and independence of solutions to (3) were explored in [5]. In [6] expressions, as well as necessary and sufficient conditions, were given for the existence of the real and pure imaginary solutions to the consistent quaternion matrix equation (3).

The high research activities on Sylvester-type matrix equations can be observed lately. In particular, we note the following papers concerning methods of their computing solutions. Liao et al. [7] established a direct method for computing its approximate solution using the generalized singular value decomposition and the canonical correlation decomposition. Efficient iterative algorithms were presented to solve a system of two generalized Sylvester matrix equations in [8] and to solve the minimum Frobenius norm residual problem for a system of Sylvester-type matrix equations over generalized reflexive matrix in [9].

Systems of periodic discrete-time coupled Sylvester quaternion matrix equations [10], systems of quaternary coupled Sylvester-type real quaternion matrix equations [11], and optimal pole assignment of linear systems by the Sylvester matrix equations [12] have been explored. Some constraint generalized Sylvester matrix equations [13, 14] were studied recently.

Special solutions to Sylvester-type quaternion matrix equations have been actively studied. Roth's solvability criteria for some Sylvester-type matrix equations were extended over the quaternion skew field with a fixed involutive automorphism in [15]. Şimşek et al. [16] established the precise solutions on the minimum residual and matrix nearness problems of the quaternion matrix equation $(\mathbf{AXB}, \mathbf{DXE}) = (\mathbf{C}, \mathbf{F})$ for centrohermitian and skew-centrohermitian matrices. Explicit solutions to some Sylvester-type quaternion matrix equations (with j-conjugation) were established by means of Kronecker map and complex representation of a quaternion matrix in [17, 18]. The expressions of the least squares solutions to some Sylvester-type matrix equations over nonsplit quaternion algebra [19] and Hermitian solutions over a split quaternion algebra [20] were derived. Solvability conditions and general solution for some generalized Sylvester real quaternion matrix equations involving η-Hermicity were given in [21, 22].

Many authors have paid attention also to the Sylvester-type matrix equation involving $*$-Hermicity

$$\mathbf{AXA}^{*} + \mathbf{BYB}^{*} = \mathbf{C}. \tag{4}$$

Chang and Wang [23] derived expressions for the general symmetric solution and the general minimum-2-norm symmetric solution to the matrix equation (4) within the real settings. Xu et al. [24] have given a representation of the least-squares Hermitian (skew-Hermitian) solution to the matrix equation (4). Zhang [25] obtained a representation of the general Hermitian nonnegative-definite (respectively positive-definite) solution to (4) within the complex settings. Yuan et al. [26] derived the expression of Hermitian solution for the matrix nearness problem associated with the quaternion matrix equation (4). Wang et al. [27] gave a necessary and sufficient condition for the existence and an expression for the re-nonnegative definite solution to (4) over $\mathbb{H}$ by using the decomposition of pairwise matrices. Wang et al. [28] established the extreme ranks for the general (skew-)Hermitian solution to (4) over $\mathbb{H}$.

Motivated by the vast application of quaternion matrices and the latest interest of Sylvester-type quaternion matrix equations, the main goal of the paper is to derive explicit determinantal representation formulas of the general, Hermitian, and skew-Hermitian solutions to (4) based on determinantal representations of the Moore-Penrose inverse.

Determinantal representation of a solution gives a direct method of its finding analogous to classical Cramer's rule that has important theoretical and practical significance. However, determinantal representations are not so unambiguous even for generalized inverses within the complex or real settings. Through looking for their more applicable explicit expressions, there are various determinantal representations of generalized inverses (see, e.g., [29–31]). By virtue of noncommutativity of quaternions, the problem for determinantal representation of generalized quaternion inverses is even more complicated, and only now it can be solved due to the theory of column-row determinants introduced in [32, 33]. Within the framework of the theory of row-column determinants, determinantal representations of various generalized quaternion inverses, namely, the Moore-Penrose inverse [34], the Drazin inverse [35], the W-weighted Drazin inverse [36], and the weighted Moore-Penrose inverse [37], have been derived by the author. These determinantal representations were used to obtain explicit representation formulas for the minimum norm least squares solutions [38] and weighted Moore-Penrose inverse solutions [39] to some quaternion matrix equations and explicit determinantal representation formulas of both Drazin and W-weighted Drazin inverse solutions to some restricted quaternion matrix equations and quaternion differential matrix equations [40–42]. Recently, determinantal representations of solutions to some systems of quaternion matrix equations [43, 44] and, in [45], two-sided generalized Sylvester matrix equation (3) have been derived by the author as well.

Other researchers also used the row-column determinants in their developments. In particular, Song derived determinantal representations of the generalized inverse $\mathbf{A}^{2}_{T,S}$ [46] and the Bott-Duffin inverse [47]. Song et al. obtained the Cramer rules for the solutions of restricted matrix equations [48] and for the generalized Stein quaternion matrix equation [49], and so forth. Moreover, Song et al. [50] have just recently considered determinantal representations of the general solution to the generalized Sylvester matrix equation (3) over $\mathbb{H}$ using row-column determinants as well. But their approach differs from ours because for determinantal representations of solutions we use only coefficient matrices of the equation, while in [50] supplementary matrices have been constructed and used.

The paper is organized as follows. In Section 2, we start with some remarkable results which have significant role during the construction of the main results of this paper. Elements of the theory of row-column determinants are given in Section 2.1, determinantal representations of the Moore-Penrose inverse and of the general solution to the quaternion matrix equation $\mathbf{AXB} = \mathbf{C}$ and its special cases are considered in Section 2.2, and the explicit determinantal representation of the general solution to (3) previously obtained within the framework of the theory of row-column determinants is in Section 2.3. The main results of the paper, namely, explicit determinantal representation formulas of the general, Hermitian, skew-Hermitian solutions to (4), are derived in Section 3. In Section 4, a numerical example to illustrate the main results is considered. Finally, in Section 5, the conclusions are drawn.

2. Preliminaries

We commence with the following preliminaries which have crucial function in the construction of the chief outcomes of the following sections.

2.1. Elements of the Theory of Row-Column Determinants. Due to noncommutativity of quaternions, a problem of defining a determinant of matrices with noncommutative entries (which is also defined as noncommutative determinants) has been unsolved for a long time. There are several versions of the definition of noncommutative determinant (see, e.g., [51–56]). But any of the previous noncommutative determinants has not fully retained those properties which it has owned for matrices with commutative entries. Moreover, if functional properties of noncommutative determinant over a ring are satisfied, then it takes on a value in its commutative subset. This dilemma can be avoided thanks to the theory of row-column determinants.

Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$. Let $\mathbf{A} \in \mathbb{H}^{n\times n}$. Row determinants of $\mathbf{A}$ along each row can be defined as follows.

Definition 1 (see [32]). *The ith row determinant* of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined for all $i = 1, \ldots, n$ by putting

$$\begin{aligned} \operatorname{r\,det}_i \mathbf{A} &= \sum_{\sigma\in S_n} (-1)^{n-r} \\ &\quad \cdot \left(a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}\right) \ldots \left(a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}\right), \\ \sigma &= \left(i i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}\right) \\ &\quad \ldots \left(i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}\right), \end{aligned} \tag{5}$$

where $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for all $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

Similarly, for a column determinant along an arbitrary column, we have the following definition.

Definition 2 (see [32]). *The jth column determinant* of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is defined for all $j = 1, \ldots, n$ by putting

$$\begin{aligned} \operatorname{c\,det}_j \mathbf{A} &= \sum_{\tau\in S_n} (-1)^{n-r} \\ &\quad \cdot \left(a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}}\right) \ldots \left(a_{j j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}\right), \\ \tau &= \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right) \\ &\quad \cdot \left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right), \end{aligned} \tag{6}$$

where $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$ and $j_{k_t} < j_{k_t+s}$ for $t = 2, \ldots, r$ and $s = 1, \ldots, l_t$.

So an arbitrary $n \times n$ quaternion matrix inducts a set from n row determinants and n column determinants that are different in general. Only for Hermitian $\mathbf{A}$, we have [32],

$$\begin{aligned} \operatorname{r\,det}_1 \mathbf{A} = \cdots = \operatorname{r\,det}_n \mathbf{A} = \operatorname{c\,det}_1 \mathbf{A} = \cdots = \operatorname{c\,det}_n \mathbf{A} \\ \in \mathbb{R}, \end{aligned} \tag{7}$$

which enables defining *the determinant of a Hermitian matrix* by putting

$$\det \mathbf{A} := \operatorname{r\,det}_i \mathbf{A} = \operatorname{c\,det}_i \mathbf{A} \tag{8}$$

for all $i = 1, \ldots, n$.

Its properties are similar to the properties of an usual (commutative) determinant and they have been completely explored in [32] by using row and column determinants that are so defined only by construction. We note the following that will be required below.

Lemma 3. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$. *Then* $\operatorname{c\,det}_i \mathbf{A}^* = \overline{\operatorname{r\,det}_i \mathbf{A}}$, $\operatorname{r\,det}_i \mathbf{A}^* = \overline{\operatorname{c\,det}_i \mathbf{A}}$.

2.2. Determinantal Representations of the Moore-Penrose Inverse with Applications to Some Quaternion Matrix Equations. For introducing determinantal representations of the Moore-Penrose inverse, the following notations will be used.

Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. $\mathbf{A}^{\alpha}_{\beta}$ denotes a submatrix of $\mathbf{A}$ whose rows are indexed by α and the columns indexed by β. So $\mathbf{A}^{\alpha}_{\alpha}$ denotes a principal submatrix of $\mathbf{A}$ with rows and columns indexed by α. If $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian, then $|\mathbf{A}|^{\alpha}_{\alpha}$ denotes the corresponding principal minor of $\det \mathbf{A}$.

Let $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \ldots, \alpha_k),\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$ denote the collection of strictly increasing sequences of k integers chosen from $\{1, \ldots, n\}$ for all $1 \leq k \leq n$. Then, for fixed $i \in \alpha$ and $j \in \beta$, the collection of sequences of row indexes that contain the index i is denoted by $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m},\ i \in \alpha\}$; similarly, the collection of sequences of column indexes that contain the index j is denote by $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n},\ j \in \beta\}$.

Let $\mathbf{a}_{.j}$ be the jth column and $\mathbf{a}_{i.}$ be the ith row of $\mathbf{A}$. Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its jth column with the column $\mathbf{b}$ and $\mathbf{A}_{i.}(\mathbf{b})$ denote the matrix obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$. Denote by $\mathbf{a}^*_{.j}$ and $\mathbf{a}^*_{i.}$ the jth column and the ith row of $\mathbf{A}^*$, respectively.

Theorem 4 (see [34]). *If* $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, *then the Moore-Penrose inverse* $\mathbf{A}^{\dagger} = (a^{\dagger}_{ij}) \in \mathbb{H}^{n\times m}$ *have the following determinantal representations,*

$$a^{\dagger}_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \operatorname{c\,det}_i \left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}^*_{.j}\right)\right)^{\beta}_{\beta}}{\sum_{\beta\in J_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|^{\beta}_{\beta}}, \tag{9}$$

$$a^{\dagger}_{ij} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \operatorname{r\,det}_j \left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{a}^*_{i.}\right)\right)^{\alpha}_{\alpha}}{\sum_{\alpha\in I_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|^{\alpha}_{\alpha}}. \tag{10}$$

Remark 5. For an arbitrary full-rank matrix $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, a column vector $\mathbf{b} \in \mathbb{H}^{n\times 1}$ and a row vector $\mathbf{c} \in \mathbb{H}^{1\times m}$ we put

$$\begin{aligned} \operatorname{c\,det}_i \left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}(\mathbf{b})\right) &= \sum_{\beta\in J_{n,n}\{i\}} \operatorname{c\,det}_i \left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}(\mathbf{b})\right)^{\beta}_{\beta}, \\ \det\left(\mathbf{A}^*\mathbf{A}\right) &= \sum_{\beta\in J_{n,n}} \left|\mathbf{A}^*\mathbf{A}\right|^{\beta}_{\beta} \quad \text{when } r = n, \\ \operatorname{r\,det}_j \left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\mathbf{c})\right) &= \sum_{\alpha\in I_{m,m}\{j\}} \operatorname{r\,det}_j \left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}(\mathbf{c})\right)^{\alpha}_{\alpha}, \\ \det\left(\mathbf{A}\mathbf{A}^*\right) &= \sum_{\alpha\in I_{m,m}} \left|\mathbf{A}\mathbf{A}^*\right|^{\alpha}_{\alpha} \quad \text{when } r = m. \end{aligned} \tag{11}$$

Remark 6. First note that $(\mathbf{A}^*)^\dagger = (\mathbf{A}^\dagger)^*$. Because of symbol equivalence, we shall use the denotation $\mathbf{A}^{\dagger,*} := (\mathbf{A}^*)^\dagger$ as well. So by Lemma 3, for the Hermitian adjoint matrix $\mathbf{A}^* \in \mathbb{H}_r^{n\times m}$ determinantal representations of its Moore-Penrose inverse $(\mathbf{A}^*)^\dagger = ((a_{ij}^*)^\dagger) \in \mathbb{H}^{m\times n}$ are

$$\left(a_{ij}^*\right)^\dagger = \overline{\left(a_{ji}\right)^\dagger} = \frac{\sum_{\alpha\in I_{r,n}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{a}_{i.}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\alpha^\alpha}, \tag{12}$$

$$\left(a_{ij}^*\right)^\dagger = \frac{\sum_{\beta\in J_{r,m}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{.i}\left(\mathbf{a}_{.j}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|_\beta^\beta}. \tag{13}$$

Since the projection matrices $\mathbf{A}^\dagger\mathbf{A} =: \mathbf{P}_A = (p_{ij})$ and $\mathbf{A}\mathbf{A}^\dagger =: \mathbf{Q}_A = (q_{ij})$ are Hermitian, then $p_{ij} = \overline{p_{ji}}$ and $q_{ij} = \overline{q_{ji}}$ for all $i \neq j$. So due to Theorem 4 and Remark 6 we have evidently the following corollaries.

Corollary 7. *If $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, then the projection matrix $\mathbf{P}_A = (p_{ij})_{n\times n}$ has the determinantal representations*

$$p_{ij} = \frac{\sum_{\beta\in J_{r,n}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\dot{\mathbf{a}}_{.j}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta} = \frac{\sum_{\alpha\in I_{r,n}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\dot{\mathbf{a}}_{i.}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\alpha^\alpha}, \tag{14}$$

where $\dot{\mathbf{a}}_{.j}$ and $\dot{\mathbf{a}}_{i.}$ are the jth column and ith row of $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$, respectively.

Corollary 8. *If $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, then the projection matrix $\mathbf{A}\mathbf{A}^\dagger =: \mathbf{Q}_A = (q_{ij})_{m\times m}$ has the determinantal representation*

$$q_{ij} = \frac{\sum_{\alpha\in I_{r,m}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\ddot{\mathbf{a}}_{i.}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|_\alpha^\alpha} = \frac{\sum_{\beta\in J_{r,m}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{.i}\left(\ddot{\mathbf{a}}_{.j}\right)\right)_\beta^\beta}{\sum_{\alpha\in J_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|_\beta^\beta}, \tag{15}$$

where $\ddot{\mathbf{a}}_{i.}$ and $\ddot{\mathbf{a}}_{.j}$ are the ith row and the jth column of $\mathbf{A}\mathbf{A}^* \in \mathbb{H}^{m\times m}$.

Determinantal representations of orthogonal projectors $\mathbf{L}_A := \mathbf{I} - \mathbf{A}^\dagger\mathbf{A}$ and $\mathbf{R}_A := \mathbf{I} - \mathbf{A}\mathbf{A}^\dagger$ induced from $\mathbf{A}$ can be derived similarly.

Theorem 9 (see [3]). *Let $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{B} \in \mathbb{H}^{r\times s}$, $\mathbf{C} \in \mathbb{H}^{m\times s}$ be known and $\mathbf{X} \in \mathbb{H}^{n\times r}$ be unknown. Then the matrix equation*

$$\mathbf{A}\mathbf{X}\mathbf{B} = \mathbf{C} \tag{16}$$

is consistent if and only if $\mathbf{A}\mathbf{A}^\dagger\mathbf{C}\mathbf{B}^\dagger\mathbf{B} = \mathbf{C}$. In this case, its general solution can be expressed as

$$\mathbf{X} = \mathbf{A}^\dagger\mathbf{C}\mathbf{B}^\dagger + \mathbf{L}_A\mathbf{V} + \mathbf{W}\mathbf{R}_B, \tag{17}$$

where $\mathbf{V}, \mathbf{W}$ are arbitrary matrices over $\mathbb{H}$ with allowable dimensions.

Theorem 10 (see [35]). *Let $\mathbf{A} \in \mathbb{H}_{r_1}^{m\times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{r\times s}$. Then the partial solution $\mathbf{X}^0 = \mathbf{A}^\dagger\mathbf{C}\mathbf{B}^\dagger = (x_{ij}^0) \in \mathbb{H}^{n\times r}$ to (16) has determinantal representations,*

$$x_{ij}^0 = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{d}_{.j}^B\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} \left|\mathbf{B}\mathbf{B}^*\right|_\alpha^\alpha}, \tag{18}$$

or

$$x_{ij}^0 = \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{B}\mathbf{B}^*\right)_{j.}\left(\mathbf{d}_{i.}^A\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} \left|\mathbf{B}\mathbf{B}^*\right|_\alpha^\alpha}, \tag{19}$$

where

$$\mathbf{d}_{.j}^B = \left[\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{B}\mathbf{B}^*\right)_{j.}\left(\tilde{\mathbf{c}}_{k.}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad k = 1,\ldots,n,$$
$$\mathbf{d}_{i.}^A = \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\tilde{\mathbf{c}}_{.l}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times r}, \quad l = 1,\ldots,r, \tag{20}$$

are the column vector and the row vector, respectively. $\tilde{\mathbf{c}}_{i.}$ and $\tilde{\mathbf{c}}_{.j}$ are the ith row and the jth column of $\widetilde{\mathbf{C}} = \mathbf{A}^*\mathbf{C}\mathbf{B}^*$.

Corollary 11. *Let $\mathbf{A} \in \mathbb{H}_k^{m\times n}$, $\mathbf{C} \in \mathbb{H}^{m\times s}$ be known and $\mathbf{X} \in \mathbb{H}^{n\times s}$ be unknown. Then the matrix equation $\mathbf{A}\mathbf{X} = \mathbf{C}$ is consistent if and only if $\mathbf{A}\mathbf{A}^\dagger\mathbf{C} = \mathbf{C}$. In this case, its general solution can be expressed as $\mathbf{X} = \mathbf{A}^\dagger\mathbf{C} + \mathbf{L}_A\mathbf{V}$, where $\mathbf{V}$ is an arbitrary matrix over $\mathbb{H}$ with an allowable dimension. The partial solution $\mathbf{X}^0 = \mathbf{A}^\dagger\mathbf{C}$ has the following determinantal representation,*

$$x_{ij}^0 = \frac{\sum_{\beta\in J_{k,n}\{i\}} \operatorname{c}\det_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\hat{\mathbf{c}}_{.j}\right)\right)_\beta^\beta}{\sum_{\beta\in J_{k,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta}. \tag{21}$$

where $\hat{\mathbf{c}}_{.j}$ is the jth column of $\widehat{\mathbf{C}} = \mathbf{A}^*\mathbf{C}$.

Corollary 12. *Let $\mathbf{B} \in \mathbb{H}_k^{r\times s}$, $\mathbf{C} \in \mathbb{H}^{n\times s}$ be given and $\mathbf{X} \in \mathbb{H}^{n\times r}$ be unknown. Then the equation $\mathbf{X}\mathbf{B} = \mathbf{C}$ is solvable if and only if $\mathbf{C} = \mathbf{C}\mathbf{B}^\dagger\mathbf{B}$ and its general solution is $\mathbf{X} = \mathbf{C}\mathbf{B}^\dagger + \mathbf{W}\mathbf{R}_B$, where $\mathbf{W}$ is any matrix with an allowable dimension. Moreover, its partial solution $\mathbf{X} = \mathbf{C}\mathbf{B}^\dagger$ has the determinantal representation,*

$$x_{ij} = \frac{\sum_{\alpha\in I_{k,r}\{j\}} \operatorname{r}\det_j\left(\left(\mathbf{B}\mathbf{B}^*\right)_{j.}\left(\hat{\mathbf{c}}_{i.}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{k,r}} \left|\mathbf{B}\mathbf{B}^*\right|_\alpha^\alpha}. \tag{22}$$

where $\hat{\mathbf{c}}_{i.}$ is the ith row of $\widehat{\mathbf{C}} = \mathbf{C}\mathbf{B}^*$.

2.3. Determinantal Representations of the General Solution to the Sylvester Matrix Equation (3)

Lemma 13 (see [3]). *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{B} \in \mathbb{H}^{r\times s}$, $\mathbf{C} \in \mathbb{H}^{m\times p}$, $\mathbf{D} \in \mathbb{H}^{q\times s}$, $\mathbf{E} \in \mathbb{H}^{m\times s}$. *Put* $\mathbf{M} = \mathbf{R}_A\mathbf{C}$, $\mathbf{N} = \mathbf{D}\mathbf{L}_B$, $\mathbf{S} = \mathbf{C}\mathbf{L}_M$. *Then the following results are equivalent.*

(i) *Eq. (3) has a pair solution* $(\mathbf{X}, \mathbf{Y})$, *where* $\mathbf{X} \in \mathbb{H}^{n\times r}$, $\mathbf{Y} \in \mathbb{H}^{p\times q}$.

(ii) $\mathbf{R}_M\mathbf{R}_A\mathbf{E} = \mathbf{0}$, $\mathbf{R}_A\mathbf{E}\mathbf{L}_D = \mathbf{0}$, $\mathbf{E}\mathbf{L}_D\mathbf{L}_N = \mathbf{0}$, $\mathbf{R}_C\mathbf{E}\mathbf{L}_B = \mathbf{0}$.

(iii) $\mathbf{Q}_M\mathbf{R}_A\mathbf{E}\mathbf{P}_D = \mathbf{R}_A\mathbf{E}$, $\mathbf{Q}_C\mathbf{E}\mathbf{L}_B\mathbf{P}_N = \mathbf{E}\mathbf{L}_B$.

(iv) $\operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{C} & \mathbf{E}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{C}\end{bmatrix}$, $\operatorname{rank}\begin{bmatrix}\mathbf{B}^* & \mathbf{D}^* & \mathbf{E}^*\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{B}^* & \mathbf{D}^*\end{bmatrix}$, $\operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{E}\\ \mathbf{0} & \mathbf{D}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix}$, $\operatorname{rank}\begin{bmatrix}\mathbf{C} & \mathbf{E}\\ \mathbf{0} & \mathbf{B}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}$.

In that case, the general solution to (3) can be expressed as

$$\mathbf{X} = \mathbf{A}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{C}\mathbf{M}^\dagger\mathbf{R}_A\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{L}_B\mathbf{N}^\dagger\mathbf{D}\mathbf{B}^\dagger - -\mathbf{A}^\dagger\mathbf{S}\mathbf{V}\mathbf{R}_N\mathbf{D}\mathbf{B}^\dagger + \mathbf{L}_A\mathbf{U} + \mathbf{Z}\mathbf{R}_B, \quad (23)$$

$$\mathbf{Y} = \mathbf{M}^\dagger\mathbf{R}_A\mathbf{E}\mathbf{D}^\dagger + \mathbf{L}_M\mathbf{S}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{L}_B\mathbf{N}^\dagger + \mathbf{L}_M\left(\mathbf{V} - \mathbf{S}^\dagger\mathbf{S}\mathbf{V}\mathbf{N}\mathbf{N}^\dagger\right) + \mathbf{W}\mathbf{R}_D, \quad (24)$$

where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Z}$, *and* $\mathbf{W}$ *are arbitrary matrices over* $\mathbb{H}$ *obeying agreeable dimensions.*

Some simplifications of (23) and (24) can be derived due to the quaternionic analog of the following proposition.

Lemma 14 (see [57]). *If* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is Hermitian and idempotent, then for any matrix* $\mathbf{B} \in \mathbb{H}^{m\times n}$ *the following equations hold*

$$\mathbf{A}(\mathbf{B}\mathbf{A})^\dagger = (\mathbf{B}\mathbf{A})^\dagger, \qquad (\mathbf{A}\mathbf{B})^\dagger\mathbf{A} = (\mathbf{A}\mathbf{B})^\dagger. \quad (25)$$

Since $\mathbf{R}_A$, $\mathbf{L}_B$, and $\mathbf{L}_M$ are projectors, then by Lemma 14 the simplifications of (23) and (24) are as follows:

$$\begin{aligned}\mathbf{X} &= \mathbf{A}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{C}\mathbf{M}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{N}^\dagger\mathbf{D}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{V}\mathbf{R}_N\mathbf{D}\mathbf{B}^\dagger + +\mathbf{L}_A\mathbf{U} + \mathbf{Z}\mathbf{R}_B,\\ \mathbf{Y} &= \mathbf{M}^\dagger\mathbf{E}\mathbf{D}^\dagger + \mathbf{P}_S\mathbf{C}^\dagger\mathbf{E}\mathbf{N}^\dagger + \mathbf{L}_M\left(\mathbf{V} - \mathbf{P}_S\mathbf{V}\mathbf{Q}_N\right) + \mathbf{W}\mathbf{R}_D.\end{aligned} \quad (26)$$

By putting $\mathbf{U}, \mathbf{V}, \mathbf{Z}$, and $\mathbf{W}$ as zero-matrices, we obtain the partial solution to (3),

$$\mathbf{X} = \mathbf{A}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{C}\mathbf{M}^\dagger\mathbf{E}\mathbf{B}^\dagger - \mathbf{A}^\dagger\mathbf{S}\mathbf{C}^\dagger\mathbf{E}\mathbf{N}^\dagger\mathbf{D}\mathbf{B}^\dagger, \quad (27)$$

$$\mathbf{Y} = \mathbf{M}^\dagger\mathbf{E}\mathbf{D}^\dagger + \mathbf{P}_S\mathbf{C}^\dagger\mathbf{E}\mathbf{N}^\dagger. \quad (28)$$

The following theorem gives determinantal representations of (27)-(28).

Theorem 15 (see [45]). *Let* $\mathbf{A} \in \mathbb{H}_{r_1}^{m\times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{r\times s}$, $\mathbf{C} \in \mathbb{H}_{r_3}^{m\times p}$, $\mathbf{D} \in \mathbb{H}_{r_4}^{q\times s}$, $\operatorname{rank}\mathbf{M} = r_5$, $\operatorname{rank}\mathbf{N} = r_6$, $\operatorname{rank}\mathbf{S} = r_7$. *Then the pair solution (27)-(28),* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times r}$, $\mathbf{Y} = (x_{gf}) \in \mathbb{H}^{p\times q}$, *to (3) by the components*

$$x_{ij} = x_{ij}^{(1)} - x_{ij}^{(2)} - x_{ij}^{(3)}, \qquad y_{gf} = y_{gf}^{(1)} + y_{gf}^{(2)}, \quad (29)$$

has the determinantal representation, as follows.

(i)

$$x_{ij}^{(1)} = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{d}_{.j}^B\right)\right)_\beta^\beta}{\sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} |\mathbf{B}\mathbf{B}^*|_\alpha^\alpha}, \quad (30)$$

or

$$x_{ij}^{(1)} = \frac{\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{r\,det}_j\left((\mathbf{B}\mathbf{B}^*)_{j.}\left(\mathbf{d}_{i.}^A\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\alpha\in I_{r_2,r}} |(\mathbf{B}\mathbf{B}^*|_\alpha^\alpha}, \quad (31)$$

where

$$\begin{aligned}\mathbf{d}_{.j}^B &= \left[\sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{r\,det}_j\left((\mathbf{B}\mathbf{B}^*)_{j.}\left(\mathbf{e}_{k.}^{(1)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad k = 1,\dots,n,\\ \mathbf{d}_{i.}^A &= \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{e}_{.l}^{(1)}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times r}, \quad l = 1,\dots,r,\end{aligned} \quad (32)$$

are the column vector and the row vector, respectively. $\mathbf{e}_{k.}^{(1)}$ *and* $\mathbf{e}_{.l}^{(1)}$ *are the kth row and the lth column of* $\mathbf{E}_1 := \mathbf{A}^*\mathbf{E}\mathbf{B}^*$.

(ii)

$$x_{ij}^{(2)} = \frac{\sum_{t=1}^{q} \varphi_{iq} \sum_{\alpha\in I_{r_2,r}\{j\}} \operatorname{r\,det}_j\left((\mathbf{B}\mathbf{B}^*)_{j.}\left(\mathbf{e}_{q.}^{(2)}\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta \sum_{\beta\in I_{r_5,m}} |\mathbf{M}\mathbf{M}^*|_\alpha^\alpha \sum_{\alpha\in I_{r_2,r}} |\mathbf{B}\mathbf{B}^*|_\alpha^\alpha}, \quad (33)$$

where $\mathbf{e}_{q.}^{(2)}$ *is qth row of* $\mathbf{E}_2 := \mathbf{E}\mathbf{B}^*$.

$$\varphi_{iq} = \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\psi_{.q}^M\right)\right)_\beta^\beta = \sum_{\alpha\in I_{r_5,m}\{q\}} \operatorname{r\,det}_q\left((\mathbf{M}\mathbf{M}^*)_{q.}\left(\psi_{i.}^A\right)\right)_\alpha^\alpha,$$

$$\psi_{.q}^M = \left[\sum_{\alpha\in I_{r_5,m}\{q\}} \operatorname{r\,det}_q\left((\mathbf{M}\mathbf{M}^*)_{q.}\left(\mathbf{c}_{f.}^{(1)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad f = 1,\dots,n,$$

$$\psi_{i.}^{A} = \left[\sum_{\beta \in J_{r_1,n}\{i\}} \mathrm{c}\det_i \left((\mathbf{A}^*\mathbf{A})_{.i} \left(\mathbf{c}_{.s}^{(1)} \right) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1\times m}, \quad s = 1, \ldots, m, \tag{34}$$

are the column vector and the row vector, respectively. $\mathbf{c}_{f.}^{(1)}$ and $\mathbf{c}_{.s}^{(1)}$ are the fth row and the sth column of $\mathbf{C}_1 := \mathbf{A}^*\mathbf{C}\mathbf{M}^*$.

(iii)

$$x_{ij}^{(3)} == \frac{\sum_{t=1}^{p}\sum_{f=1}^{q}\sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{c}\det_i \left((\mathbf{A}^*\mathbf{A})_{.i} \left(\mathbf{s}_{.t}^{(1)} \right) \right)_{\beta}^{\beta} \sum_{\beta\in J_{r_3,p}\{t\}} \mathrm{c}\det_t \left((\mathbf{C}^*\mathbf{C})_{.t} \left(\mathbf{e}_{.f}^{(3)} \right) \right)_{\beta}^{\beta} \eta_{fj}}{\sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta} \sum_{\beta\in J_{r_3,p}} |\mathbf{C}^*\mathbf{C}|_{\beta}^{\beta} \sum_{\alpha\in J_{r_6,s}} |\mathbf{N}^*\mathbf{N}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_2,r}} |\mathbf{B}\mathbf{B}^*|_{\alpha}^{\alpha}}, \tag{35}$$

where $\mathbf{s}_{.t}^{(1)}$ is the tth column of $\mathbf{S}_1 := \mathbf{A}^*\mathbf{S}$, $\mathbf{e}_{.f}^{(3)}$ is the fth column of $\mathbf{E}_3 := \mathbf{C}^*\mathbf{E}$, and

$$\eta_{fj} = \sum_{\alpha\in I_{r_2,r}\{t\}} \mathrm{r}\det_j \left((\mathbf{B}\mathbf{B}^*)_{j.} \left(\zeta_{f.}^{N} \right) \right)_{\alpha}^{\alpha} = \sum_{\beta\in J_{r_6,s}\{f\}} \mathrm{c}\det_f \left((\mathbf{N}^*\mathbf{N})_{.f} \left(\zeta_{.j}^{B} \right) \right)_{\beta}^{\beta},$$

$$\zeta_{f.}^{N} = \left[\sum_{\beta\in J_{r_6,s}\{f\}} \mathrm{c}\det_f \left((\mathbf{N}^*\mathbf{N})_{.f} \left(\mathbf{d}_{.k}^{(1)} \right) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1\times r}, \quad k = 1, \ldots, r,$$

$$\zeta_{.j}^{B} = \left[\sum_{\alpha\in I_{r_2,r}\{j\}} \mathrm{r}\det_j \left((\mathbf{B}\mathbf{B}^*)_{j.} \left(\mathbf{d}_{l.}^{(1)} \right) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{s\times 1}, \quad l = 1, \ldots, s, \tag{36}$$

are the row vector and the column vector, respectively. $\mathbf{d}_{.k}^{(1)}$ and $\mathbf{d}_{l.}^{(1)}$ are the kth column and the lth row of $\mathbf{D}_1 = \mathbf{N}^*\mathbf{D}\mathbf{B}^*$.

(iv)

$$y_{gf}^{(1)} = \frac{\sum_{\beta\in J_{r_5,p}\{g\}} \mathrm{c}\det_g \left((\mathbf{M}^*\mathbf{M})_{.g} \left(\mathbf{d}_{.f}^{D} \right) \right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_5,p}} |\mathbf{M}^*\mathbf{M}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_4,q}} |\mathbf{D}\mathbf{D}^*|_{\alpha}^{\alpha}}, \tag{37}$$

or

$$y_{gf}^{(1)} = \frac{\sum_{\alpha\in I_{r_4,q}\{f\}} \mathrm{r}\det_f \left((\mathbf{D}\mathbf{D}^*)_{f.} \left(\mathbf{d}_{g.}^{M} \right) \right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_5,p}} |\mathbf{M}^*\mathbf{M}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_4,q}} |\mathbf{D}\mathbf{D}^*|_{\alpha}^{\alpha}}, \tag{38}$$

where

$$\mathbf{d}_{.f}^{D} = \left[\sum_{\alpha\in I_{r_4,q}\{f\}} \mathrm{r}\det_f \left((\mathbf{D}\mathbf{D}^*)_{f.} \left(\mathbf{e}_{k.}^{(4)} \right) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{p\times 1}, \quad k = 1, \ldots, p,$$

$$\mathbf{d}_{g.}^{M} = \left[\sum_{\beta\in J_{r_5,p}\{g\}} \mathrm{c}\det_g \left((\mathbf{M}^*\mathbf{M})_{.g} \left(\mathbf{e}_{.l}^{(4)} \right) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1\times q}, \quad l = 1, \ldots, q, \tag{39}$$

are the column vector and the row vector, respectively. $\mathbf{e}_{k.}^{(4)}$ and $\mathbf{e}_{.l}^{(4)}$ are the kth row and the lth column of $\mathbf{E}_4 := \mathbf{M}^*\mathbf{E}\mathbf{D}^*$.

(v)

$$y_{gf}^{(2)} = \frac{\sum_{t=1}^{p}\sum_{\beta\in J_{r_7,p}\{g\}} \mathrm{c}\det_g \left((\mathbf{S}^*\mathbf{S})_{.g} \left(\ddot{\mathbf{s}}_{.t} \right) \right)_{\beta}^{\beta} \xi_{tf}}{\sum_{\beta\in J_{r_7,p}} |\mathbf{S}^*\mathbf{S}|_{\beta}^{\beta} \sum_{\beta\in J_{r_3,p}} |\mathbf{C}^*\mathbf{C}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_6,q}} |\mathbf{N}\mathbf{N}^*|_{\alpha}^{\alpha}}, \tag{40}$$

where

$$\xi_{tf} = \sum_{\alpha\in I_{r_6,q}\{t\}} \mathrm{r}\det_f \left((\mathbf{N}\mathbf{N}^*)_{f.} \left(\phi_{t.}^{C} \right) \right)_{\alpha}^{\alpha} = \sum_{\beta\in J_{r_3,p}\{t\}} \mathrm{c}\det_t \left((\mathbf{C}^*\mathbf{C})_{.t} \left(\phi_{.f}^{N} \right) \right)_{\beta}^{\beta},$$

$$\phi_{t.}^{C} = \left[\sum_{\beta\in J_{r_3,p}\{t\}} \mathrm{c}\det_t \left((\mathbf{C}^*\mathbf{C})_{.t} \left(\mathbf{e}_{.k}^{(5)} \right) \right)_{\beta}^{\beta} \right] \in \mathbb{H}^{1\times q}, \quad k = 1, \ldots, q, \tag{41}$$

$$\phi_{.f}^{N} = \left[\sum_{\alpha\in I_{r_6,q}\{j\}} \mathrm{r}\det_f \left((\mathbf{N}\mathbf{N}^*)_{f.} \left(\mathbf{e}_{l.}^{(5)} \right) \right)_{\alpha}^{\alpha} \right] \in \mathbb{H}^{s\times 1}, \quad l = 1, \ldots, p,$$

are the row vector and the column vector, respectively. $\mathbf{e}_{.k}^{(5)}$ and $\mathbf{e}_{l.}^{(5)}$ are the kth column and the lth row of $\mathbf{E}_5 = \mathbf{C}^*\mathbf{E}\mathbf{N}^*$.

3. Determinantal Representations of the General and (Skew-)Hermitian Solutions to (4)

Now consider (4). Since for an arbitrary matrix $\mathbf{A}$ it is evident that $\mathbf{P}_{A^*} = (\mathbf{A}^*)^{\dagger}\mathbf{A}^* = (\mathbf{A}\mathbf{A}^{\dagger})^* = \mathbf{Q}_A$, so $\mathbf{Q}_{A^*} = \mathbf{P}_A$, $\mathbf{L}_{A^*} = \mathbf{I} - \mathbf{P}_{A^*} = \mathbf{I} - \mathbf{Q}_A = \mathbf{R}_A$, and $\mathbf{R}_{A^*} = \mathbf{L}_A$. Due to the above, $\mathbf{M} = \mathbf{R}_A\mathbf{B}$ and $\mathbf{N} = \mathbf{B}^*\mathbf{L}_{A^*} = \mathbf{B}^*\mathbf{R}_A = (\mathbf{R}_A\mathbf{B})^* = \mathbf{M}^*$, and we obtain the following analog of Lemma 13.

Lemma 16. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{B} \in \mathbb{H}^{m\times k}$, $\mathbf{C} \in \mathbb{H}^{m\times m}$. *Put* $\mathbf{M} = \mathbf{R}_A\mathbf{B}$, $\mathbf{S} = \mathbf{B}\mathbf{L}_M$. *Then the following results are equivalent.*

(i) *Equation (4) has a pair solution* $(\mathbf{X}, \mathbf{Y})$, *where* $\mathbf{X} \in \mathbb{H}^{n\times n}$, $\mathbf{Y} \in \mathbb{H}^{k\times k}$.

(ii) $\mathbf{R}_M\mathbf{R}_A\mathbf{C} = \mathbf{0}$, $\mathbf{R}_A\mathbf{C}\mathbf{R}_B = \mathbf{0}$, $\mathbf{C}\mathbf{R}_B\mathbf{R}_M = \mathbf{0}$, $\mathbf{R}_B\mathbf{C}\mathbf{R}_A = \mathbf{0}$.

(iii) $\mathbf{Q}_M\mathbf{R}_A\mathbf{C}\mathbf{Q}_B = \mathbf{R}_A\mathbf{C}$, $\mathbf{Q}_B\mathbf{C}\mathbf{R}_A\mathbf{Q}_M = \mathbf{C}\mathbf{R}_A$.

(iv) $\operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathbf{C}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{B}\end{bmatrix}$, $\operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{0} & \mathbf{B}^*\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}^*\end{bmatrix}$, $\operatorname{rank}\begin{bmatrix}\mathbf{B} & \mathbf{C}\\ \mathbf{0} & \mathbf{A}^*\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbf{B} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^*\end{bmatrix}$.

In that case, the general solution to (4) can be expressed as follows:

$$\begin{aligned}\mathbf{X} &= \mathbf{A}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger}\\ &\quad - \mathbf{A}^\dagger\mathbf{S}\mathbf{B}^{*,\dagger}\mathbf{C}\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{S}\mathbf{V}\mathbf{L}_M\mathbf{B}^*\mathbf{A}^{*,\dagger}\\ &\quad + \mathbf{L}_A\mathbf{U} + \mathbf{Z}\mathbf{L}_A,\\ \mathbf{Y} &= \mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger} + \mathbf{P}_S\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger} + \mathbf{L}_M\left(\mathbf{V} - \mathbf{P}_S\mathbf{V}\mathbf{P}_M\right)\\ &\quad + \mathbf{W}\mathbf{L}_B.\end{aligned} \tag{42}$$

where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Z}$, *and* $\mathbf{W}$ *are arbitrary matrices over* $\mathbb{H}$ *with allowable dimensions.*

By putting $\mathbf{U}$, $\mathbf{V}$, $\mathbf{Z}$, and $\mathbf{W}$ as zero-matrices with compatible dimensions, we obtain the following partial solution to (4),

$$\begin{aligned}\mathbf{X} &= \mathbf{A}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger}\\ &\quad - \mathbf{A}^\dagger\mathbf{S}\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger},\end{aligned} \tag{43}$$

$$\mathbf{Y} = \mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger} + \mathbf{P}_S\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}. \tag{44}$$

The following theorem gives determinantal representations of (43)-(44).

Theorem 17. *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{B} \in \mathbb{H}^{m\times k}_{r_2}$, $\operatorname{rank}\mathbf{M} = r_3$, $\operatorname{rank}\mathbf{S} = r_4$. *Then the partial pair solution (43)-(44) to (4),* $\mathbf{X} = (x_{ij}) \in \mathbb{H}^{n\times n}$, $\mathbf{Y} = (y_{pg}) \in \mathbb{H}^{k\times k}$, *by the components*

$$\begin{aligned}x_{ij} &= x_{ij}^{(1)} - x_{ij}^{(2)} - x_{ij}^{(3)},\\ y_{pg} &= y_{pg}^{(1)} + y_{pg}^{(2)},\end{aligned} \tag{45}$$

possesses the following determinantal representations:

(i)

$$x_{ij}^{(1)} = \frac{\sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{v}_{i.}\right)\right)_\alpha^\alpha}{\left(\sum_{\alpha\in I_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\alpha^\alpha\right)^2} \tag{46}$$

or

$$x_{ij}^{(1)} = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{v}_{.j}\right)\right)_\beta^\beta}{\left(\sum_{\beta\in J_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta\right)^2}, \tag{47}$$

where

$$\mathbf{v}_{i.} = \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{c}_{.s}^{(1)}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times n}, \quad s = 1, \dots, n, \tag{48}$$

$$\mathbf{v}_{.j} = \left[\sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{c}_{f.}^{(1)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad f = 1, \dots, n \tag{49}$$

are the row vector and the column vector, respectively; $\mathbf{c}_{.s}^{(1)}$ *and* $\mathbf{c}_{f.}^{(1)}$ *are the sth column and the fth row of* $\mathbf{C}_1 = \mathbf{A}^*\mathbf{C}\mathbf{A}$.

(ii)

$$x_{ij}^{(2)} = \frac{\sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\widetilde{\phi}_{i.}\right)\right)_\alpha^\alpha}{\left(\sum_{\beta\in J_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta\right)^2 \sum_{\alpha\in I_{r_3,m}} \left|\mathbf{M}\mathbf{M}^*\right|_\alpha^\alpha}, \tag{50}$$

where $\widetilde{\phi}_{i.}$ *is the ith row of* $\widetilde{\mathbf{\Phi}} := \mathbf{\Phi}\mathbf{C}\mathbf{A}$ *and* $\mathbf{\Phi} = (\phi_{iq}) \in \mathbb{H}^{n\times m}$ *is such that*

$$\begin{aligned}\phi_{iq} &= \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\eta_{.q}^M\right)\right)_\beta^\beta\\ &= \sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{rdet}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\eta_{i.}^A\right)\right)_\alpha^\alpha,\end{aligned} \tag{51}$$

$$\begin{aligned}\eta_{.q}^M &= \left[\sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{rdet}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\mathbf{b}_{f.}^{(1)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad f = 1, \dots, n,\\ \eta_{i.}^A &= \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{b}_{.s}^{(1)}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times m}, \quad s = 1, \dots, m,\end{aligned} \tag{52}$$

are the column vector and the row vector, respectively. $\mathbf{b}_{f.}^{(1)}$ *and* $\mathbf{b}_{.s}^{(1)}$ *are the fth row and the sth column of* $\mathbf{B}_1 = \mathbf{A}^*\mathbf{B}\mathbf{M}^*$ *and* $\mathbf{c}_{q.}^{(2)}$ *is the qth row of* $\mathbf{C}_2 = \mathbf{C}\mathbf{A}$.

(iii)

$$x_{ij}^{(3)} = \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\widetilde{\upsilon}_{.j}\right)\right)_\beta^\beta}{\left(\sum_{\beta\in J_{r_1,n}} \left|\mathbf{A}^*\mathbf{A}\right|_\beta^\beta\right)^2 \sum_{\beta\in J_{r_2,k}} \left|\mathbf{B}^*\mathbf{B}\right|_\beta^\beta \sum_{\beta\in J_{r_3,m}} \left|\mathbf{M}\mathbf{M}^*\right|_\beta^\beta}, \tag{53}$$

where $\widetilde{\upsilon}_{.j}$ *is the jth column of* $\widetilde{\mathbf{\Upsilon}} = \mathbf{A}^*\mathbf{S}\mathbf{\Upsilon}$, *the matrix* $\mathbf{\Upsilon} = (\upsilon_{pj}) \in \mathbb{H}^{k\times n}$ *such that*

$$\upsilon_{pj} = \sum_{\beta\in J_{r_2,k}\{p\}} \operatorname{cdet}_p\left(\left(\mathbf{B}^*\mathbf{B}\right)_{.p}\left(\widetilde{\mathbf{c}}_{.j}\right)\right)_\beta^\beta, \tag{54}$$

where $\widetilde{\mathbf{c}}_{.j}$ *is the jth column of* $\widetilde{\mathbf{C}} = \mathbf{B}^*\mathbf{C}\Phi^*$ *and* Φ^* *is Hermitian adjoint to* $\Phi = (\phi_{iq})$ *from (51).*

(iv)

$$y_{pg}^{(1)} = \frac{\sum_{\beta\in J_{r_3,k}\{p\}} \operatorname{c}\det_p\left((\mathbf{M}^*\mathbf{M})_{.p}\left(\mathbf{d}_{.g}^{B}\right)\right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_3,k}} |\mathbf{M}^*\mathbf{M}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_2,k}} |\mathbf{B}^*\mathbf{B}|_{\alpha}^{\alpha}}, \quad (55)$$

or

$$y_{pg}^{(1)} = \frac{\sum_{\alpha\in I_{r_2,k}\{g\}} \operatorname{r}\det_g\left((\mathbf{B}^*\mathbf{B})_{g.}\left(\mathbf{d}_{p.}^{M}\right)\right)_{\alpha}^{\alpha}}{\sum_{\beta\in J_{r_3,k}} |\mathbf{M}^*\mathbf{M}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_2,k}} |\mathbf{B}^*\mathbf{B}|_{\alpha}^{\alpha}}, \quad (56)$$

where

$$\mathbf{d}_{.g}^{B} = \left[\sum_{\alpha\in I_{r_2,k}\{g\}} \operatorname{r}\det_g\left((\mathbf{B}^*\mathbf{B})_{g.}\left(\mathbf{c}_{q.}^{(4)}\right)\right)_{\alpha}^{\alpha}\right] \in \mathbb{H}^{k\times 1}, \quad q = 1,\dots,k,$$
$$\mathbf{d}_{p.}^{M} = \left[\sum_{\beta\in J_{r_3,k}\{p\}} \operatorname{c}\det_p\left((\mathbf{M}^*\mathbf{M})_{.p}\left(\mathbf{c}_{.l}^{(4)}\right)\right)_{\beta}^{\beta}\right] \in \mathbb{H}^{1\times k}, \quad l = 1,\dots,k, \quad (57)$$

are the column vector and the row vector, respectively. $\mathbf{c}_{q.}^{(4)}$ *and* $\mathbf{c}_{.l}^{(4)}$ *are the qth row and the lth column of* $\mathbf{C}_4 := \mathbf{M}^*\mathbf{C}\mathbf{B}$.

(v)

$$y_{pg}^{(2)} = \frac{\sum_{\beta\in J_{r_4,k}\{p\}} \operatorname{c}\det_p\left((\mathbf{S}^*\mathbf{S})_{.p}\left(\widetilde{\omega}_{.g}\right)\right)_{\beta}^{\beta}}{\sum_{\beta\in J_{r_4,k}} |\mathbf{S}^*\mathbf{S}|_{\beta}^{\beta} \sum_{\beta\in J_{r_2,k}} |\mathbf{B}^*\mathbf{B}|_{\beta}^{\beta} \sum_{\alpha\in I_{r_3,k}} |\mathbf{M}^*\mathbf{M}|_{\alpha}^{\alpha}}, \quad (58)$$

where $\widetilde{\mathbf{\Omega}} = \mathbf{S}^*\mathbf{S}\mathbf{\Omega}$ *and* $\mathbf{\Omega} = (\omega_{tg})$ *such that*

$$\omega_{tg} = \sum_{\beta\in J_{r_2,k}\{t\}} \operatorname{c}\det_t\left((\mathbf{B}^*\mathbf{B})_{.t}\left(\mathbf{d}_{.g}^{M}\right)\right)_{\beta}^{\beta} = \sum_{\alpha\in I_{r_3,k}\{g\}} \operatorname{r}\det_g\left((\mathbf{M}^*\mathbf{M})_{g.}\left(\mathbf{d}_{t.}^{B}\right)\right)_{\alpha}^{\alpha},$$
$$\mathbf{d}_{.g}^{M} = \left[\sum_{\alpha\in I_{r_3,k}\{g\}} \operatorname{r}\det_g\left((\mathbf{M}^*\mathbf{M})_{g.}\left(\mathbf{c}_{q.}^{(4,*)}\right)\right)_{\alpha}^{\alpha}\right] \in \mathbb{H}^{k\times 1}, \quad q = 1,\dots,k, \quad (59)$$
$$\mathbf{d}_{t.}^{B} = \left[\sum_{\beta\in J_{r_3,k}\{t\}} \operatorname{c}\det_t\left((\mathbf{B}^*\mathbf{B})_{.t}\left(\mathbf{c}_{.l}^{(4,*)}\right)\right)_{\beta}^{\beta}\right] \in \mathbb{H}^{1\times k}, \quad l = 1,\dots,k,$$

are the column vector and the row vector, respectively. $\mathbf{c}_{q.}^{(4,*)}$ *and* $\mathbf{c}_{.l}^{(4,*)}$ *are the qth row and the lth column of* $\mathbf{C}_4^* := \mathbf{M}^*\mathbf{C}\mathbf{B}$.

Proof. The proof evidently follows from the proof of Theorem 15 by substitution corresponding matrices. For a better understanding more complete proof will be made in some points, and a few comments will be done in others.

(i) For the first term of (43), $\mathbf{X}_1 = \mathbf{A}^{\dagger}\mathbf{C}(\mathbf{A}^*)^{\dagger} = (x_{ij}^{(1)})$, we have

$$x_{ij}^{(1)} = \sum_{l=1}^{m}\sum_{t=1}^{m} a_{il}^{\dagger} c_{lt} a_{tj}^{*,\dagger}. \quad (60)$$

By using determinantal representations (9) and (13) of the Moore-Penrose inverses $\mathbf{A}^{\dagger}$ and $(\mathbf{A}^*)^{\dagger}$, respectively, we obtain

$$x_{ij}^{(1)} = \frac{\sum_{l=1}^{m}\sum_{t=1}^{m}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{a}_{.l}^{*}\right)\right)_{\beta}^{\beta} c_{lt} \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r}\det_j\left((\mathbf{A}^*\mathbf{A})_{j.}\left(\mathbf{a}_{t.}\right)\right)_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_{\alpha}^{\alpha} \sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}}. \quad (61)$$

Suppose $\mathbf{e}_{l.}$ and $\mathbf{e}_{.l}$ are the unit row vector and the unit column vector, respectively, such that all their components are 0, except the lth components, which are 1. Denote $\mathbf{C}_1 := \mathbf{A}^*\mathbf{C}\mathbf{A}$. Since $\sum_{l=1}^{m}\sum_{t=1}^{m} a_{fl}^{*} c_{lt} a_{ts} = c_{fs}^{(1)}$, then

$$x_{ij}^{(1)} = \frac{\sum_{f=1}^{n}\sum_{s=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{e}_{.f}\right)\right)_{\beta}^{\beta} c_{fs}^{(1)} \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r}\det_j\left((\mathbf{A}^*\mathbf{A})_{j.}\left(\mathbf{e}_{s.}\right)\right)_{\alpha}^{\alpha}}{\sum_{\alpha\in I_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_{\alpha}^{\alpha} \sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}} \quad (62)$$

If we denote by

$$v_{is} := \sum_{f=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{e}_{.f}\right)\right)_{\beta}^{\beta} c_{fs}^{(1)} = \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c}\det_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{c}_{.s}^{(1)}\right)\right)_{\beta}^{\beta} \quad (63)$$

the sth component of a row vector $\mathbf{v}_{i.} = [v_{i1},\dots,v_{in}]$, then

$$\sum_{s=1}^{m} v_{is} \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r}\det_j\left((\mathbf{A}^*\mathbf{A})_{j.}\left(\mathbf{e}_{s.}\right)\right)_{\alpha}^{\alpha} = \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r}\det_j\left((\mathbf{A}^*\mathbf{A})_{j.}\left(\mathbf{v}_{i.}\right)\right)_{\alpha}^{\alpha}. \quad (64)$$

Further, it is evident that $\sum_{\beta\in J_{r_1,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta = \sum_{\alpha\in I_{r_1,n}}|\mathbf{A}^*\mathbf{A}|_\alpha^\alpha$, so the first term of (43) has the determinantal representation (46), where $\mathbf{v}_{i.}$ is (48).

If we denote by

$$\begin{aligned} v_{fj}^{(2)} &\coloneqq \sum_{s=1}^{n} c_{fs}^{(1)} \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{e}_{s.}\right)\right)_\alpha^\alpha \\ &== \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{c}_{f.}^{(1)}\right)\right)_\alpha^\alpha \end{aligned} \tag{65}$$

the fth component of a column vector $\mathbf{v}_{.j} = [v_{1j},\ldots,v_{nj}]$, then

$$\begin{aligned} &\sum_{f=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{e}_{.f}\right)\right)_\beta^\beta v_{fj} \\ &= \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{v}_{.j}\right)\right)_\beta^\beta. \end{aligned} \tag{66}$$

So another determinantal representation of the first term of (43) is (47), where $\mathbf{v}_{.j}$ is (49).

(ii) For the second term $\mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} \coloneqq \mathbf{X}_2 = (x_{ij}^{(2)})$ of (43), we have

$$x_{ij}^{(2)} = \sum_{l=1}^{m}\sum_{p=1}^{k}\sum_{q=1}^{m}\sum_{t=1}^{m} a_{il}^\dagger b_{lp} m_{pq}^\dagger c_{qt} a_{tj}^{*,\dagger}. \tag{67}$$

Using determinantal representations (9) for the Moore-Penrose inverse $\mathbf{A}^\dagger$, (10) for $\mathbf{M}^\dagger = (m_{pq}^\dagger)$, and (13) for $(\mathbf{A}^*)^\dagger$, respectively, we obtain

$$\begin{aligned} x_{ij}^{(2)} &= \sum_{l=1}^{m}\sum_{p=1}^{k}\sum_{q=1}^{m}\sum_{t=1}^{m} \frac{\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.l}^*\right)\right)_\beta^\beta b_{lp} \sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{r\,det}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\mathbf{m}_{p.}^*\right)\right)_\alpha^\alpha}{\sum_{\beta\in J_{r_1,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta \mid \sum_{\alpha\in I_{r_3,m}}|\mathbf{M}\mathbf{M}^*|_\alpha^\alpha} \\ &\quad\times\times \frac{c_{qt}\sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{a}_{t.}\right)\right)_\alpha^\alpha}{\sum_{\alpha\in I_{r_1,n}}|\mathbf{A}^*\mathbf{A}|_\alpha^\alpha}. \end{aligned} \tag{68}$$

Further, thinking as above in the point (i), we obtain

$$\begin{aligned} \phi_{iq} &\coloneqq \sum_{l=1}^{m}\sum_{p=1}^{k}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}_{.l}^*\right)\right)_\beta^\beta b_{lp} \\ &\cdot \sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{r\,det}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\mathbf{m}_{p.}^*\right)\right)_\alpha^\alpha \\ &= \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\eta_{.q}^M\right)\right)_\beta^\beta \\ &= \sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{r\,det}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\eta_{i.}^A\right)\right)_\alpha^\alpha, \end{aligned} \tag{69}$$

where

$$\eta_{.q}^M = \left[\sum_{\alpha\in I_{r_3,m}\{q\}} \operatorname{r\,det}_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{q.}\left(\mathbf{b}_{f.}^{(1)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{n\times 1}, \quad f = 1,\ldots,n,$$

$$\eta_{i.}^A = \left[\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{b}_{.s}^{(1)}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times m}, \quad s = 1,\ldots,m, \tag{70}$$

are the column vector and the row vector, respectively. $\mathbf{b}_{f.}^{(1)}$ and $\mathbf{b}_{.s}^{(1)}$ are the fth row and the sth column of $\mathbf{B}_1 = \mathbf{A}^*\mathbf{B}\mathbf{M}^*$. Construct the matrix $\mathbf{\Phi} = (\phi_{iq}) \in \mathbb{H}^{n\times m}$ such that ϕ_{iq} are obtained by (69), and denote $\widetilde{\mathbf{\Phi}} \coloneqq \mathbf{\Phi}\mathbf{C}\mathbf{A}$. Since

$$\begin{aligned} &\sum_{q=1}^{m}\sum_{t=1}^{m}\phi_{iq}c_{qt}\sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{a}_{t.}\right)\right)_\alpha^\alpha \\ &= \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\widetilde{\phi}_{i.}\right)\right)_\alpha^\alpha, \end{aligned} \tag{71}$$

where $\widetilde{\phi}_{i.}$ is the ith row of $\widetilde{\mathbf{\Phi}}$, then we have (50).

(iii) For the third term $\mathbf{A}^\dagger\mathbf{S}\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger} \coloneqq \mathbf{X}_3 = (x_{ij}^{(3)})$ of (43), we use the determinantal representation (9) to $\mathbf{A}^\dagger$ and $(\mathbf{B})^\dagger$. Then by Corollary 11 and taking into account the fact that $\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger} = (\mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger)^*$, we have

$$x_{ij}^{(3)} == \frac{\sum_{p=1}^{k}\sum_{t=1}^{m}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{s}_{.p}^{(1)}\right)\right)_\beta^\beta \sum_{\beta\in J_{r_2,k}\{p\}} \operatorname{c\,det}_p\left(\left(\mathbf{B}^*\mathbf{B}\right)_{.p}\left(\mathbf{c}_{.t}^{(3)}\right)\right)_\beta^\beta \phi_{tj}^*}{\left(\sum_{\beta\in J_{r_1,n}}|\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2 \sum_{\beta\in J_{r_2,k}}|\mathbf{B}^*\mathbf{B}|_\beta^\beta \sum_{\beta\in J_{r_3,m}}|\mathbf{M}\mathbf{M}^*|_\beta^\beta}, \tag{72}$$

where $\mathbf{s}_{.p}^{(1)}$ is the pth column of $\mathbf{S}_1 := \mathbf{A}^*\mathbf{S}$, $\mathbf{c}_{.t}^{(3)}$ is the tth column of $\mathbf{C}_3 := \mathbf{B}^*\mathbf{C}$, ϕ_{tj}^* is the tjth entry of Φ^* that is Hermitian adjoint to $\Phi = (\phi_{iq})$ from (51). Denote $\mathbf{C}_3\Phi^* = \mathbf{B}^*\mathbf{C}\Phi^* = \widetilde{\mathbf{C}}$. Then,

$$\begin{aligned}\sum_{t=1}^{m}\sum_{\beta\in J_{r_2,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{B}^*\mathbf{B})_{.p}\left(\mathbf{c}_{.t}^{(3)}\right)\right)_\beta^\beta \phi_{tj}^* \\ = \sum_{\beta\in J_{r_2,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{B}^*\mathbf{B})_{.p}\left(\widetilde{\mathbf{c}}_{.j}\right)\right)_\beta^\beta.\end{aligned} \tag{73}$$

Construct the matrix $\Upsilon = (\upsilon_{pj}) \in \mathbb{H}^{k\times n}$ such that

$$\upsilon_{pj} = \sum_{\beta\in J_{r_2,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{B}^*\mathbf{B})_{.p}\left(\widetilde{\mathbf{c}}_{.j}\right)\right)_\beta^\beta. \tag{74}$$

Denote $\mathbf{S}_1\Upsilon = \mathbf{A}^*\mathbf{S}\Upsilon =: \widetilde{\Upsilon} = (\widetilde{\upsilon}_{ij}) \in \mathbb{H}^{n\times n}$. Since

$$\begin{aligned}\sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\mathbf{s}_{.p}^{(1)}\right)\right)_\beta^\beta \upsilon_{pj} \\ = \sum_{\beta\in J_{r_1,n}\{i\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\widetilde{\upsilon}_{.j}\right)\right)_\beta^\beta,\end{aligned} \tag{75}$$

it follows (53).

(iv) Due to Theorem 10 and similarly as above for the first term $\mathbf{Y}_1 = \mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger} = (y_{pg}^{(1)})$ of (44), we have the determinantal representations (55) and (56).

(v) Finally, for the second term $\mathbf{Y}_2 = \mathbf{P}_S\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger} = (y_{pg}^{(2)})$ of (44) using (14) for a determinantal representation of $\mathbf{P}_S$, and due to Theorem 10 for $\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}$, we obtain

$$y_{pg}^{(2)} = \frac{\sum_{t=1}^{k}\sum_{\beta\in J_{r_4,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{S}^*\mathbf{S})_{.p}\left(\ddot{\mathbf{s}}_{.t}\right)\right)_\beta^\beta \omega_{tg}}{\sum_{\beta\in J_{r_4,k}} |\mathbf{S}^*\mathbf{S}|_\beta^\beta \sum_{\beta\in J_{r_2,k}} |\mathbf{B}^*\mathbf{B}|_\beta^\beta \sum_{\alpha\in I_{r_3,k}} |\mathbf{M}^*\mathbf{M}|_\alpha^\alpha}, \tag{76}$$

where ϕ_{tg} are

$$\begin{aligned}\omega_{tg} &= \sum_{\beta\in J_{r_2,k}\{t\}} \operatorname{c\,det}_t\left((\mathbf{B}^*\mathbf{B})_{.t}\left(\mathbf{d}_{.g}^{M}\right)\right)_\beta^\beta \\ &= \sum_{\alpha\in I_{r_3,k}\{g\}} \operatorname{r\,det}_g\left((\mathbf{M}^*\mathbf{M})_{g.}\left(\mathbf{d}_{t.}^{B}\right)\right)_\alpha^\alpha,\end{aligned} \tag{77}$$

$$\begin{aligned}\mathbf{d}_{.g}^{M} &= \left[\sum_{\alpha\in I_{r_3,k}\{g\}} \operatorname{r\,det}_g\left((\mathbf{M}^*\mathbf{M})_{g.}\left(\mathbf{c}_{q.}^{(4,*)}\right)\right)_\alpha^\alpha\right] \in \mathbb{H}^{k\times 1}, \\ &\qquad q = 1,\ldots,k, \\ \mathbf{d}_{t.}^{B} &= \left[\sum_{\beta\in J_{r_3,k}\{t\}} \operatorname{c\,det}_t\left((\mathbf{B}^*\mathbf{B})_{.t}\left(\mathbf{c}_{.l}^{(4,*)}\right)\right)_\beta^\beta\right] \in \mathbb{H}^{1\times k}, \\ &\qquad l = 1,\ldots,k,\end{aligned} \tag{78}$$

are the column vector and the row vector, respectively. $\mathbf{c}_{q.}^{(4,*)}$ and $\mathbf{c}_{.l}^{(4,*)}$ are the qth row and the lth column of $\mathbf{C}_4^* := \mathbf{M}^*\mathbf{C}\mathbf{B}$.

Construct the matrix $\boldsymbol{\Omega} = (\omega_{tg}) \in \mathbb{H}^{k\times k}$ such that ω_{tg} are obtained by (77), and denote $\widetilde{\boldsymbol{\Omega}} := \mathbf{S}^*\mathbf{S}\boldsymbol{\Omega}$. Since

$$\begin{aligned}\sum_{t=1}^{k}\sum_{\beta\in J_{r_4,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{S}^*\mathbf{S})_{.p}\left(\ddot{\mathbf{s}}_{.t}\right)\right)_\beta^\beta \omega_{tg} \\ = \sum_{\beta\in J_{r_4,k}\{p\}} \operatorname{c\,det}_p\left((\mathbf{S}^*\mathbf{S})_{.p}\left(\widetilde{\omega}_{.g}\right)\right)_\beta^\beta,\end{aligned} \tag{79}$$

it follows (58). □

Due to [24], the following lemma can be generalized to $\mathbb{H}$.

Lemma 18. *Suppose that matrices $\mathbf{A} \in \mathbb{H}^{m\times n}$ and $\mathbf{B} \in \mathbb{H}^{m\times m}$ and $\mathbf{C} \in \mathbb{H}^{m\times m}$ are given with $\mathbf{C} = \mathbf{C}^* = (-\mathbf{C}^*)$. Then when (4) is solvable, it must have Hermitian (skew-Hermitian) solutions.*

The general Hermitian solution to (4) can be expressed as $\widehat{\mathbf{X}} = (1/2)(\mathbf{X} + \mathbf{X}^*)$, $\widehat{\mathbf{Y}} = (1/2)(\mathbf{Y} + \mathbf{Y}^*)$, where $(\mathbf{X}, \mathbf{Y})$ is an arbitrary solution to (4). Since by Lemma 18 the existence of Hermitian solutions to (4) needs that $\mathbf{C}$ is Hermitian, then

$$\begin{aligned}\mathbf{X}^* &= \mathbf{A}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger} \\ &\quad - \mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger}\mathbf{S}^*\mathbf{A}^{*,\dagger}, \\ \mathbf{Y}^* &= \mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger} + \mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger}\mathbf{P}_S.\end{aligned} \tag{80}$$

It is evident that the determinantal representations of $\widehat{\mathbf{X}} = (\widehat{x}_{ij})$ and $\widehat{\mathbf{Y}} = (\widehat{y}_{ij})$ can be obtained as $\widehat{x}_{ij} = (1/2)(x_{ij} + \overline{x_{ji}})$ for all $i, j = 1,\ldots,n$ and $\widehat{y}_{pg} = (1/2)(y_{pg} + \overline{y_{gp}})$ for all $p, g = 1,\ldots,k$, where x_{ij} and y_{pg} are determined by Theorem 15 and

$$\begin{aligned}\overline{x_{ji}} &= \overline{x_{ji}^{(1)}} - \overline{x_{ji}^{(2)}} - \overline{x_{ji}^{(3)}}, \\ \overline{y_{gp}} &= \overline{y_{gp}^{(1)}} + \overline{y_{gp}^{(2)}}.\end{aligned} \tag{81}$$

Moreover, $\overline{x_{ji}^{(\gamma)}}$ for all $\gamma = 1, 2, 3$ has the following determinantal representations.

(i) $\overline{x_{ji}^{(1)}} = x_{ij}^{(1)}$.

(ii)

$$\overline{x_{ji}^{(2)}} = \frac{\sum_{\beta\in J_{r_1,n}\{j\}} \operatorname{c\,det}_i\left((\mathbf{A}^*\mathbf{A})_{.i}\left(\widetilde{\phi}_{.j}^*\right)\right)_\beta^\beta}{\left(\sum_{\beta\in J_{r_1,n}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta\right)^2 \sum_{\beta\in J_{r_3,k}} |\mathbf{M}^*\mathbf{M}|_\beta^\beta}, \tag{82}$$

where $\widetilde{\phi}_{.j}^*$ is the jth column of $\widetilde{\boldsymbol{\Phi}}^* = \mathbf{A}^*\mathbf{C}\boldsymbol{\Phi}^*$ that is Hermitian adjoint to $\widetilde{\boldsymbol{\Phi}}$ from the point (i) of Theorem 17, and $\boldsymbol{\Phi}^* = (\phi_{qj}^*) \in \mathbb{H}^{n\times m}$ is such that

$$\begin{aligned}\phi_{qj}^* &= \sum_{\alpha\in I_{r_1,n}\{j\}} \operatorname{r\,det}_j\left((\mathbf{A}^*\mathbf{A})_{j.}\left(\zeta_{q.}^{M}\right)\right)_\alpha^\alpha \\ &= \sum_{\beta\in J_{r_3,m}\{q\}} \operatorname{c\,det}_q\left((\mathbf{M}\mathbf{M}^*)_{.q}\left(\zeta_{.j}^{A}\right)\right)_\beta^\beta,\end{aligned} \tag{83}$$

$$\zeta^M_{q.} = \left[\sum_{\beta\in J_{r_3,m}\{q\}} \mathrm{c}\det_q\left(\left(\mathbf{M}\mathbf{M}^*\right)_{.q}\left(\mathbf{b}^{(1,*)}_{.f}\right)\right)^\beta_\beta\right] \in \mathbb{H}^{1\times n},$$
$$f = 1,\ldots,n, \quad (84)$$
$$\zeta^A_{.j} = \left[\sum_{\alpha\in I_{r_1,n}\{j\}} \mathrm{r}\det_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{b}^{(1,*)}_{s.}\right)\right)^\alpha_\alpha\right] \in \mathbb{H}^{1\times m},$$
$$s = m,\ldots,1,$$

are the row vector and the column vector, respectively. $\mathbf{b}^{(1,*)}_{.f}$ and $\mathbf{b}^{(1)}_{s.}$ are the fth column and the sth row of $\mathbf{B}^*_1 = \mathbf{M}\mathbf{B}^*\mathbf{A}$.

(iii)

$$\overline{x^{(3)}_{ji}} = \frac{\sum_{\alpha\in I_{r_1,n}\{j\}} \mathrm{r}\det_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\widetilde{\upsilon}^*_{i.}\right)\right)^\alpha_\alpha}{\left(\sum_{\alpha\in I_{r_1,n}} |\mathbf{A}^*\mathbf{A}|^\alpha_\alpha\right)^2 \sum_{\beta\in J_{r_2,k}} |\mathbf{B}^*\mathbf{B}|^\alpha_\alpha \sum_{\beta\in I_{r_3,m}} |\mathbf{M}\mathbf{M}^*|^\alpha_\alpha}, \quad (85)$$

where $\widetilde{\upsilon}^*_{i.}$ is the ith row of $\widetilde{\mathbf{\Upsilon}}^* = \mathbf{\Upsilon}^*\mathbf{S}^*\mathbf{A}$, the matrix $\mathbf{\Upsilon}^* = (\upsilon^*_{ip}) \in \mathbb{H}^{n\times k}$ such that

$$\upsilon^*_{ip} = \sum_{\alpha\in I_{r_2,k}\{p\}} \mathrm{r}\det_p\left(\left(\mathbf{B}^*\mathbf{B}\right)_{p.}\left(\widetilde{\mathbf{c}}^*_{i.}\right)\right)^\alpha_\alpha, \quad (86)$$

where $\widetilde{\mathbf{c}}^*_{i.}$ is the ith row of $\widetilde{\mathbf{C}}^* = \Phi\mathbf{C}\mathbf{B}$, and Φ is obtained by (51).

Similarly, $\overline{y^{(\delta)}_{gp}}$ for all $\delta = 1, 2$ has the following determinantal representations.

(i)

$$\overline{y^{(1)}_{gp}} = \frac{\sum_{\alpha\in I_{r_3,k}\{g\}} \mathrm{r}\det_g\left(\left(\mathbf{M}^*\mathbf{M}\right)_{g.}\left(\mathbf{d}^B_{p.}\right)\right)^\alpha_\alpha}{\sum_{\alpha\in I_{r_3,k}} |\mathbf{M}^*\mathbf{M}|^\alpha_\alpha \sum_{\beta\in J_{r_2,k}} |\mathbf{B}^*\mathbf{B}|^\beta_\beta} = \frac{\sum_{\beta\in J_{r_2,k}\{p\}} \mathrm{c}\det_p\left(\left(\mathbf{B}^*\mathbf{B}\right)_{.p}\left(\mathbf{d}^M_{.g}\right)\right)^\beta_\beta}{\sum_{\alpha\in I_{r_3,k}} |\mathbf{M}^*\mathbf{M}|^\alpha_\alpha \sum_{\beta\in J_{r_2,k}} |\mathbf{B}^*\mathbf{B}|^\beta_\beta}, \quad (87)$$

where

$$\mathbf{d}^B_{p.} = \left[\sum_{\beta\in J_{r_2,k}\{p\}} \mathrm{c}\det_p\left(\left(\mathbf{B}^*\mathbf{B}\right)_{.p}\left(\mathbf{c}^{(4,*)}_{.q}\right)\right)^\beta_\beta\right] \in \mathbb{H}^{1\times k},$$
$$q = 1,\ldots,k, \quad (88)$$
$$\mathbf{d}^M_{.g} = \left[\sum_{\alpha\in I_{r_3,k}\{g\}} \mathrm{r}\det_g\left(\left(\mathbf{M}^*\mathbf{M}\right)_{g.}\left(\mathbf{c}^{(4,*)}_{l.}\right)\right)^\alpha_\alpha\right] \in \mathbb{H}^{k\times 1},$$
$$l = 1,\ldots,k,$$

are the row vector and the column vector, respectively. $\mathbf{c}^{(4,*)}_{.q}$ and $\mathbf{c}^{(4,*)}_{l.}$ are the qth column and the lth row of $\mathbf{C}^*_4 := \mathbf{B}^*\mathbf{C}\mathbf{M}$.

(ii)

$$\overline{y^{(2)}_{gp}} = \frac{\sum_{\alpha\in I_{r_4,k}\{p\}} \mathrm{r}\det_p\left(\left(\mathbf{S}^*\mathbf{S}\right)_{p.}\left(\widetilde{\omega}^*_{g.}\right)\right)^\alpha_\alpha}{\sum_{\alpha\in I_{r_4,k}} |\mathbf{S}^*\mathbf{S}|^\alpha_\alpha \sum_{\beta\in J_{r_2,m}} |\mathbf{B}^*\mathbf{B}|^\beta_\beta \sum_{\alpha\in I_{r_3,m}} |\mathbf{M}\mathbf{M}^*|^\alpha_\alpha}, \quad (89)$$

where $\widetilde{\mathbf{\Omega}}^* = \mathbf{\Omega}^*\mathbf{S}^*\mathbf{S}$ and $\mathbf{\Omega}^* = (\omega^*_{gt})$ such that

$$\omega^*_{gt} = \sum_{\beta\in J_{r_3,k}\{g\}} \mathrm{c}\det_g\left(\left(\mathbf{M}^*\mathbf{M}\right)_{.g}\left(\mathbf{d}^B_{.t}\right)\right)^\beta_\beta = \sum_{\alpha\in I_{r_2,k}\{t\}} \mathrm{r}\det_t\left(\left(\mathbf{B}^*\mathbf{B}\right)_{t.}\left(\mathbf{d}^M_{g.}\right)\right)^\alpha_\alpha,$$

$$\mathbf{d}^B_{.t} = \left[\sum_{\alpha\in I_{r_2,k}\{t\}} \mathrm{r}\det_t\left(\left(\mathbf{B}^*\mathbf{B}\right)_{t.}\left(\mathbf{c}^{(4)}_{q.}\right)\right)^\alpha_\alpha\right] \in \mathbb{H}^{1\times k},$$
$$q = 1,\ldots,k, \quad (90)$$
$$\mathbf{d}^M_{g.} = \left[\sum_{\beta\in J_{r_3,k}\{p\}} \mathrm{c}\det_g\left(\left(\mathbf{M}^*\mathbf{M}\right)_{.g}\left(\mathbf{c}^{(4)}_{.l}\right)\right)^\beta_\beta\right] \in \mathbb{H}^{k\times 1},$$
$$l = 1,\ldots,k,$$

are the row vector and the column vector, respectively. $\mathbf{c}^{(4)}_{.q}$ and $\mathbf{c}^{(4)}_{l.}$ are the qth column and the lth row of $\mathbf{C}_4 := \mathbf{M}^*\mathbf{C}\mathbf{B}$.

Remark 19. By Lemma 18, if $\mathbf{C} = -\mathbf{C}^*$ and (4) is solvable, then it has skew-Hermitian solutions. The general skew-Hermitian solution to (4) can be expressed as $\widetilde{\mathbf{X}} = (1/2)(\mathbf{X} - \mathbf{X}^*)$, $\widetilde{\mathbf{Y}} = (1/2)(\mathbf{Y} - \mathbf{Y}^*)$, where $(\mathbf{X}, \mathbf{Y})$ is an arbitrary solution to (4). So due to the above one we can obtain corresponding determinantal representations of skew-Hermitian solution.

4. An Example

In this section, we give an example to illustrate our results. Let us consider the matrix equation

$$\mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{B}\mathbf{Y}\mathbf{B}^* = \mathbf{C}. \quad (91)$$

where

$$\mathbf{A} = \begin{bmatrix} -\mathbf{j}+\mathbf{k} & 1+\mathbf{i} \\ 1+\mathbf{i} & \mathbf{j}-\mathbf{k} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1-\mathbf{k} \\ 2+\mathbf{i} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{k} & \mathbf{i} \\ -\mathbf{i} & \mathbf{k} \end{bmatrix}. \quad (92)$$

Since $\det \mathbf{A}^*\mathbf{A} = \det \begin{bmatrix} 4 & -4\mathbf{k} \\ 4\mathbf{k} & 4 \end{bmatrix} = 0$, then $\operatorname{rank} \mathbf{A} = 1$, and, evidently, $\operatorname{rank} \mathbf{B} = 1$. By Theorem 4, one can find

$$\mathbf{A}^\dagger = \frac{1}{8}\begin{bmatrix} \mathbf{j}-\mathbf{k} & 1-\mathbf{i} \\ 1-\mathbf{i} & -\mathbf{j}+\mathbf{k} \end{bmatrix},$$
$$\mathbf{B}^\dagger = \frac{1}{7}\begin{bmatrix} 1+\mathbf{k} & 2-\mathbf{i} \end{bmatrix}, \qquad (93)$$
$$\mathbf{M} = \frac{1}{8}\begin{bmatrix} 0.5+\mathbf{j}-\mathbf{k} \\ 1+\mathbf{i}-0.5\mathbf{j} \end{bmatrix},$$

and $\mathbf{S} = \mathbf{0}$. It is easy to check that (91) is consistent. First, we can find the solution to (91) by direct calculation. By (43),

$$\begin{aligned} \mathbf{X} &= \mathbf{A}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{B}\mathbf{M}^\dagger\mathbf{C}\mathbf{A}^{*,\dagger} - \mathbf{A}^\dagger\mathbf{S}\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger}\mathbf{B}^*\mathbf{A}^{*,\dagger} \\ &= \mathbf{0} - \frac{1}{72}\begin{bmatrix} -2+4\mathbf{i}+3\mathbf{j}-4\mathbf{k} & -2-3\mathbf{i}+2\mathbf{j}+2\mathbf{k} \\ 4-3\mathbf{i}+4\mathbf{j}-2\mathbf{k} & -2-4\mathbf{i}-3\mathbf{j}-4\mathbf{k} \end{bmatrix} \\ &\quad - \mathbf{0} = \frac{1}{72}\begin{bmatrix} 2-4\mathbf{i}-3\mathbf{j}+4\mathbf{k} & 2+3\mathbf{i}-2\mathbf{j}-2\mathbf{k} \\ -4+3\mathbf{i}-4\mathbf{j}+2\mathbf{k} & 2+4\mathbf{i}+3\mathbf{j}+4\mathbf{k} \end{bmatrix}, \end{aligned} \qquad (94)$$

and by (43) $\mathbf{Y} = \mathbf{M}^\dagger\mathbf{C}\mathbf{B}^{*,\dagger} + \mathbf{P}_S\mathbf{B}^\dagger\mathbf{C}\mathbf{M}^{*,\dagger} = (1/63)[-4 + 8\mathbf{j} + 10\mathbf{k}]$.

Now, we find the solution to (91) by determinantal representations. So,

$$\mathbf{M}\mathbf{M}^* = \frac{9}{4}\begin{bmatrix} 1 & \mathbf{j} \\ -\mathbf{j} & 1 \end{bmatrix},$$
$$\mathbf{C}_2 := \mathbf{C}\mathbf{A} = 2\begin{bmatrix} -1+\mathbf{i} & \mathbf{j}+\mathbf{k} \\ \mathbf{j}+\mathbf{k} & 1-\mathbf{i} \end{bmatrix}, \qquad (95)$$
$$\begin{aligned} \mathbf{B}_1 &= \mathbf{A}^*\mathbf{B}\mathbf{M}^* \\ &= \frac{1}{2}\begin{bmatrix} 6-2\mathbf{i}+\mathbf{j}+7\mathbf{k} & -1-7\mathbf{i}+6\mathbf{j}-2\mathbf{k} \\ -7-\mathbf{i}-2\mathbf{j}+6\mathbf{k} & 2-6\mathbf{i}-7\mathbf{j}-\mathbf{k} \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} \varphi_{11} &= 3-\mathbf{i}+0.5\mathbf{j}+3.5\mathbf{k}, \\ \varphi_{12} &= -0.5-3.5\mathbf{i}+3\mathbf{j}-\mathbf{k}, \end{aligned} \qquad (96)$$

and $(\sum_{\beta\in J_{1,2}} |\mathbf{A}^*\mathbf{A}|_\beta^\beta)^2 = 64$, $\sum_{\alpha\in I_{1,2}} |\mathbf{M}\mathbf{M}^*|_\alpha^\alpha = 4.5$, then

$$\begin{aligned} x_{11} &= -\frac{(6-2\mathbf{i}+\mathbf{j}+7\mathbf{k})(-1+\mathbf{i}) + (-1-7\mathbf{i}+6\mathbf{j}-2\mathbf{k})(\mathbf{j}+\mathbf{k})}{288} \\ &= \frac{1}{36} - \frac{\mathbf{i}}{18} - \frac{\mathbf{j}}{24} + \frac{\mathbf{k}}{18}. \end{aligned} \qquad (97)$$

So x_{11} obtained by Cramer's rule and the matrix method (94) are equal.

Similarly, we can obtain for all x_{ij}, $i, j = 1, 2$ and y_{11}.

5. Conclusions

Within the framework of the theory of row-column determinants, we have derived explicit determinantal representation formulas (analogs of Cramer's rule) of the general, Hermitian, and skew-Hermitian solutions to the Sylvester-type matrix equation $\mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{B}\mathbf{Y}\mathbf{B}^* = \mathbf{C}$ over the quaternion skew field. To accomplish that goal, we have used determinantal representations of the Moore-Penrose matrix inverse, which were previously introduced by the author.

Data Availability

The data used to support the findings of this study are included within the article titled "Determinantal Representations of General Solutions to the Generalized Sylvester Quaternion Matrix Equation and Its Type". The prior studies (and datasets) are cited at relevant places within the text.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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